3 Gauss map and continued fractions

In this lecture we will introduce the Gauss map, which is very important for its connection with continued fractions in number theory.

The *Gauss map* $G : [0,1] \to [0,1]$ is the following map:

$$G(x) = \begin{cases} 0 & \text{if } x = 0 \\ \left\{\frac{1}{x}\right\} = \frac{1}{x} \mod 1 & \text{if } 0 < x \le 1 \end{cases}$$

Here $\{x\}$ denotes the *fractional part* of x. We can write $\{x\} = x - [x]$ where $[x]$ is the integer part. Equivalently, $\{x\} = x \mod 1$.

Remark that

$$\left[\frac{1}{x}\right] = n \quad \Leftrightarrow \quad n \le \frac{1}{x} < n+1 \quad \Leftrightarrow \quad \frac{1}{n+1} < x \le \frac{1}{n}.$$

Thus, explicitely, one has the following expression (see the graph in Figure 1):

$$G(x) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{x} - n & \text{if } \frac{1}{n+1} < x \le \frac{1}{n} \end{cases} \qquad \text{for } n \in \mathbb{N}.$$

The rescrition of G to an interval of the form $(1/n+1, 1/n]$ is called *branch*. Each *branch* $G : (1/n+1, 1/n] \to [0,1)$ is monotone, surjective (onto $[0,1)$) and invertible (see Figure 1).

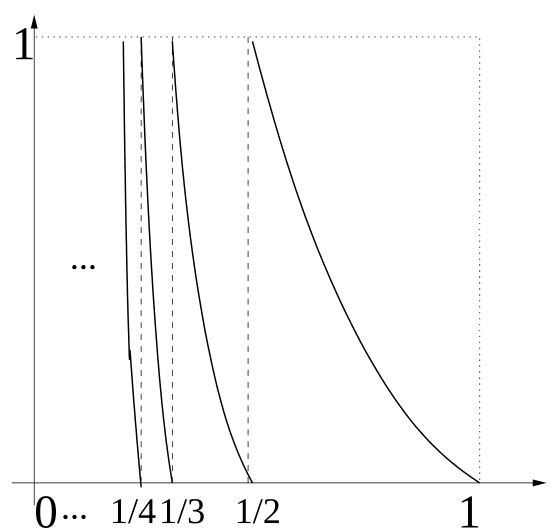

Figure 1: The first branches of the graph of the Gauss map.

The Gauss map is important for its connections with continued fractions.

A *finite continued fraction* (CF will be used as shortening for Continued Fraction) is an expression of the form

$$\frac{1}{a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots \frac{1}{a_n}}}}} \tag{1}$$

where $a_0, a_1, a_2, \dots, a_n \in \mathbb{N}\backslash\{0\}$ are called *entries* of the continued fraction expansion. We will denote the finite continued fraction expansion by $[a_0, a_1, a_2, \dots, a_n]$.

Every finite continued fraction expansion correspond to a rational number p/q (which can be obtained by clearing out denominators).

Example 3.1. *For example*

$$\frac{1}{2+\frac{1}{3}} = \frac{1}{\frac{2\cdot 3+1}{3}} = \frac{3}{7}.$$

Conversely, all rational numbers in $[0,1]$ admit a representation as a finite continued fraction[1].

Example 3.2. *For example*

$$\frac{3}{4} = \frac{1}{1+\frac{1}{3}}, \qquad \frac{49}{200} = \frac{1}{3+\frac{1}{4+\frac{1}{12+\frac{1}{4}}}}.$$

Every *irrational* number $x \in (0,1)$ can be expressed through a (unique) *infinite* continued fraction[2], that we denote by

$$[a_0, a_1, a_2, a_3, \dots] = \frac{1}{a_0 + \frac{1}{a_1+\frac{1}{a_2+\frac{1}{a_3+\dots}}}}.$$

Example 3.3. *For example*

$$\pi = 3 + \frac{1}{7+\frac{1}{15+\frac{1}{1+\frac{1}{293+\dots}}}},$$

$$\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\dots}}}} = \frac{\sqrt{5}-1}{2}.$$

The number $(\sqrt{5}-1)/2$ is known as golden mean[3] *and it has the lowest possible continued fraction entries, all entries equal to one. Similarly, the number whose CF entries are all equal to* 2 *is known as* silver mean.

One can see that a number is rational if and only if the continued fraction expansion is finite.

If x is an irrational number whose infinite continued fraction expansion is $[a_1, a_2, a_3, \dots]$, one can *truncate* the continued fraction expansion at level n and obtain a rational number that we denote p_n/q_n

$$\frac{p_n}{q_n} = [a_0, a_1, a_2, \dots, a_n].$$

These numbers p_n/q_n are called *convergents* of the continued fraction.

Two of the important properties of convergents are the following:

1. One can prove that p_n/q_n converge to x exponentially fast, i.e.

$$\lim_{n\to\infty} \frac{p_n}{q_n} = x \quad \text{and} \quad \left|\frac{p_n}{q_n} - x\right| \le \frac{1}{(\sqrt{2})^n}. \tag{2}$$

   Thus, the fractions p_n/q_n give *rational approximations* of x.

[1] This representatin is not unique: if the last digit a_n of a finite CF is 1, then $[a_0, \dots, a_{n-1}, 1] = [a_0, \dots, a_{n-1}+1]$. If one requires that the last entry is different than one, though, then one can prove that the representation as finite continued fraction is unique.

[2] To be precise, when we write such an infinite continued fraction expression, its value is the limit of the finite continued fraction expansion truncations $[a_0, a_1, a_2, a_3, \dots, a_n]$, each of which is a well defined rational number. One should first prove that this limit exist, see (2).

[3] The inverse of the golden mean is $\frac{\sqrt{5}+1}{2}$, known as *golden ratio*. It appears often in art and in nature since it is considered aesthetically pleasing: for example, the ratio of the width and height of the facade of the Partenon in Athens is exactly the golden ratio and a whole Renessaince treaty, Luca Pacioli's *De divina proportione*, written in 1509, is dedicated to the golden ratio in arts, science and architecture.

2. Convergents give *best approximations* among all rational approximations with denominator up to q_n, that is

$$\left|x - \frac{p_n}{q_n}\right| \leq \left|x - \frac{p}{q}\right|, \quad \forall \ p \in \mathbb{Z}, \quad 0 \leq q \leq q_n.$$

One can also see that the continued fraction expansion of an irrational number is unique.

To find the continued fraction expansion of a number, we will exploit the relation with the symbolic coding of the Gauss map, in the same way that binary expansions are related to the symbolic coding of the doubling map.

Let P_n be the subintervals of $[0,1)$ naturally determined by the domains of the branches of the Gauss map:

$$P_1 = \left(\frac{1}{2}, 1\right], \quad P_2 = \left(\frac{1}{3}, \frac{1}{2}\right], \quad P_3 = \left(\frac{1}{4}, \frac{1}{3}\right], \quad \dots, P_n = \left(\frac{1}{n+1}, \frac{1}{n}\right], \dots$$

Remark that P_n accumulate towards 0 as n increases If we add $P_0 = \{0\}$, the collection $\{P_0, P_1, \dots, P_n, \dots\}$ is a (countable) partition[4] of $[0,1]$.

Theorem 3.1. *Let x be irrational. Let $a_0, a_1, \dots, a_n, \dots$ be the itinerary of $\mathcal{O}_G^+(x)$ with respect to the partition $\{P_0, P_1, P_2, \dots, P_n, \dots\}$, i.e.*

$$x \in P_{a_0}, G(x) \in P_{a_1}, \dots, G^2(x) \in P_{a_2}, \dots, G^k(x) \in P_{a_k}, \dots,$$

Then $x = [a_0, a_1, a_2, \dots, a_n, \dots]$. Thus, itineraries of the Guass map give the entries of the continued fraction expansions.

Remark 3.1. *If x is rational, then there exists n such that $G^n(x) = 0$ and hence $G^m(x) = 0$ for all $m \geq n$. In this case, $G^m(x) \in P_0$ for all $m \geq n$ so the itinerary is eventually zero. The theorem is still true if we consider the beginning of the itineary: the finite itinerary before the tail of 0 gives the entries of the finite continued fraction expansion of x.*

Proof. Let us first remark that

$$x \in P_n \quad \Leftrightarrow \quad \frac{1}{n+1} < x \leq \frac{1}{n} \quad \Leftrightarrow \quad n \leq \frac{1}{x} < n+1 \quad \Leftrightarrow \quad \left[\frac{1}{x}\right] = n. \tag{3}$$

In particular, $a_0 = [1/x]$ since $x \in P_{a_0}$. Thus,

$$G(x) = \left\{\frac{1}{x}\right\} = \frac{1}{x} - \left[\frac{1}{x}\right] = \frac{1}{x} - a_0 \quad \Leftrightarrow \quad x = \frac{1}{a_0 + G(x)}.$$

Let us prove by induction that

$$a_n = \left[\frac{1}{G^n(x)}\right] \quad \text{and} \quad x = \frac{1}{a_0 + \frac{1}{a_1 + \dots \frac{1}{a_n + G^{n+1}(x)}}} = [a_0, a_1, \dots, a_n + G^{n+1}(x)]. \tag{4}$$

We have already shown that this is true for $n = 0$. Assume that it is proved for n and consider $n+1$. Since $G^{n+1}(x) \in P_{a_{n+1}}$ by definition of itinerary, we have $a_{n+1} = \left[\frac{1}{G^n(x)}\right]$ by (3). This proves the first part of (4) for $n+1$. Then, recalling the definition of G we have

$$G^{n+2}(x) = \frac{1}{G^{n+1}(x)} - \left[\frac{1}{G^{n+1}(x)}\right] = \frac{1}{G^{n+1}(x)} - a_{n+1} \qquad \Leftrightarrow \qquad G^{n+1}(x) = \frac{1}{a_{n+1} + G^{n+2}(x)}$$

[4] Recall that a partition is a collection of disjoint sets whose union is the whole space.

so that, plugging that in the second part of the inductive assumption (4) we get

$$x = \frac{1}{a_0 + \dots \frac{1}{a_n + G^{n+1}(x)}} = \frac{1}{a_0 + \dots \frac{1}{a_n + \frac{1}{a_{n+1} + G^{n+2}(x)}}},$$

which proves the second part of (4) for $n+1$. Thus, recursively, the itinerary is producing[5] the infinite continued fraction expansion of x. □

From the proof of the previous theorem, one can see the following.

Remark 3.2. *The Gauss map acts on the digits of the CF expansion as the one-sided shift, that is*

$$\text{if } x = [a_0, a_1, a_2, \dots, a_n, \dots]$$
$$\text{then } G(x) = [a_1, a_2, a_3, \dots, a_{n+1}, \dots].$$

One can characterize in terms of orbits of the Gauss map various class of numbers. For example:

1. *Rational numbers* are exactly the numbers x which have *finite* continued fraction expansion or equivalently such that there exists $n \in \mathbb{N}$ such that $G^n(x) = 0$ (*eventually mapped to zero by the Gauss map*).

2. *Quadratic irrationals*, that is numbers of the form $\frac{a+b\sqrt{c}}{d}$, where a, b, c, d are integers[6], are exactly numbers which have a *eventually periodic* continued fraction expansion or equivalently are *pre-periodic points for the Gauss map.*

In number theory (and in particular in Diophantine approximation) other class of numbers (for example Badly approximable numbers) can be characterized in terms of their continued fraction expansion[7].

Example 3.4. *We have already seen two examples of quadratic irrationals, the golden mean g and the silver mean s:*

$$g = \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \dots}}}} = \frac{\sqrt{5} - 1}{2}, \qquad s = \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \dots}}}} = \sqrt{2} - 1.$$

Both the golden mean and the silver mean are fixed points of the Gauss map: $G(g) = g$, $G(s) = s$. Similarly all other fixed points correspond to numbers whose continued fraction entries are all equal.

Example 3.5. *Let $\alpha = \frac{-3+3\sqrt{5}}{2}$. Then one can check that $\alpha = [2, 3, 2, 3, 2, 3, \dots]$, so that the entries are periodic and the period is 2. Thus $G^2(\alpha) = \alpha$. Explicitely, since we know the itinerary of α, we can write down the equation satisfied by α. We know that*

$$G(\alpha) = \frac{1}{\alpha} - 2, \quad \text{since} \left[\frac{1}{\alpha}\right] = 2, \quad \text{and} \quad G(G(\alpha)) = \frac{1}{G(\alpha)} - 3 \quad \text{since} \quad \left[\frac{1}{G(\alpha)}\right] = 3,$$

[5] One should still prove that the finite continued fractions in (4) do converge, as n tends to infinity and that the limit is x. This can be done by the same method that one can use to show that convergents tend to x exponentially fast.

[6] Equivalently, one can define quadratic irrationals as solutions of equations of degree two with integer coefficients.

[7] One can defined Badly approximable numbers as the numbers for which there exists a number A such that all entries a_n of their continued fraction expansion are bounded by A. In particular, quadratic irrationals are badly approximable.

so that the equation $G^2(\alpha) = \alpha$ becomes

$$\frac{1}{\frac{1}{\alpha} - 2} - 3 = \alpha.$$

Using the ideas in the previous exercise, one can produce quadratic irrationals with any given periodic sequence of CF entries.

Exercise 3.1. *Prove that if $G^n(x) = 0$ then x has a representation as a finite continued fraction expansion and thus it is rational.*

Exercise 3.2. *Prove that if $G^n(x) = x$ then x satisfies an equation of degree two with integer entries. Conclude that x is a quadratic irrational.*