# Ladder diagrams and Wilson loop correlators

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Based on
JHEP 05 (2018) 168, with P. Pisani, A. Rios Fukelman
Work in progress with P. Pisani, A. Rios Fukelman, K.Zarembo

Workshop on Supersymmetric Localization and Holography: Black Hole Entropy and Wilson Loops

Trieste - 9th of July 2018

# Motivations to study Wilson loops in $\mathcal{N} = 4$ SYM

⊛ WL are observables with valuable physical interpretation in any gauge theory

- $q\bar{q}$ potential $\leftrightarrow$ confinement
- Bremsstrahlung function $\leftrightarrow$ Radiation of accelerated charges
- Related to gluon scattering amplitudes

⊛ We consider them in $\mathcal{N} = 4$ SYM

- Gauge theory in the prototypical AdS/CFT example
- Some WL VEVs and correlators can be computed exactly $\Rightarrow$ precision tests of AdS/CFT correspondence

⊛ Most exact results have been obtained with Integrability and Localization methods. In this talk I will focus on the **resummation of ladder diagrams**

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# Outline

- Wilson loops in $\mathcal{N} = 4$ SYM
- Dyson Equation for ladder contribution to a Wilson loop VEV
- Dyson Equation for the ladder contribution to the correlator of 2 Wilson loops
- Consider an internal space separation $\gamma$, between the 2 loops
  - For some critical $\gamma^*$ the correlator is susy. Dyson eqs. resolution $\equiv$ Localization results
  - In same parametric limit $\cos\gamma \gg 1$ Ladder diagrams dominate and the strong coupling limit of the ladder resummation is in agreement with the string theory computations.
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# Wilson loop operators

In any gauge theory:

$$W(\mathcal{R},\mathcal{C}) = \mathrm{tr}_{\mathcal{R}}\mathcal{P}\left(e^{i\oint_{\mathcal{C}} A_\mu \dot{x}^\mu d\tau}\right) := \mathrm{tr}_{\mathcal{R}}\prod_{\tau\in\mathcal{C}}(1 + iA_\mu \dot{x}^\mu d\tau)$$

its VEV measures the phase of an external particle

For a given theory a WL depends on:

- The trajectory of the particle, $\mathcal{C} : x^\mu(\tau)$
- The type/charge of the particle, $\mathcal{R}$ : gauge group rep

Interesting physical interpretations depending of the choice of $\mathcal{C}$



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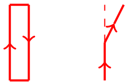

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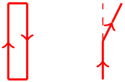

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# Wilson loops in $\mathcal{N} = 4$ SYM

- In $\mathcal{N} = 4$ SYM the external charge can also couple, in addition to the gauge potential, to the 6 scalar fields

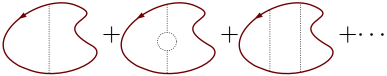
$$W(\mathcal{R}, \mathcal{C}, \vec{n}) = \mathrm{tr}_{\mathcal{R}} \mathcal{P} \left( e^{\oint_{\mathcal{C}} \left( iA_\mu \dot{x}^\mu + \vec{n} \cdot \vec{\Phi} |\dot{x}^\mu| \right) d\tau} \right)$$

with $\vec{n}(\tau)$ a unit vector in $\mathbb{R}^6$

- Wilson loop VEVs are functions of the coupling

$$\langle W(\mathcal{R}, \mathcal{C}, \vec{n}) \rangle = f(\lambda)$$

  - If $\lambda \ll 1$ we rely on perturbative exp. to account for $f(\lambda)$



  - If $\lambda \gg 1$ we can appeal to the AdS/CFT correspondence

# Gravity dual of Wilson loops

AdS/CFT prescription: Loop $\mathcal{C}$ provides a boundary condition

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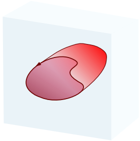
[Maldacena 98];[Rey, Yee 98]: The VEVs of Wilson loops in the fund. rep. are given by the string partition function with contour $\mathcal{C}$ as the boundary condition at the boundary of AdS

$$\langle W(\mathcal{C})\rangle = \int_{X|_{bdry}=\mathcal{C}} [\mathcal{D}X\ldots]\, e^{-S_{\text{string}}}$$

$$\simeq e^{-\text{Area}}$$



- Semiclassical approximation is valid for large $\lambda = g_{YM}^2 N$. The crucial point is that the effective string tension becomes

$$T_{eff} = \frac{R^2}{2\pi\alpha'} \sim \sqrt{\lambda} \qquad \text{with } R \text{ the AdS radius}$$

- Going beyond amounts to consider quantum world-sheet fluctuations $\rightarrow$ Silva's talk in the afternoon

# Precision test of AdS/CFT

Comparing dual points of view is challenging:

field theory results are valid for $\lambda \ll 1$

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Ladder diagrams: The resummation of Feynman diagrams with no interaction vertices is a tractable problem in many cases, but in general it accounts for $\langle W \rangle$ only partially

- For some supersymmetric Wilson loops ladder diagrams are the only contribution

- For certain parametric limit ladder diagrams are the leading order contribution.

In such cases the strong coupling limit of the resummation of ladder diagrams can be contrasted with dual string theory computations.

# Ladders for Wilson loops in $\mathcal{N} = 4$ SYM

Given some curve $x^\mu(t)$ we define the non-abelian phase factor

$$\vec{U}(t_1, t_2) = \vec{\mathcal{P}} \exp \int_{t_1}^{t_2} \mathcal{O}(t) dt \qquad \mathcal{O} = iA_\mu \dot{x}^\mu + \Phi_6 |\dot{x}^\mu|$$

to compute

$$W(t_2 - t_1) := \frac{1}{N} \langle \mathrm{tr} \vec{U}(t_1, t_2) \rangle \qquad \text{[gauge inv. for } x^\mu(t_1) = x^\mu(t_2)\text{]}$$

we expand the exponential and contract with propagators

$$\langle \Phi^i_{A,j}(x) \Phi^k_{B,l}(y) \rangle = \frac{\lambda}{8\pi^2 N} \frac{\delta^i_l \delta^k_j \delta_{AB}}{(x-y)^2} \quad \langle A^i_{\mu,j}(x) A^k_{\nu,l}(y) \rangle = \frac{\lambda}{8\pi^2 N} \frac{\delta^i_l \delta^k_j \delta_{\mu\nu}}{(x-y)^2}$$

Combination of gluon & scalar prop. leads to an effective prop.

$$= \langle \mathcal{O}^i_j(t) \mathcal{O}^k_l(t') \rangle = \frac{\delta^i_l \delta^k_j}{N} G(t, t')$$




$$G(t, t') = \frac{\lambda}{8\pi^2} \frac{|\dot{x}(t)||\dot{x}(t')| - \dot{x}^\mu(t) \dot{x}_\mu(t')}{(x(t) - x(t'))^2}$$

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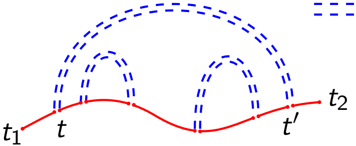


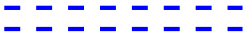

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# An integral equation for $W(t)$

The non-abelian phase factor satisfies a recursion relation

$$\vec{U}(t_1, t_2) = \mathbf{1} + \int_{t_1}^{t_2} dt\, \vec{U}(t_1, t)\mathcal{O}(t)$$

Doing Wick contractions and applying large $N$ factorization

$$\begin{aligned}
\langle \mathrm{tr}\vec{U}(0,t)\rangle &= N + \int_0^t dt' \langle \mathrm{tr}\vec{U}(0,t')\mathcal{O}(t')\rangle \\
&= N + \frac{1}{N}\int_0^t dt' \int_0^{t'} dt'' \langle \mathrm{tr}\vec{U}(0,t'')\mathrm{tr}\vec{U}(t'',t')\rangle G(t',t'') \\
&\simeq N + \frac{1}{N}\int_0^t dt' \int_0^{t'} dt'' \langle \mathrm{tr}\vec{U}(0,t'')\rangle\langle \mathrm{tr}\vec{U}(t'',t')\rangle G(t',t'')
\end{aligned}$$

Therefore

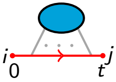
$$W(t) = 1 + \int_0^t dt' \int_0^{t'} dt''\, W(t'-t'')W(t'')G(t',t'')$$

This integral equation

$$W(t) = 1 + \int_0^t dt' \int_0^{t'} dt'' W(t' - t'') W(t'') G(t', t'')$$

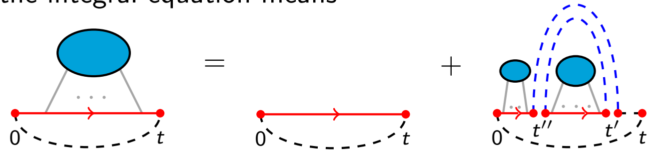
can be understood diagrammatically

Schematically we represent $W(t)$ as follows




$$= W(t)\,\delta^i_j$$

Thus, the integral equation means




- No propagator
- $t'$ is the rightmost point contracted with propagator

# Exact solution for the circular loop

$$x^\mu = (R\cos t, R\sin t, 0, 0) \quad \Rightarrow \quad G(t,t') = \frac{\lambda}{16\pi^2}$$

Doing a Laplace transform

$$\tilde{W}(z) = \int_0^\infty dt\, e^{-zt} W(t)$$

The integral eq. is translated into a quadratic algebraic eq.

$$\tilde{W}(z) = \frac{1}{z} + \frac{\lambda}{16\pi^2}\frac{\tilde{W}(z)^2}{z} \quad \Rightarrow \quad \tilde{W}(z) = \frac{8\pi^2}{\lambda}\left(z - \sqrt{z^2 - \frac{\lambda}{4\pi^2}}\right)$$

Anti-transforming

$$W(t) = \frac{4\pi}{\sqrt{\lambda}t} I_1\left(\frac{\sqrt{\lambda}t}{2\pi}\right)$$

Planar exact circular loop vev is (non-ladder diagrams cancel out)

$$\langle W_{\text{circle}}\rangle = W(2\pi) = \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}) \simeq e^{\sqrt{\lambda} - \frac{3}{4}\log\lambda + \frac{1}{2}\log\frac{2}{\pi}}$$

[Erickson, Semenoff, Zarembo 00], [Drukker, Gross 00]

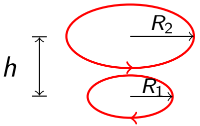
# Connected correlator of 2 circular loops

The connected correlator

$$\langle W(\mathcal{C}_1, \mathcal{C}_2)\rangle_c = \langle W(\mathcal{C}_1) W(\mathcal{C}_2)\rangle - \langle W(\mathcal{C}_1)\rangle\langle W(\mathcal{C}_2)\rangle$$

for two concentric circles given by

$$\mathcal{C}_1 \;:\; x_1^\mu(t) = (R_1 \cos t, R_1 \sin t, 0, 0), \qquad \vec{n}_1 = (1,0,0,0,0,0)$$
$$\mathcal{C}_2 \;:\; x_2^\mu(t) = (R_2 \cos t, -R_2 \sin t, h, 0), \qquad \vec{n}_2 = (\cos\gamma, \sin\gamma, 0,0,0,0)$$



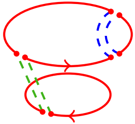

$\gamma$ parametrizes the difference

in the internal space orientation

- For $\gamma = 0$ this was originally considered in [Zarembo 01]
- By turn on $\gamma$ $\begin{cases} \text{We can consider the susy correlator} \\ \text{We can implement a ladder limit} \end{cases}$

Now, we have 2 sorts of propagators

$$\langle \mathcal{O}^i_{A,j}(t)\mathcal{O}^k_{B,l}(t')\rangle = \frac{\delta^i_l \delta^k_j}{N} G_{AB}(t,t')$$



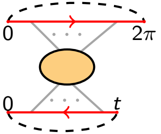
Rainbows $G_{11} = G_{22} = \frac{\lambda}{16\pi^2}$

Ladders $G_{12} = G_{21} = \frac{\lambda}{16\pi^2}\frac{\cos\gamma + \cos(t-t')}{\frac{R_1^2+R_2^2+h^2}{2R_1R_2} - \cos(t-t')}$

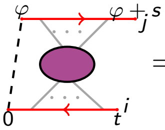
The connected correlator is given in terms of

$$K(t) = \langle \mathrm{tr}\overleftarrow{U}_1(0,t)\mathrm{tr}\vec{U}_2(0,2\pi)\rangle_{\mathrm{conn}} \quad \Gamma(t,s|\varphi) = \frac{1}{N}\langle \mathrm{tr}\overleftarrow{U}_1(0,t)\vec{U}_2(\varphi,\varphi+s)\rangle$$

schematically represented as follows




$= K(t)$

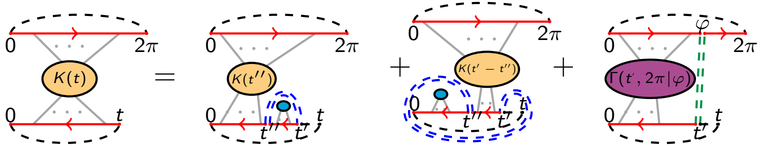



$= \Gamma(t,s|\varphi)\delta^i_j$

# A Dyson equation for $K(t)$

As before, using $U$-recursion relation, Wick contractions and large $N$-factorization we get an integral equation

Schematically




- $t'$ is the rightmost point in $(0, t)$ contracted with a propagator
- When contracted with $t''$ we have a rainbow, $W$ and $K$
- When contracted with $\varphi$ we have a ladder and $\Gamma$

$$K(t) = 2g\int_0^t dt' \int_0^{t'} dt''\, W(t'-t'')K(t'') + \int_0^t dt' \int_0^{2\pi} d\varphi\, G(\varphi - t')\Gamma(t', 2\pi|\varphi)$$

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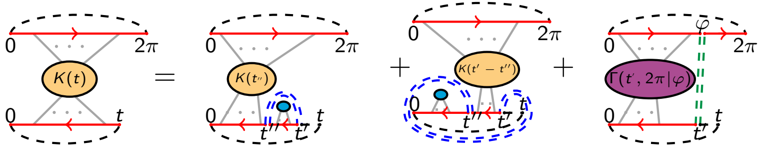


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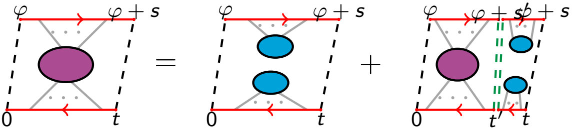
# A Dyson equation for $\Gamma(t, s, |\varphi)$

The auxiliary function $\Gamma(t, s|\varphi)$ also satisfy an integral equation

Schematically we have




$$\Gamma(t,s|\varphi) = W(t)W(s) + \int_0^t dt' \int_0^s ds'\, W(t-t')W(s-s')G(\varphi+s'-t')\Gamma(t',s'|\varphi)$$

How to proceed?

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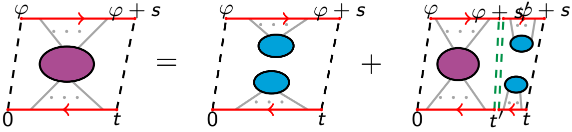


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How to proceed?

If we solve first the integral equation for $\Gamma$ and then the integral equation for $K$, we will obtain the exact contribution of ladder diagrams to the connected correlator in the planar limit

$$\langle W(\mathcal{C}_1, \mathcal{C}_2)\rangle_c^{\text{ladder}} = K(2\pi)$$

Still tricky in the general case...

# Exact ladder resummation in a critical case

Defining

$$\cosh\beta = \frac{R_1^2 + R_2^2 + h^2}{2R_1R_2}$$

there is a critical relation rendering the ladder propagator constant

$$G(t,t') = \frac{\lambda}{16\pi^2}\frac{\cos\gamma + \cos(t-t')}{\cosh\beta - \cos(t-t')} \to -\frac{\lambda}{16\pi^2} \quad \text{for} \quad \cos\gamma = -\cosh\beta$$

This simplifies the integral equations, which can be exactly solved by doing a Laplace transform

$$\tilde{\Gamma}(z,w|\varphi) = \frac{\tilde{W}(z)\tilde{W}(w)}{1+g\tilde{W}(z)\tilde{W}(w)} \quad \Rightarrow \quad \Gamma(t,s|\varphi) = W(t-s)$$

which gives

$$\begin{aligned} K(2\pi) &= \langle W(C_1)W(C_2)\rangle_c^{\text{ladders}} \\ &= 2\lambda\, I_1^2(\sqrt{\lambda}) + \frac{\sqrt{\lambda}}{2}\, I_0(\sqrt{\lambda}) I_1(\sqrt{\lambda}) - 2\lambda\, I_0^2(\sqrt{\lambda}) \\ &\simeq -e^{2\sqrt{\lambda}-\frac{1}{2}\log\lambda - \log 8\pi} \end{aligned}$$

# Critical case is supersymemtric

- Circles with different internal space orientation preserve different sets of susy's. However, the share some susy's when

$$\cos\gamma = -\cosh\beta = \cos\gamma^*$$

- Then, connected correlators computed from the matrix model obtained with supersymmetric localization, whose partition function is [Pestun 07]

$$\mathcal{Z} = \int dM e^{-\frac{N}{2g}\mathrm{tr}(M^2)}$$

In analogy with: $\langle W(C_1)\rangle = \langle \mathrm{tr} e^{2\pi M}\rangle_{\mathrm{M.M.}} \simeq \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda})$

The connected correlator [Giombi,Pestun,Ricci 09]

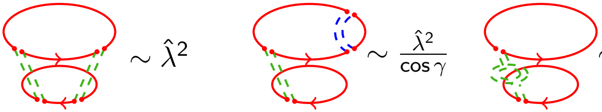
$$\begin{aligned}\langle W(C_1)W(C_2)\rangle_{\mathrm{conn}} &= \langle \mathrm{tr} e^{2\pi M}\mathrm{tr} e^{-2\pi M}\rangle_{\mathrm{M.M.}} \\ &\simeq 2\lambda\, I_1^2(\sqrt{\lambda}) + \frac{\sqrt{\lambda}}{2} I_0(\sqrt{\lambda}) I_1(\sqrt{\lambda}) - 2\lambda\, I_0^2(\sqrt{\lambda})\end{aligned}$$

# Ladder limit

- Limit in which ladder diagrams dominate over the rest
- It requires an analytical continuation of the internal space separation $\gamma$ [DC, Henn, Maldacena,Sever 12]

$$\cos\gamma \to \infty \quad \& \quad \lambda \to 0 \qquad \text{such that} \qquad \hat{\lambda} := \lambda \cos\gamma \quad \text{is fixed}$$

For instance, at 2-loop order



$\sim \hat{\lambda}^2$ $\qquad \sim \frac{\hat{\lambda}^2}{\cos\gamma}$ $\qquad \sim \frac{\hat{\lambda}^2}{\cos\gamma}$

- Diagrams with vertices can be dismissed
- Diagrams with rainbows can be dismissed $(W(t) \to 1)$

$$\Gamma(t,s|\varphi) = 1 + \int_0^t dt' \int_0^s ds' \, G(\varphi + s' - t')\Gamma(t', s'|\varphi)$$

Taking partial derivatives we get

$$\partial_t \partial_s \Gamma(t,s|\varphi) = G(\varphi + s - t)\Gamma(t,s|\varphi)$$

Change of coordinates & ansatz

$$x = s - t, \quad y = s + t, \qquad \Gamma(t,s|\varphi) = \Psi(x)e^{\omega y}$$

we obtain a Schrödinger problem

$$-\Psi''(x) - G(\varphi + x)\Psi(x) = -\omega^2 \Psi(x)$$

In the strong coupling limit the potential is very deep:

$$\Gamma(t,s|\varphi) \approx \sum_n e^{\omega_0(t+s)}\delta(\varphi+s-t+2\pi n) \quad \omega_0^2 = G(0) \approx \frac{\frac{\lambda}{16\pi^2}\cos\gamma}{\frac{R_1^2+R_2^2+h^2}{2R_1R_2} - 1}$$

Plugging in the equation for $K(2\pi)$

$$\langle W(C_1)W(C_2)\rangle_c \simeq e^{4\pi\omega_0} \simeq e^{\sqrt{\lambda\cos\gamma}\sqrt{\frac{2R_1R_2}{R_1^2+R_2^2+h^2-2R_1R_2}}}$$

# String dual the correlator of Wilson loops

We need a string configuration in $AdS_5 \times S^5$, whose worldsheet extends between two concentric circles.

- For $\gamma = 0$ done by [Olesen, Zarembo 00]
- For $h = 0$ and $\gamma \neq 0$ done by [Drukker, Fiol 05]

For the background metric we consider

$$ds^2 = \frac{L^2}{z^2}(dz^2 + dr^2 + r^2 d\varphi^2 + dx^2) + L^2 d\phi^2$$

and the ansatz for the string embedding is

$$x = \sigma\,, \qquad \varphi = \tau\,, \qquad r = r(x)\,, \qquad z = z(x)\,, \qquad \phi = \phi(x)$$

with boundary conditions

$$r(0) = R_1, \qquad r(h) = R_2, \qquad z(0) = z(h) = 0$$

- Solving the EOMS from the Nambu-Goto action we get configurations characterized by 2 constants of motion $(s, t)$
- $(s, t)$ are related to the physical and internal space separations $(h, \gamma)$ through elliptic functions

$$\gamma = 2\frac{\sqrt{1-s-t}}{\sqrt{1+t}} K\left(\tfrac{s+t}{1+t}\right)$$

$$\log\left(\frac{(a+c+h)(a-c)}{(a-c-h)(a+c)}\right) = \frac{4\sqrt{t}}{\sqrt{s}}\frac{1}{\sqrt{1+t}}\left[K\left(\tfrac{s+t}{1+t}\right) - (1-s)\,\Pi\left(s\,\middle|\,\tfrac{s+t}{1+t}\right)\right]$$

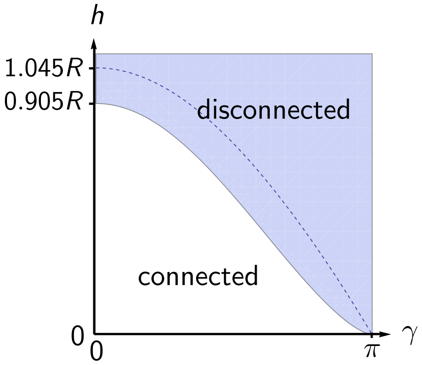
$$\text{for } c = \frac{R_1^2 - R_2^2}{2h} - \frac{h}{2} \qquad a = \sqrt{c^2 + R_1^2}$$

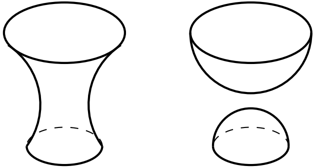
The regularized action is also given by elliptic functions

$$S^{\mathrm{reg}} = -\frac{2\sqrt{\lambda}}{\sqrt{s}}\frac{1}{\sqrt{1+t}}\left[(1+t)\,E\left(\tfrac{s+t}{1+t}\right) - (1-s)\,K\left(\tfrac{s+t}{1+t}\right)\right]$$

# Gross-Ouguri phase transition

- For a given $\gamma$, the regularized area computed by $S^{\mathrm{reg}}$ grows with $h$
- Beyond some critical value $S^{\mathrm{reg}} > -2\sqrt{\lambda}$
- This indicates a phase transition between configuration with connected and world-sheets



Area depends on $h$ Area indep. of $h$

# Minimal area string in the ladder limit

We can implement the ladder limit by considering $0 \leq s \leq 1$ and $t \to \infty$

$$\cos\gamma \simeq \frac{8t}{1-s}$$

In this limit, $s$ is simply related to the space-time parameters

$$s = \frac{h^2 + (R_1 - R_2)^2}{h^2 + (R_1 + R_2)^2}$$

The regularized action is

$$S^{\mathrm{reg}} \simeq -2\sqrt{\lambda}\sqrt{\frac{t}{s}} \simeq -\sqrt{\lambda}\sqrt{\cos\gamma}\sqrt{\frac{1-s}{2s}}$$

which gives

$$\langle W(C_1)W(C_2)\rangle_c \simeq e^{\sqrt{\lambda\cos\gamma}\sqrt{\frac{2R_1R_2}{R_1^2+R_2^2+h^2-2R_1R_2}}}$$

# Summary of main results

- Resummation of ladder diagrams can be obtained by solving Dyson integral equations

- The was a critical case in the connected correlator, $\cos\gamma = -\frac{R_1^2+R_2^2+h^2}{2R_1R_2}$, for which the resummation of ladders can be computed exactly

- This case turns out to be supersymmetric and the corresponding interaction diagrams cancel. Ladders give the full answer

- There is a parametric limit, $\cos\gamma \gg 1$ for which ladder diagrams dominate over all other diagrams.

- The resummation in the ladder limit and strong coupling exactly matches the minimal area of the dual string. Precision test in non-susy case

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