

# The topologically twisted index on $S^2 \times T^2$ and black strings in AdS

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9 July 2018

JTL, L. A. Pando Zayas, V. Rathee and W. Zhao

JHEP 1801 (2018) 026 [arXiv:1707.04197]; PRL 120 (2018) 221602 [arXiv:1711.01076]

Junho Hong and JTL, JHEP 1807 (2018) 018 [arXiv:1804.04592]

JTL, L. A. Pando Zayas and S. Zhou, in progress



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# Supersymmetric partition functions and precision holography

# Supersymmetric partition functions from localization

- ▶ Localization is a very powerful tool for computing supersymmetric partition functions and observables
  - $S^n$  partition functions, Wilson loop observables
  - $S^n \times S^1$  partition functions and supersymmetric indices
- ▶ Generically, the partition function takes the form

$$Z_{\text{susy}} = \int d\Phi Z_{\text{classical}} Z_{1\text{-loop}}$$

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$$Z_{\text{susy}} = \int d\Phi Z_{\text{classical}} Z_{1\text{-loop}}$$

- ▶ We explore the connection between the topologically twisted index on  $S^2 \times T^2$  and  $\text{AdS}_5$  black string microstates

[F. Benini, K. Hristov and A. Zaffaroni, arXiv:1504.03698]

[S. M. Hosseini, A. Nedelin and A. Zaffaroni, arXiv:1611.09374]

## AdS/CFT and black hole/black string entropy

- AdS/CFT allows us to compare observables on both sides of the duality

Global AdS  $\leftrightarrow$  partition function on  $S^n$

Black holes in AdS  $\leftrightarrow$  partition function on  $S^{n-1} \times S^1$

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- ▶ Consider a BPS black string in  $\text{AdS}_5$

	boundary		near horizon
AdS	$\text{AAdS}_5$	$\longrightarrow$	$\text{AdS}_3 \times S^2$
	$\downarrow$		$\downarrow$
CFT	$S^2 \times T^2$	$\longrightarrow$	$T^2$

## The topologically twisted index on $S^2 \times T^n$

- ▶ The topologically twisted index was introduced by [F. Benini and A. Zaffaroni, arXiv:1504.03698](#)
- ▶ Take an object with magnetic flux on  $S^2$  in AdS
  - magnetic black hole in  $\text{AdS}_4$
  - magnetic black string in  $\text{AdS}_5$
- ▶ What do we do on the field theory side?
  - Background  $R$  symmetry flux on  $S^2$
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- ▶ This topologically twisted index on  $S^2 \times S^1$  is conjectured to count the black hole microstates [[Benini, Hristov, Zaffaroni](#)]
  - Many general features are now known
  - Extended to dyonic black holes, black holes with hyperbolic horizons, [magnetic black strings, ...](#)

## Counting black hole/black string microstates

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  1. Magnetic black holes in M-theory on  $\text{AdS}_4 \times S^7/\mathbb{Z}_k$   
Dual to ABJM theory
  2. Magnetic black holes in massive IIA on  $\text{AdS}_4 \times S^6$   
Dual to  $\mathcal{N} = 2$  Chern-Simons-matter theory

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- ▶ Then we focus on black strings in  $\text{AdS}_5$ 
  - 3 Magnetic black strings in IIB on  $\text{AdS}_5 \times S^5$   
Dual to  $\mathcal{N} = 4$  super-Yang-Mills

# The $S^2 \times S^1$ index and AdS black holes

## Building blocks of the $S^2 \times S^1$ index

- ▶ Consider three-dimensional  $\mathcal{N} = 2$  Chern-Simons-matter theories on  $S^2 \times S^1$
- ▶ The topologically twisted index receives contributions from:
  - Vector multiplets:

$$Z_{\text{vector}} = \prod_i \frac{dx_i}{2\pi i x_i} x_i^{km_i} \prod_{\alpha \in G} (1 - x^\alpha)$$

- Chiral multiplets:

$$Z_{\text{chiral}} = \prod_{\mu \in R} \left( \frac{\sqrt{x^\mu y^{\mu_f}}}{1 - x^\mu y^{\mu_f}} \right)^{\mu(\mathfrak{m}) + \mu_f(\mathfrak{n}) - q + 1}$$

- ▶ These elements can be combined to construct the index for various models

## M-theory on $AdS_4 \times S^7/Z_k$

- The field theory dual is ABJM theory

Chern-Simons-matter with  $U(N)_k \times U(N)_{-k}$  gauge groups and bi-fundamental matter  $A_i, B_j$

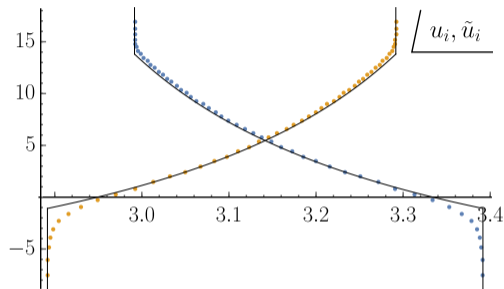
- The topologically twisted index is given by

$$Z(y_a, n_a) = \frac{1}{(N!)^2} \sum_{\mathbf{m}, \tilde{\mathbf{m}}} \int \prod_i \frac{dx_i}{2\pi i x_i} x_i^{k m_i} \prod_{i \neq j} \left(1 - \frac{x_i}{x_j}\right) \\ \int \prod_i \frac{d\tilde{x}_i}{2\pi i \tilde{x}_i} \tilde{x}_i^{-k \tilde{m}_i} \prod_{i \neq j} \left(1 - \frac{\tilde{x}_i}{\tilde{x}_j}\right) \\ \prod_{i,j} \prod_a \left( \frac{\sqrt{\frac{x_i}{x_j} y_a}}{1 - \frac{x_i}{x_j} y_a} \right)^{m_i - \tilde{m}_j - n_a + 1} \prod_{i,j} \prod_b \left( \frac{\sqrt{\frac{\tilde{x}_i}{\tilde{x}_j} y_b}}{1 - \frac{\tilde{x}_i}{\tilde{x}_j} y_b} \right)^{\tilde{m}_j - m_i - n_b + 1}$$

- The index can be evaluated using the Jeffrey-Kirwan residue  
[Benini, Hristov, Zaffaroni]

# Eigenvalue distribution

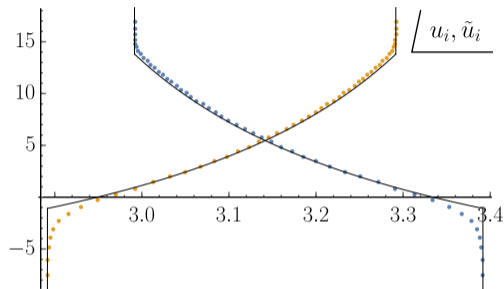
- Single solution to the BAE up to permutations



Solution for  $\Delta_a = \{0.3, 0.4, 0.5, 2\pi - 1.2\}$  and  $N = 50$

# Eigenvalue distribution

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Solution for  $\Delta_a = \{0.3, 0.4, 0.5, 2\pi - 1.2\}$  and  $N = 50$

- ▶ Large- $N$  behavior

$$\text{Re log } Z \sim \boxed{f_0 N^{3/2}} + f_1 N^{1/2} - \frac{1}{2} \log N + \dots$$

- ▶ Subleading terms are difficult to extract analytically

Tails in the distribution lead to complications

## Massive IIA theory on $AdS_4 \times S^6$

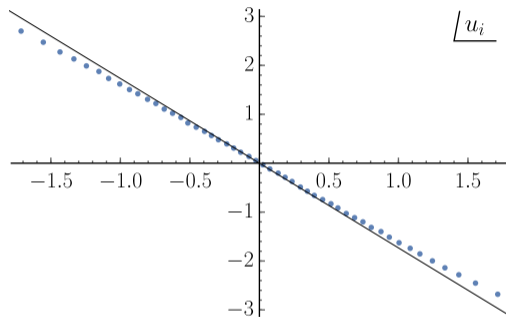
- ▶ The dual field theory is an  $\mathcal{N} = 2$  Chern-Simons-matter theory with  $SU(N)_k$  gauge group and adjoint matter  $X, Y, Z$  [Guarino, Jafferis and Varela, arXiv:1504.08009]
- ▶ Here the topologically twisted index is given by

$$Z(y_a, \mathbf{n}_a) = \frac{(-1)^N}{N!} \sum_{\mathbf{m}} \int \prod_i \frac{dx_i}{2\pi i x_i} x_i^{k m_i} \prod_{i \neq j} \left( 1 - \frac{x_i}{x_j} \right) \prod_{i,j} \prod_a \left( \frac{\sqrt{\frac{x_i}{x_j} y_a}}{1 - \frac{x_i}{x_j} y_a} \right)^{m_i - m_j + n_a + 1}$$

- ▶ Once again, the index is evaluated using the Jeffrey-Kirwan residue [Benini, Khachatryan and Milan 1707.06886; Hosseini, Hristov and Passias 1707.06884]

## Eigenvalue distribution

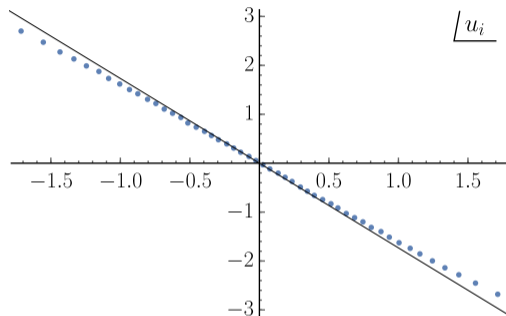
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$$\text{Re log } Z \sim \boxed{f_0 N^{5/3}} + f_1 N^{2/3} + f_2 N^{1/3} + f_3 \log N + \dots$$

- ▶ Can we understand the subleading behavior?  
No tails, but still have to deal with endpoints

The  $S^2 \times T^2$  index and AdS  
black strings

## Building blocks of the $S^2 \times T^2$ index

- ▶ We now turn to the topologically twisted index on  $S^2 \times T^2$  where  $T^2$  is parametrized by  $q = e^{2\pi i \tau}$ 
  - Four-dimensional Yang-Mills theory on  $S^2 \times T^2$

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  - Four-dimensional Yang-Mills theory on  $S^2 \times T^2$
- ▶ Work in an  $\mathcal{N} = 1$  language
  - Vector multiplets:

$$Z_{\text{vector}} = (-1)^{2\rho(\mathfrak{m})} \prod_{i \in G} \frac{dx_i}{2\pi i x_i} \eta(q)^2 \prod_{\alpha \in G} \left( \frac{\theta_1(x^\alpha, q)}{i\eta(q)} \right)$$

- Chiral multiplets:

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## IIB on $AdS_5 \times S^5$

- ▶ The standard AdS/CFT setup
  - IIB on  $AdS_5 \times S^5 \longleftrightarrow \mathcal{N} = 4$  SYM with  $SU(N)$  gauge group
- ▶ On the gravity side [Benini and Bobev, arXiv:1302.4451]
  - Consider the  $SU(4)_R \supset U(1)^3$  truncation — ie the STU model
  - Magnetic string solutions with magnetic charges  $\mathbf{n}_a$
  - Near horizon  $AdS_3 \times S^2$  with  $\mathbf{n}_1 + \mathbf{n}_2 + \mathbf{n}_3 = 2$

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- ▶ On the field theory side
  - $\mathcal{N} = 4$  SYM on  $T^2 \times S^2$  with modular parameter  $q = e^{2\pi i \tau}$
  - Magnetic fluxes on  $S^2$  (given by  $\mathbf{n}_a$ ) enforce the topological twisting
  - We also turn on chemical potentials  $\Delta_a$
- ▶ The topologically twisted index is then a function of these parameters —  $Z(\Delta_a, \mathbf{n}_a; \tau)$

# The topologically twisted index

- Compute the topologically twisted index for  $\mathcal{N} = 4$  SYM with gauge group  $SU(N)$  on  $S^2 \times T^2$

One **vector** and three **chiral** multiplets in the  $\mathcal{N} = 1$  language

- The result is [Hosseini, Nedelin and Zaffaroni, arXiv:1611.09374]

$$Z = \frac{1}{N!} \sum_{\mathbf{m}} \int \prod_i \frac{dx_i}{2\pi i x_i} \eta(q)^2 \prod_{i \neq j} \left( \frac{\theta_1(\frac{x_i}{x_j}, q)}{i\eta(q)} \right) \prod_a \prod_{i,j} \left( \frac{i\eta(q)}{\theta_1(\frac{x_i}{x_j} y_a, q)} \right)^{m_i - m_j - n_a + 1}$$
$$x_i = e^{iu_i}, \quad y_a = e^{i\Delta_a}, \quad q = e^{2\pi i \tau}$$

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$$x_i = e^{iu_i}, \quad y_a = e^{i\Delta_a}, \quad q = e^{2\pi i \tau}$$

- After evaluating the Jeffrey-Kirwan residue

$$Z(\Delta_a, \mathbf{n}_a; \tau) = \mathcal{A} \sum_{I \in \text{BAEs}} \frac{1}{\det \mathbb{B}} \prod_{i \neq j} \left[ \frac{\theta_1(\frac{x_i}{x_j}, q)}{i\eta(q)} \prod_a \left( \frac{i\eta(q)}{\theta_1(\frac{x_i}{x_j} y_a, q)} \right)^{1 - n_a} \right]$$

## Solving the BAE

- ▶ The BAEs that we need to solve are

$$1 = e^{iB_i} \equiv e^{iv} \prod_j \prod_a \frac{\theta_1(e^{i(u_j - u_i + \Delta_a)}, q)}{\theta_1(e^{i(u_i - u_j + \Delta_a)}, q)}$$

- ▶ How do we obtain the  $u_j$ 's?

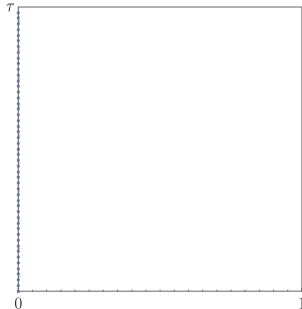
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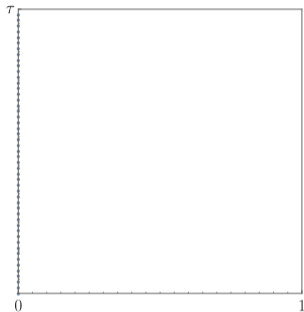
- How do we obtain the  $u_j$ 's?

Hosseini, Nedelin, Zaffaroni obtained  $u_j = \bar{u} + 2\pi \frac{\tau}{N} j$  in the high-temperature limit  $\beta \rightarrow 0^+$  where  $\tau = i\beta/2\pi$



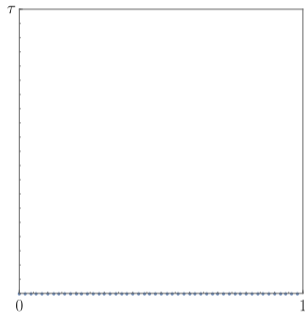
## Multiple solutions to the BAE

- Evenly distributed eigenvalues  $\Rightarrow$  good solution for any  $q$ !



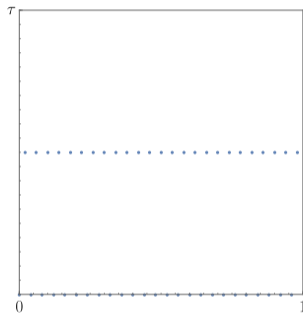
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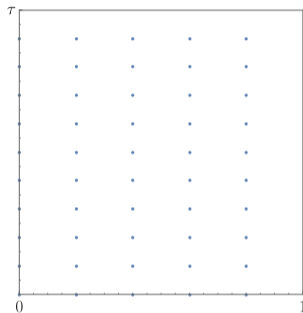
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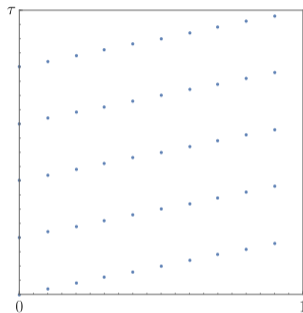
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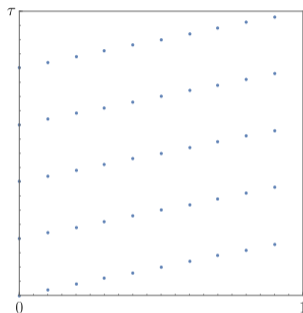
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- ▶ We find a family of exact solutions specified by  $\{m, n, r\}$  where  $N = mn$  and  $r = 0, 1, \dots, n-1$

$$u_{jk} = \bar{u} + 2\pi \frac{j + k\tilde{\tau}}{m} \quad \tilde{\tau} = \frac{m\tau + r}{n}$$

with  $j = 0, 1, \dots, m-1$  and  $k = 0, 1, \dots, n-1$

## The sum over sectors

- The topologically twisted index for  $\mathcal{N} = 4$  SYM on  $S^2 \times T^2$  can be written as  $Z = \sum_{\{m,n,r\}} Z_{\{m,n,r\}}$  where

$$Z_{\{m,n,r\}}(\Delta_a, \mathbf{n}_a; \tau) = \frac{i^{N-1}}{\det \mathbb{B}_{\{m,n,r\}}} \prod_a \left[ \psi(\Delta_a, \tau) \left( \frac{m}{\psi(m\Delta_a, \tilde{\tau})} \right)^N \right]^{1-n_a}$$

and

$$\psi(u, \tau) = \frac{\theta_1(u, \tau)}{\eta^3(\tau)} = \sqrt{\varphi_{-2,1}(u, \tau)}$$

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- The sum over sectors is crucial for modularity of the index  
Two modular parameters:  $\tau : T^2$  and  $\tilde{\tau} : T^2/\mathbb{Z}_m \times \mathbb{Z}_n$

## The index as an elliptic genus

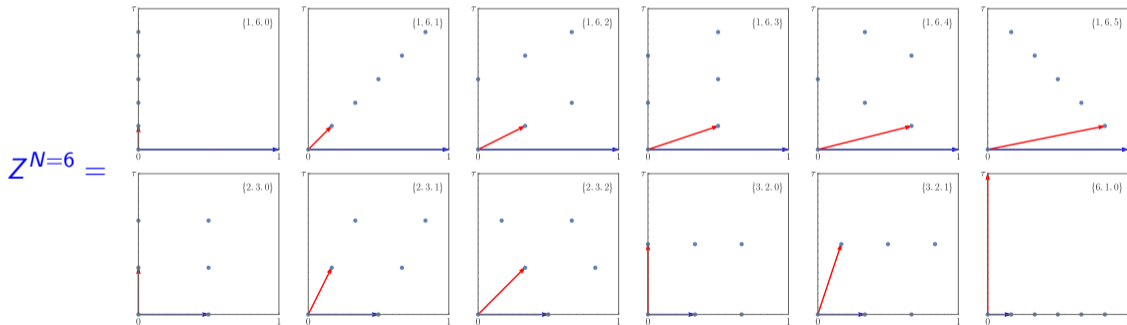
- ▶ The index computes the elliptic genus of the  $\mathcal{N} = (0, 2)$  SCFT obtained by reducing on  $S^2$   
Transforms under  $SL(2, \mathbb{Z})$  as a weak Jacobi form of weight 0

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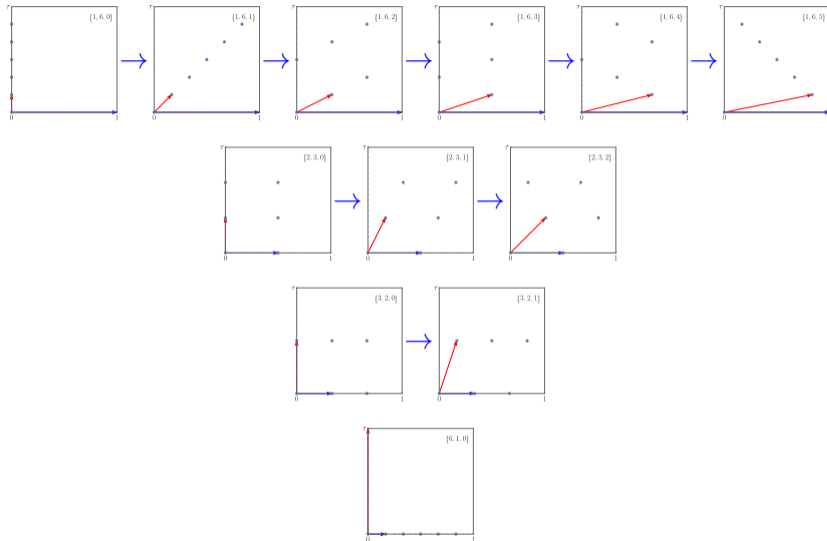
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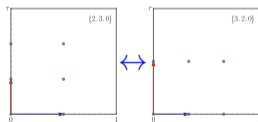
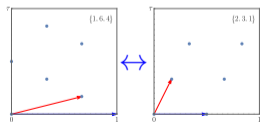
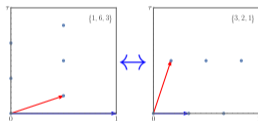
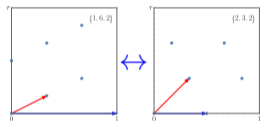
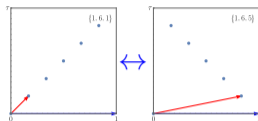
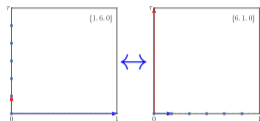
- Consider, for example, the case  $N = 6$



# The transformation $T : \tau \rightarrow \tau + 1$



# The transformation $S : \tau \rightarrow -1/\tau$



The high-temperature limit

## The high-temperature limit of the index

- We can take the Cardy limit of the index in order to compare with the black string in AdS

Expect  $\log Z \sim \frac{\pi^2}{6\beta} c_r$  as  $\beta \rightarrow 0^+$  where  $\tau = \frac{i\beta}{2\pi}$

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- This is obtained by performing a modular transformation  $\tau \rightarrow -1/\tau$

$$\begin{aligned} Z(\Delta_a, \mathbf{n}_a; \tau) &= e^{\frac{i}{2\pi\tau'} \sum_a m_a \Delta_a'^2} Z(\Delta'_a, \mathbf{n}_a; \tau') \\ &= e^{\frac{i}{2\pi\tau'} \sum_a m_a \Delta_a'^2} \sum_{\{m', n', r'\}} Z_{\{m', n', r'\}}(\Delta'_a, \mathbf{n}_a; \tau') \end{aligned}$$

where

$$\tau' = -\frac{1}{\tau} = \frac{2\pi i}{\beta}, \quad \Delta'_a = \frac{\Delta_a}{\tau} = -\frac{2\pi i \Delta_a}{\beta}$$

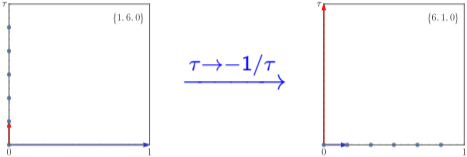
and

$$m_a = -\frac{N^2 - 1}{2}(1 - \mathbf{n}_a)$$

are indices of the Jacobi form

# The high-temperature limit of the index

- We expect the index to be dominated by a single sector



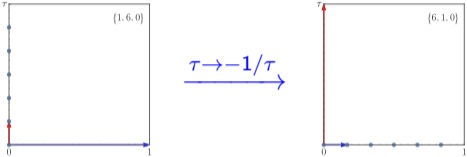
$$Z_{\{1,N,0\}}(\Delta_a, \mathbf{n}_a; \tau) \longrightarrow Z_{\{N,1,0\}}(\Delta_a/\tau_a, \mathbf{n}_a; -1/\tau)$$

- This is the sector considered in Hosseini, Nedelin, Zaffaroni

$$\log Z(\Delta_a, \mathbf{n}_a; i\beta/2\pi) \Big|_{\beta \rightarrow 0^+} \sim \frac{\pi^2}{6\beta} c_r(\Delta_a, \mathbf{n}_a)$$

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- But can we really ignore the other sectors?

## High-temperature limit in the sector $\{m, n, r\}$

- Expanding the theta functions for  $\tau' \rightarrow i\infty$  gives

$$\log Z_{\{m,n,r\}} = \frac{2\pi^2}{\beta} \left[ \sum_a (1 - n_a) (d_a(1 - d_a) - m'^2 x_a(1 - x_a)) \right. \\ \left. + 2(n' - 1) \alpha_{\{m',n',r'\}}(d_a) \right] + \mathcal{O}(1)$$

where

$$d_a = \frac{\Delta_a}{2\pi} \pmod{1}, \quad x_a = \frac{n' \Delta_a}{2\pi} \pmod{1}$$

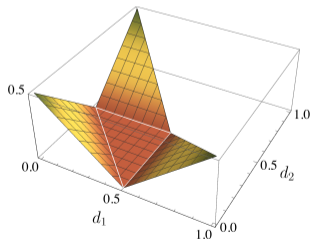
- Here  $\alpha_{\{m',n',r'\}}(d_a)$  parametrizes the high temperature behavior of the Jacobian factor  $\det \mathbb{B}$

$$\log \det \mathbb{B}_{\{m',n',r'\}} \sim -\frac{4\pi^2}{\beta} (n' - 1) \alpha_{\{m',n',r'\}}(d_a)$$

# The determinant factor $\alpha(d_a)$

- ▶ We have been unable to find a general expression for  $\alpha_{\{m',n',r'\}}(d_a)$ 
  - Complicated dependence on the chemical potentials  $d_a = \Delta_a/2\pi \pmod{1}$
- ▶ However
  - We find  $\det \mathbb{B}$  is either  $\mathcal{O}(1)$  ( $\alpha = 0$ ) or approaches zero ( $\alpha > 0$ )
  - In general,  $\alpha$  is piecewise linear

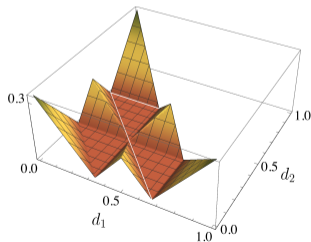
$$N = 2 : \quad \alpha_{\{2,1,0\}} = 0, \quad \alpha_{\{1,2,r'\}} = \max(0, d_a - 1/2)$$



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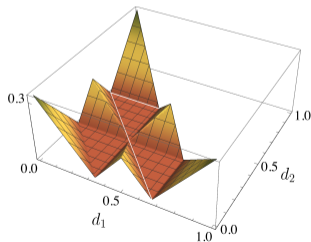
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- ▶ The behavior is more complicated when  $N$  is composite

## Domination by $Z_{\{1,N,0\}}$

- Hosseini, Nedlin, Zaffaroni suggest that the high-temperature limit is dominated by

$$\log Z_{\{1,N,0\}} \sim -\frac{2\pi^2}{\beta}(N^2 - 1) \sum_a (1 - n_a) d_a (1 - d_a)$$

- This is the case provided  $\log Z_{\{1,N,0\}} \geq \log Z_{\{m,n,r\}}$ , or

$$\sum_a (1 - n_a) \left( \frac{x_a(1 - x_a)}{n'^2} - d_a(1 - d_a) \right) \geq \frac{2(n' - 1)}{N^2} \alpha_{\{m',n',r'\}}(d_a)$$

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- Not universally true, but can be studied
  - Subdominant sectors will be exponentially suppressed
  - Expect  $Z_{\{1,N,0\}}$  to dominate whenever a good string dual exists

## Extremization and the central charge

- ▶ Assuming  $Z_{\{1,N,0\}}$  dominates, we obtain

$$\begin{aligned}\log Z(\Delta_a, \mathbf{n}_a; i\beta/2\pi) \Big|_{\beta \rightarrow 0^+} &\sim -\frac{2\pi^2}{\beta} (N^2 - 1) \sum_a (1 - n_a) d_a (1 - d_a) \\ &\sim \frac{\pi^2}{6\beta} c_r(\Delta_a, \mathbf{n}_a)\end{aligned}$$

- ▶ For fixed charges  $\mathbf{n}_a$ , we extremize the trial right-moving central charge  $c_r$  with respect to the potentials  $d_a$

$$\begin{aligned}\bar{d}_a &= \frac{n_a(n_a - 1)}{2\Theta}, \quad \Theta = 1 - (n_1 n_2 + n_2 n_3 + n_3 n_1) \\ \Rightarrow \quad &\boxed{c_r(\mathbf{n}_a) = 3(N^2 - 1) \frac{n_1 n_2 n_3}{\Theta}}\end{aligned}$$

(Unitarity demands  $n_1 n_2 n_3 > 0$ , and supersymmetry demands  $n_1 + n_2 + n_3 = 2$ )

Final thoughts

## What about the large- $N$ limit?

- In the Cardy limit, we expect  $N^2$  behavior

$$\log Z \sim \frac{N^2}{\beta} \quad \text{ie} \quad c_r = \mathcal{O}(N^2)$$

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- ▶ Is there a  $\log(N)$  correction?

And if so, is it universal? Can it be reproduced in the AdS black string dual?

- ▶ At finite temperature we expect modular covariance

$$Z \sim N^2 \psi(\Delta_a, \mathfrak{n}_a, \tau)$$

- ▶ Can we study the elliptic genus at large- $N$ ?  
And on the AdS side of the duality?

# Summary

- ▶ We have explored the topologically twisted index for  $\mathcal{N} = 4$  SYM on  $S^2 \times T^2$ 
  - Conjectured to count black string microstates in the holographic dual
- ▶ Main result: There are multiple solutions to the BAE for the index on  $S^2 \times T^2$ 
  - Needed for modular covariance of the index
  - But in the Cardy limit, only a single sector dominates
- ▶ Much remains to be understood in the precision counting of AdS black hole/black string microstates