$f : X \to X$, X metric space

μ probability measure, $\mathcal{B}$ = Borel prob.

μ f-invariant if $\mu(f^{-1}(A)) = \mu(A)$ $\forall A \in \mathcal{B}$

thm (Poincare Recurrence, 1890's) ~~Let ...~~ $\mu(A) > 0$, then for μ a.e. x $\exists$ $k = k(x) > 0$ s.t. $f^{k(x)}(x) \in A$.

Proof Let $A_0 := \{x \in A : f^n(x) \notin A \ \forall n \geq 1\}$.

~~Let for~~ $n \geq 0$, let $A_n = f^{-n}(A_0)$

Exercise) $m \neq n \Rightarrow A_m \cap A_n = \emptyset$.

then

$$1 = \mu(X) \geq \mu\left(\bigcup_{n=0}^{\infty} A_n\right) = \sum_{n=0}^{\infty} \mu(A_n) \overset{\text{invariance of } \mu}{=} \sum_{n=0}^{\infty} \mu(A_0) \Rightarrow \mu(A_0) = 0 \ \square$$

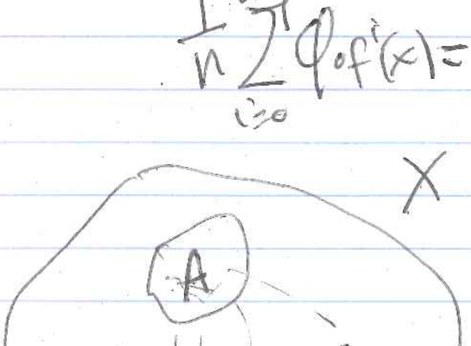
Thm (Birkhoff, 1930's) $\forall \varphi \in L^1(\mu)$, μ a.e. x,

$$\bar{\varphi}(x) := \lim_{n\to\infty} \frac{1}{n} \sum \varphi \circ f^i(x) \quad \text{exists.}$$

Rmk 1 In general the limit depends very much on x. $X = [0,1]$, $f(x) = x$ identity. Then $\bar{\varphi}(x) = \varphi(x)$.

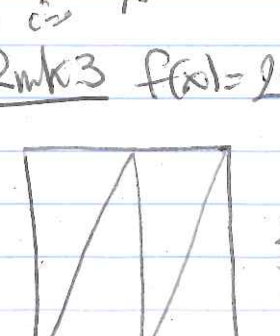
Rmk 2 If $A \in \mathcal{B}$, $\varphi = \mathbb{1}_A$ then

$$\frac{1}{n}\sum_{i=0}^{n-1} \varphi \circ f^i(x) = \frac{1}{n}\sum_{i=0}^{n-1} \mathbb{1}_A \circ f^i(x) = \frac{1}{n} \#\{0 \leq i \leq n-1 : f^i(x) \in A\}$$



Rmk 3 $f(x) = 2x \bmod 1$

$\varphi = \mathbb{1}_{[0,1/2]}$



Exercise 2 Find x_0 s.t. ~~φ ...~~ limit $\bar{\varphi}(x)$ does not exist.

Does every dynamical system have an invariant measure?

Ex 1 $X = [0,1]$ $f(x) = \frac{1}{2}x$

then $f(0) = 0 \Rightarrow \delta_0$ is f-invt

$f^n(x) \to 0$ $\forall x$ $\Rightarrow \delta_0$ is the only f-invt measure.

Ex 2 $X = (0,1)$ $f(x) = \frac{1}{2}x$. No invt measure.

Theorem (Krylov-Bogoliubov, 1937)

X compact metric space, $f: X \to X$ cts $\Rightarrow \exists\, \mu$ f-inv. prob.

$\mathcal{M} = \{$probability measures on $X\}$.

Weak-star topology on $\mathcal{M}$: $\mu_n \to \mu \Leftrightarrow \int \varphi\, d\mu_n \to \int \varphi\, d\mu$ $\forall \varphi$ cts

1. If X is compact then $\mathcal{M}$ is compact in weak-star top.
2. If f is continuous, then $f_*: \mathcal{M} \to \mathcal{M}$ is continuous.

Proof Let $\mu_0 \in \mathcal{M}$. (e.g. $\mu_0 = \delta_x$ for arbitrary $x \in X$)

Let $\mu_n = \frac{1}{n}\sum_{i=0}^{n-1} f^i_*(\mu_0)$ $\left(= \frac{1}{n}\sum_{i=0}^{n-1} \delta_{f^i(x)} \text{ if } \mu_0 = \delta_x\right)$

By compactness of $\mathcal{M}$, $\exists\, \mu \in \mathcal{M}$, $n_k \to \infty$ s.t

$$\mu_{n_k} \to \mu.$$

Exercise 3 Show that $f_* \mu_{n_k} \to \mu$.

Then by continuity of f_*, $f_*\mu_{n_k} \to f_*\mu$

and so $f_*\mu = \mu \Rightarrow \mu$ is f-invt.

Introdução to Ergodic theory 3

C1 Let $A_0 := \{x \in A : f^n(x) \notin A \ \forall n \geq 1\}$, $A_n = f^{-n}(A_0)$
then $m \neq n \Rightarrow A_m \cap A_n = \emptyset$.

Proof Suppose $\exists\, n > m \geq 0$, $A_n \cap A_m \neq \emptyset$, $x \in A_n \cap A_m$.
Then

$$f^n(x) \in f^n(A_n \cap A_m) = f^n(f^{-n}(A_0) \cap f^{-m}(A_0)) = A_0 \cap f^{n-m}(A_0)$$

contradicts defn of A_0.

C2 $f(x) = 2x \bmod 1$, $\varphi = \mathbb{1}_{[0,1/2]}$. Choose x with base 2 expansion containing long blocks of 0's and 1's of lengths increasing by an order of magnitude.

C3

$$f_*\mu_{n_k} = f_*\left(\frac{1}{n_k}\sum_{i=0}^{n_k-1} f_*^i \mu_0\right) = \frac{1}{n_k}\sum_{i=0}^{n_k-1} f_*^{i+1}\mu_0$$

$$= \frac{1}{n_k}\left(\sum_{i=0}^{n_k-1} f_*^i \mu_0 - \mu_0 + f_*^{n_k}\mu_0\right)$$

$$= \frac{1}{n_k}\sum_{i=0}^{n_k-1} f_*^i \mu_0 - \frac{1}{n_k}\mu_0 + \frac{1}{n_k} f_*^{n_k}\mu_0$$

$$= \mu_{n_k} + \underbrace{\frac{\mu_0}{n_0} - \frac{f_*^{n_k}\mu_0}{n_k}}_{\downarrow 0} \longrightarrow \mu.$$