Dynamical systems

Expanding maps on the circle. Classification

Jana Rodriguez Hertz

ICTP

2018

Index

expanding maps are factors of σ

theorem

expanding maps are factors of σ

theorem

- $f : \mathbb{S}^1 \to \mathbb{S}^1$ expanding map

expanding maps are factors of σ

theorem

- $f : \mathbb{S}^1 \to \mathbb{S}^1$ expanding map
- $\deg(f) = 2$

expanding maps are factors of σ

theorem

- $f : \mathbb{S}^1 \to \mathbb{S}^1$ expanding map
- $\deg(f) = 2$
- $\Rightarrow f$ is a factor of σ on Σ_2^+

expanding maps are factors of σ

theorem

- $f : \mathbb{S}^1 \to \mathbb{S}^1$ expanding map
- $\deg(f) = 2$
- $\Rightarrow$ f is a factor of σ on Σ_2^+
- $\exists\, h : \Sigma_2^+ \to \mathbb{S}^1$ such that $f^n(h(\underline{x})) \in \Delta_{x_n}$ for all $n \geq 0$

Index

points of non-injectivity

points of non-injectivity

points of non-injectivity

points of non-injectivity

- if $h(\underline{x}) = h(\underline{y}) = x$

points of non-injectivity

points of non-injectivity

- if $h(\underline{x}) = h(\underline{y}) = x$
- then there exists $n \geq 0$

points of non-injectivity

points of non-injectivity

- if $h(\underline{x}) = h(\underline{y}) = x$
- then there exists $n \geq 0$
- such that

$$f^n(x) = p$$

points of non-injectivity

points of non-injectivity

- if $h(\underline{x}) = h(\underline{y}) = x$
- then there exists $n \geq 0$
- such that

$$f^n(x) = p$$

- where $f(p) = p$

proof

comments on the proof - points of non-injectivity

proof

comments on the proof - points of non-injectivity

- f is injective on Δ_i^o

proof

comments on the proof - points of non-injectivity

- f is injective on Δ_i^o
- $f(\partial\Delta_i) = p$

proof

comments on the proof - points of non-injectivity

- f is injective on Δ_i^o
- $f(\partial\Delta_i) = p$
- if $x \in \Delta_0^o \cup \Delta_1^o$

proof

comments on the proof - points of non-injectivity

- f is injective on Δ_i^o
- $f(\partial\Delta_i) = p$
- if $x \in \Delta_0^o \cup \Delta_1^o$
- then first symbol of $\underline{x}$ such that $h(\underline{x}) = x$

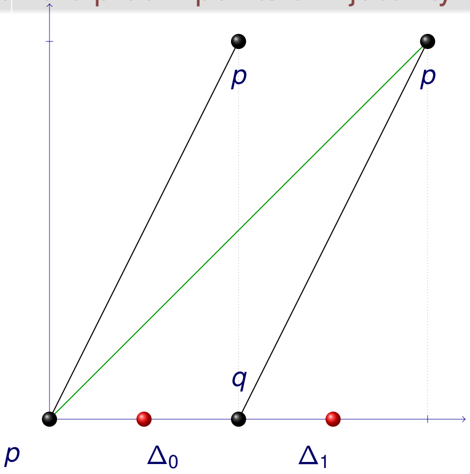
proof

comments on the proof - points of non-injectivity

- f is injective on Δ_i^o
- $f(\partial\Delta_i) = p$
- if $x \in \Delta_0^o \cup \Delta_1^o$
- then first symbol of $\underline{x}$ such that $h(\underline{x}) = x$
- is 0 or 1 (no ambiguity)

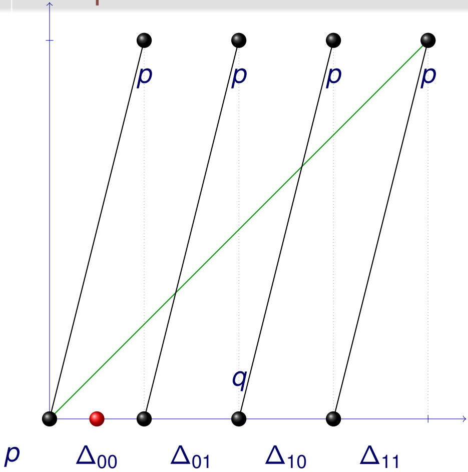
proof

comments on the proof - points of injectivity



proof

comments on the proof - f^2



proof

proof - points of injectivity

proof

proof - points of injectivity

- let $h(\underline{x}) = h(\underline{y}) = x$ with $\underline{x} \neq \underline{y}$

proof

proof - points of injectivity

- let $h(\underline{x}) = h(\underline{y}) = x$ with $\underline{x} \neq \underline{y}$
- let N be the first integer such that $x_N \neq y_N$

proof

proof - points of injectivity

- let $h(\underline{x}) = h(\underline{y}) = x$ with $\underline{x} \neq \underline{y}$
- let N be the first integer such that $x_N \neq y_N$
- then

$$h(\underline{x}) = h(\underline{y}) = x \in \bigcap_{n=0}^{N-1} f^{-n}(\Delta_{x_n})$$

proof

proof - points of injectivity

- let $h(\underline{x}) = h(\underline{y}) = x$ with $\underline{x} \neq \underline{y}$
- let N be the first integer such that $x_N \neq y_N$
- then

$$h(\underline{x}) = h(\underline{y}) = x \in \bigcap_{n=0}^{N-1} f^{-n}(\Delta_{x_n})$$

- which is an interval $\Delta_{x_0 \dots x_{N-1}}$

proof

proof - points of injectivity

proof

proof - points of injectivity

- now N is the first integer such that $x_N \neq y_N$

proof

proof - points of injectivity

- now N is the first integer such that $x_N \neq y_N$
- then
$$h(\underline{x}) = h(\underline{y}) = x \in f^{-N}(\Delta_0) \cap f^{-N}(\Delta_1)$$

proof

proof - points of injectivity

- now N is the first integer such that $x_N \neq y_N$
- then
$$h(\underline{x}) = h(\underline{y}) = x \in f^{-N}(\Delta_0) \cap f^{-N}(\Delta_1)$$
- $\Rightarrow x \in f^{-N}(\Delta_0 \cap \Delta_1)$

proof

proof - points of injectivity

- now N is the first integer such that $x_N \neq y_N$
- then
$$h(\underline{x}) = h(\underline{y}) = x \in f^{-N}(\Delta_0) \cap f^{-N}(\Delta_1)$$
- $\Rightarrow x \in f^{-N}(\Delta_0 \cap \Delta_1)$
- but $\Delta_0 \cap \Delta_1 = \{p, q\}$

proof

proof - points of injectivity

- now N is the first integer such that $x_N \neq y_N$
- then
$$h(\underline{x}) = h(\underline{y}) = x \in f^{-N}(\Delta_0) \cap f^{-N}(\Delta_1)$$
- $\Rightarrow x \in f^{-N}(\Delta_0 \cap \Delta_1)$
- but $\Delta_0 \cap \Delta_1 = \{p, q\}$
- $\Rightarrow f^{N+1}(x) = p$

proof

proof - points of injectivity

- now N is the first integer such that $x_N \neq y_N$
- then
$$h(\underline{x}) = h(\underline{y}) = x \in f^{-N}(\Delta_0) \cap f^{-N}(\Delta_1)$$
- $\Rightarrow x \in f^{-N}(\Delta_0 \cap \Delta_1)$
- but $\Delta_0 \cap \Delta_1 = \{p, q\}$
- $\Rightarrow f^{N+1}(x) = p$ $\square$

Index

classification

theorem

classification

theorem

- let $f, g : \mathbb{S}^1 \to \mathbb{S}^1$ be expanding maps

classification

theorem

- let $f, g : \mathbb{S}^1 \to \mathbb{S}^1$ be expanding maps
- such that $\deg(f) = \deg(g) = 2$

classification

theorem

- let $f, g : \mathbb{S}^1 \to \mathbb{S}^1$ be expanding maps
- such that $\deg(f) = \deg(g) = 2$
- $\Rightarrow$ f and g are topologically conjugate

Index

corollary

classification

corollary

classification

- $f : \mathbb{S}^1 \to \mathbb{S}^1$ expanding map

corollary

classification

- $f : \mathbb{S}^1 \to \mathbb{S}^1$ expanding map
- $\deg(f) = 2$

corollary

classification

- $f : \mathbb{S}^1 \to \mathbb{S}^1$ expanding map
- $\deg(f) = 2$
- $\Rightarrow f$ topologically conjugate to E_2

Index

proof

proof - theorem

proof

proof - theorem

- $f, g : \mathbb{S}^1 \to \mathbb{S}^1$ expanding maps

proof

proof - theorem

- $f, g : \mathbb{S}^1 \to \mathbb{S}^1$ expanding maps
- with $\deg(f) = \deg(g) = 2$

proof

proof - theorem

- $f, g : \mathbb{S}^1 \to \mathbb{S}^1$ expanding maps
- with $\deg(f) = \deg(g) = 2$
- $\Rightarrow \exists\ h_f, h_g : \Sigma_2^+ \to \mathbb{S}^1$ semiconjugacies

proof

proof - theorem

- $f, g : \mathbb{S}^1 \to \mathbb{S}^1$ expanding maps
- with $\deg(f) = \deg(g) = 2$
- $\Rightarrow \exists\ h_f, h_g : \Sigma_2^+ \to \mathbb{S}^1$ semiconjugacies
- such that

proof

proof - theorem

- $f, g : \mathbb{S}^1 \to \mathbb{S}^1$ expanding maps
- with $\deg(f) = \deg(g) = 2$
- $\Rightarrow \exists\ h_f, h_g : \Sigma_2^+ \to \mathbb{S}^1$ semiconjugacies
- such that
  1. $f \circ h_f = h_f \circ \sigma$

proof

proof - theorem

- $f, g : \mathbb{S}^1 \to \mathbb{S}^1$ expanding maps
- with $\deg(f) = \deg(g) = 2$
- $\Rightarrow \exists\ h_f, h_g : \Sigma_2^+ \to \mathbb{S}^1$ semiconjugacies
- such that
  1. $f \circ h_f = h_f \circ \sigma$
  2. $g \circ h_g = h_g \circ \sigma$

proof

definition of h

proof

definition of *h*

- let us define $h : \mathbb{S}^1 \to \mathbb{S}^1$ such that

proof

definition of h

- let us define $h : \mathbb{S}^1 \to \mathbb{S}^1$ such that
- $$h(x) = h_g(h_f^{-1}(x))$$

proof

h is well-defined, case 1

proof

h is well-defined, case 1

- Case 1: $h_f^{-1}(x)$ consists of a single point

proof

h is well-defined, case 1

- Case 1: $h_f^{-1}(x)$ consists of a single point
- $\Rightarrow h$ is well-defined

proof

h is well-defined, case 1

- Case 1: $h_f^{-1}(x)$ consists of a single point
- $\Rightarrow$ h is well-defined $\surd$

proof

h is well-defined, case 2

proof

h is well-defined, case 2

- $\exists \underline{x} \neq \underline{y}$ such that $h_f(\underline{x}) = h_f(\underline{y}) = x$

proof

h is well-defined, case 2

- $\exists \underline{x} \neq \underline{y}$ such that $h_f(\underline{x}) = h_f(\underline{y}) = x$
- $\Rightarrow \exists N$ such that $f^N(x) = p_f$, with $f(p_f) = p_f$

h is well-defined, case 2

h is well-defined, case 2

- $f^N(x) = p_f$

h is well-defined, case 2

- $f^N(x) = p_f$
- $h_f(\underline{x}) = x$

h is well-defined, case 2

- $f^N(x) = p_f$
- $h_f(\underline{x}) = x$
- $\iff \sigma^N(\underline{x}) = 0000000\ldots$ or $\sigma^N(\underline{x}) = 1111111\ldots$

h is well-defined, case 2

- $f^N(x) = p_f$
- $h_f(\underline{x}) = x$
- $\iff \sigma^N(\underline{x}) = 0000000\ldots$ or $\sigma^N(\underline{x}) = 1111111\ldots$
- otherwise
$$f^n(x) = p \in \Delta_{01} \cup \Delta_{10}$$
for some $n \geq N$

h is well-defined, case 2

- $f^N(x) = p_f$
- $h_f(\underline{x}) = x$
- $\iff \sigma^N(\underline{x}) = 0000000\ldots$ or $\sigma^N(\underline{x}) = 1111111\ldots$
- otherwise

$$f^n(x) = p \in \Delta_{01} \cup \Delta_{10}$$

  for some $n \geq N \rightarrow$ contradiction

proof

h is well defined, case 2

proof

h is well defined, case 2

- $h_f(\underline{x}) = h_f(\underline{y}) = x$ with $\underline{x} \neq \underline{y}$

proof

h is well defined, case 2

- $h_f(\underline{x}) = h_f(\underline{y}) = x$ with $\underline{x} \neq \underline{y}$
- $N \geq 0$ the first such that $f^N(x) = p$

proof

h is well defined, case 2

- $h_f(\underline{x}) = h_f(\underline{y}) = x$ with $\underline{x} \neq \underline{y}$
- $N \geq 0$ the first such that $f^N(x) = p$
- $\Rightarrow x_n = 0$ and $y_n = 1$ for all $n \geq N$

proof

h is well defined, case 2

- $h_f(\underline{x}) = h_f(\underline{y}) = x$ with $\underline{x} \neq \underline{y}$
- $N \geq 0$ the first such that $f^N(x) = p$
- $\Rightarrow x_n = 0$ and $y_n = 1$ for all $n \geq N$
- $x_{N-1} = 1$ and $y_{N-1} = 0$

proof

h is well defined, case 2

- $h_f(\underline{x}) = h_f(\underline{y}) = x$ with $\underline{x} \neq \underline{y}$
- $N \geq 0$ the first such that $f^N(x) = p$
- $\Rightarrow x_n = 0$ and $y_n = 1$ for all $n \geq N$
- $x_{N-1} = 1$ and $y_{N-1} = 0$
- $x_n = y_n$ for all $n \leq N - 2$

proof

h is well defined, case 2

proof

h is well defined, case 2

- let us show $h_g(\underline{x}) = h_g(\underline{y})$

proof

h is well defined, case 2

- let us show $h_g(\underline{x}) = h_g(\underline{y})$
- $h_g(\underline{x}) \in \Delta^g_{x_0 \ldots x_{N-2} 10000\ldots}$

proof

h is well defined, case 2

- let us show $h_g(\underline{x}) = h_g(\underline{y})$
- $h_g(\underline{x}) \in \Delta^g_{x_0 \ldots x_{N-2} 10000\ldots}$
- $h_g(\underline{y}) \in \Delta^g_{x_0 \ldots x_{N-2} 011111\ldots}$

proof

h is well defined, case 2

- let us show $h_g(\underline{x}) = h_g(\underline{y})$
- $h_g(\underline{x}) \in \Delta^g_{x_0 \dots x_{N-2} 10000\dots}$
- $h_g(\underline{y}) \in \Delta^g_{x_0 \dots x_{N-2} 011111\dots}$
- $\Rightarrow h_g(\underline{x}), h_g(\underline{y}) \in \Delta_{x_1 \dots x_{N-2}} = [a_{N-2}, b_{N-2}]$

proof

h is well defined, case 2

- let us show $h_g(\underline{x}) = h_g(\underline{y})$
- $h_g(\underline{x}) \in \Delta^g_{x_0 \dots x_{N-2} 10000\dots}$
- $h_g(\underline{y}) \in \Delta^g_{x_0 \dots x_{N-2} 011111\dots}$
- $\Rightarrow h_g(\underline{x}), h_g(\underline{y}) \in \Delta_{x_1 \dots x_{N-2}} = [a_{N-2}, b_{N-2}]$
- g^{N-1} is injective in (a_{N-2}, b_{N-2})

proof

h is well defined, case 2

- let us show $h_g(\underline{x}) = h_g(\underline{y})$
- $h_g(\underline{x}) \in \Delta^g_{x_0 \dots x_{N-2} 10000\dots}$
- $h_g(\underline{y}) \in \Delta^g_{x_0 \dots x_{N-2} 011111\dots}$
- $\Rightarrow h_g(\underline{x}), h_g(\underline{y}) \in \Delta_{x_1 \dots x_{N-2}} = [a_{N-2}, b_{N-2}]$
- g^{N-1} is injective in (a_{N-2}, b_{N-2})
- $\exists!\ r \in (a_{N-2}, b_{N-2})$ such that $g^{N-1}(r) = q_g$

proof

h is well defined, case 2

- let us show $h_g(\underline{x}) = h_g(\underline{y})$
- $h_g(\underline{x}) \in \Delta^g_{x_0 \ldots x_{N-2} 10000\ldots}$
- $h_g(\underline{y}) \in \Delta^g_{x_0 \ldots x_{N-2} 011111\ldots}$
- $\Rightarrow h_g(\underline{x}), h_g(\underline{y}) \in \Delta_{x_1 \ldots x_{N-2}} = [a_{N-2}, b_{N-2}]$
- g^{N-1} is injective in (a_{N-2}, b_{N-2})
- $\exists!\ r \in (a_{N-2}, b_{N-2})$ such that $g^{N-1}(r) = q_g$
- $h_g(\underline{x}) \in [r, b_{N-2}]$ and $h_g(\underline{y}) \in [a_{N-2}, r]$

proof

h is well defined, case 2

proof

h is well defined, case 2

- $h_g(\underline{x}) \in [r, b_{N-2}] \cap \bigcap_{n \geq N} g^{-N}(\Delta_0)$

proof

h is well defined, case 2

- $h_g(\underline{x}) \in [r, b_{N-2}] \cap \bigcap_{n \geq N} g^{-N}(\Delta_0) = r$

proof

h is well defined, case 2

- $h_g(\underline{x}) \in [r, b_{N-2}] \cap \bigcap_{n \geq N} g^{-N}(\Delta_0) = r$
- $h_g(\underline{y}) \in [a_{N/2}, r] \cap \bigcap_{n \geq N} g^{-N}(\Delta_1)$

proof

h is well defined, case 2

- $h_g(\underline{x}) \in [r, b_{N-2}] \cap \bigcap_{n \geq N} g^{-N}(\Delta_0) = r$
- $h_g(\underline{y}) \in [a_{N/2}, r] \cap \bigcap_{n \geq N} g^{-N}(\Delta_1) = r$

proof

h is well defined, case 2

- $h_g(\underline{x}) \in [r, b_{N-2}] \cap \bigcap_{n \geq N} g^{-N}(\Delta_0) = r$
- $h_g(\underline{y}) \in [a_{N/2}, r] \cap \bigcap_{n \geq N} g^{-N}(\Delta_1) = r$
- $\Rightarrow$ *h* is well-defined.

proof

h is continuous

proof

h is continuous

- let x be such that $f^n(x) \neq p_f$ for all $n \geq 0$

proof

h is continuous

- let x be such that $f^n(x) \neq p_f$ for all $n \geq 0$
- take $N > 0$ such that $d(\underline{x}, \underline{y}) < \frac{1}{3^N} \Rightarrow d(h_g(\underline{x}), h_g(\underline{y})) < \varepsilon$

proof

h is continuous

- let x be such that $f^n(x) \neq p_f$ for all $n \geq 0$
- take $N > 0$ such that $d(\underline{x}, \underline{y}) < \frac{1}{3^N} \Rightarrow d(h_g(\underline{x}), h_g(\underline{y})) < \varepsilon$
- $x = \bigcap_{n=0}^{\infty} f^{-n}(\Delta_{x_n})$ is in the interior of $\bigcap_{n=0}^{N} f^{-n}(\Delta_{x_n})$

proof

h is continuous

- let x be such that $f^n(x) \neq p_f$ for all $n \geq 0$
- take $N > 0$ such that $d(\underline{x}, \underline{y}) < \frac{1}{3^N} \Rightarrow d(h_g(\underline{x}), h_g(\underline{y})) < \varepsilon$
- $x = \bigcap_{n=0}^{\infty} f^{-n}(\Delta_{x_n})$ is in the interior of $\bigcap_{n=0}^{N} f^{-n}(\Delta_{x_n})$
- $\Rightarrow$ there is $\delta > 0$ such that $d(x, y) < \delta \Rightarrow d(h(x), h(y)) < \varepsilon$

proof

h is continuous

proof

h is continuous

- let x be such that $f^K(x) = p_f$ for some $K > 0$

proof

h is continuous

- let x be such that $f^K(x) = p_f$ for some $K > 0$
- $\Rightarrow h_f^{-1}(x) = \{\underline{x}, \underline{y}\}$ such that

proof

h is continuous

- let x be such that $f^K(x) = p_f$ for some $K > 0$
- $\Rightarrow h_f^{-1}(x) = \{\underline{x}, \underline{y}\}$ such that
- $\underline{x} = x_0 \dots x_{K-2}011111\dots$ and $\underline{y} = x_0 \dots x_{K-2}100000\dots$

proof

h is continuous

- let x be such that $f^K(x) = p_f$ for some $K > 0$
- $\Rightarrow h_f^{-1}(x) = \{\underline{x}, \underline{y}\}$ such that
- $\underline{x} = x_0 \dots x_{K-2}011111\dots$ and $\underline{y} = x_0 \dots x_{K-2}100000\dots$
- take $\varepsilon > 0$ and $N > 0$ and take $y > x$

proof

h is continuous

- let x be such that $f^K(x) = p_f$ for some $K > 0$
- $\Rightarrow h_f^{-1}(x) = \{\underline{x}, \underline{y}\}$ such that
- $\underline{x} = x_0 \dots x_{K-2}011111\dots$ and $\underline{y} = x_0 \dots x_{K-2}100000\dots$
- take $\varepsilon > 0$ and $N > 0$ and take $y > x$
- if $y \in \Delta_{x_0 \dots x_{K-2}1000}$ (with N subsymbols)

proof

h is continuous

- let x be such that $f^K(x) = p_f$ for some $K > 0$
- $\Rightarrow h_f^{-1}(x) = \{\underline{x}, \underline{y}\}$ such that
- $\underline{x} = x_0 \dots x_{K-2}011111 \dots$ and $\underline{y} = x_0 \dots x_{K-2}100000 \dots$
- take $\varepsilon > 0$ and $N > 0$ and take $y > x$
- if $y \in \Delta_{x_0 \dots x_{K-2}1000}$ (with N subsymbols)
- then $d(h(x), h(y)) < \varepsilon$

proof

h is continuous

proof

h is continuous

- analogously, we take $y < x$

proof

h is continuous

- analogously, we take $y < x$
- if $y \in \Delta_{x_0 \ldots x_{K-2}011111}$ (with N subsymbols)

proof

h is continuous

- analogously, we take $y < x$
- if $y \in \Delta_{x_0 \dots x_{K-2} 011111}$ (with N subsymbols)
- then $d(h_f^{-1}(y), \underline{y}) < \frac{1}{3^N}$ and then

proof

h is continuous

- analogously, we take $y < x$
- if $y \in \Delta_{x_0 \ldots x_{K-2} 011111}$ (with N subsymbols)
- then $d(h_f^{-1}(y), \underline{y}) < \frac{1}{3^N}$ and then
- $d(h(y), h(x)) = d(h_g(h_f^{-1}(y)), h_g(\underline{y})) < \varepsilon$

proof

conclusion

proof

conclusion

- it is easy to see that h^{-1} is well defined and continuous.

proof

conclusion

- it is easy to see that h^{-1} is well defined and continuous.
- and

$$h \circ f =$$

proof

conclusion

- it is easy to see that h^{-1} is well defined and continuous.
- and

$$h \circ f = h_g \circ h_f^{-1} \circ f =$$

proof

conclusion

- it is easy to see that h^{-1} is well defined and continuous.
- and

$$h \circ f = h_g \circ h_f^{-1} \circ f = h_g \circ \sigma \circ h_f^{-1} =$$

proof

conclusion

- it is easy to see that h^{-1} is well defined and continuous.
- and

$$h \circ f = h_g \circ h_f^{-1} \circ f = h_g \circ \sigma \circ h_f^{-1} = g \circ h_g \circ h_f^{-1} =$$

proof

conclusion

- it is easy to see that h^{-1} is well defined and continuous.
- and

$$h \circ f = h_g \circ h_f^{-1} \circ f = h_g \circ \sigma \circ h_f^{-1} = g \circ h_g \circ h_f^{-1} = g \circ h$$

proof

conclusion

- it is easy to see that h^{-1} is well defined and continuous.
- and

$$h \circ f = h_g \circ h_f^{-1} \circ f = h_g \circ \sigma \circ h_f^{-1} = g \circ h_g \circ h_f^{-1} = g \circ h$$

□