Localization, Wilson Loops and Precision Tests

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Workshop on Supersymmetric Localization and Holography:
Black Hole Entropy and Wilson Loops
ICTP, Trieste, July 9th, 2018

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AIM

Make precision tests of AdS/CFT

Gain insights into string perturbation theory

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WHAT DO WE MEAN BY AdS/CFT?

The dual pairs,

$\mathcal{N}=4$ SYM in d $=4$ $G=U(N)$ gauge group	$\longleftrightarrow$	IIB string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$
$\mathcal{N}=6$ SCSM in d $=3$ $G=U_k(N)\times U_{-k}(N)$ gauge group	$\longleftrightarrow$	IIA string theory on $\mathrm{AdS}_4 \times \mathbb{CP}^3$

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Outline

1. **Exact results:**

   Susy Localization

2. **Gauge theory Wilson loops bestiary:**

   Straight lines, circles and cusps

3. **AdS/CFT parameters**

4. **String theory and semiclassical expansion:**

   String duals to Wilson loops. Classical worldsheet.

   Semiclassical expansion of string partition function

   Zero modes: CKV,...

   1001 Determinants: Zeta function, Gelfand-Yaglom, ...

5. **Comparison of String/Gauge, Summary & Open Questions**

1 Exact results in gauge theory: Susy Localization

Exact results in QFT are rare and difficult to obtain, but...

The situation improves if enough symmetry is present, and this is the case for

$\mathcal{N}$=4 SYM and $\mathcal{N}$=6 SCSM.

On the gauge theory side two techniques have been exploited to obtain exact results:

- Integrability: exact results in planar level limit and beyond see recent work by P Vieira 4pt
- Localization: exact results $\forall\ \lambda, N$, but only applicable for susy objects.

2 Wilson loops bestiary: straight lines, circles, ...

Susy Wilson loops require additional couplings to scalars and fermions.

$\mathcal{N}$=4 SYM WLs depend on two data: $\mathcal{C}, \vec{n}$

$$W_{\mathsf{R}}[\mathcal{C},\vec{n}] = \frac{1}{\mathsf{dim[R]}} \mathsf{Tr}_{\mathsf{R}}\, \mathcal{P} \exp\left[\oint (i A_\mu \dot{x}^\mu + |\dot{x}|\, \vec{\phi}\cdot\vec{n})\, ds\right]$$

$\mathcal{C}$: $x^\mu(s)$ curve on spacetime. timelike/spacelike/null (additional $i's$ required)

$\vec{n}$: $\vec{n}(s)$ maps every point in $\mathcal{C}$ to $\mathbb{R}^6$. Dictates scalar field coupling along $\mathcal{C}$.

R: representation for charged particle. A_μ, ϕ^i in adjoint of $U(N)$, e.g.: $A_\mu = A_\mu^a T_R^a$ $(a = 1, \ldots N^2)$

Supersymmetry demands $\vec{n}^2 = 1$ $(\vec{n} \in S^5)$ and a relation

$$\dot{x}^\mu(s) \quad \longleftrightarrow \quad \vec{n}(s)$$

[Maldacena, Rey-Yee, Drukker-Gross-Ooguri, Erickson-Semenoff-Zarembo, Drukker-Giombi-Ricci-Trancanelli, ...]

$\mathcal{N}=4$ SYM WLs

- $\mathcal{C}$: straight line, $\vec{n}=\vec{n}_0$ constant $\Rightarrow$ $\frac{1}{2}$-**BPS** (16 susies)

$$\langle W_{\mathsf{R}}(\text{line}, \vec{n}_0)\rangle = 1, \quad \text{independent of } \lambda, \mathsf{N}, \mathsf{R}$$

When computing vev perturbatively, the result arises from an exact cancelation between gauge field and scalar propagators.

Localization technique allows to compute exact vevs of some WLs:

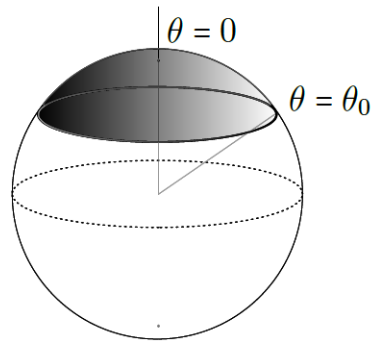
- $\mathcal{C}$: circle, $\vec{n}=\vec{n}_0$ constant $\Rightarrow$ $\frac{1}{2}$-**BPS**. Answer is non-trivial:

For $\mathsf{R}=\Box$

$$\begin{aligned}
\langle W_{\Box}(\text{circle}, \vec{n}_0)\rangle &= \frac{1}{N} L^1_{N-1}\left(-\frac{\lambda}{4N}\right)\exp\left[\frac{\lambda}{8N}\right], && \forall\lambda, N\\
&= \boldsymbol{\frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda})} + \frac{\lambda}{48N^2} I_2(\sqrt{\lambda}) + \frac{\lambda^2}{1280N^4} I_4(\sqrt{\lambda}) + \ldots && N \gg 1\\
&= \frac{e^{\sqrt{\lambda}}}{\lambda^{3/4}}\left(\sqrt{\frac{2}{\pi}} - \frac{3}{4}\frac{1}{\sqrt{2\pi}}\frac{1}{\lambda^{1/2}} - \frac{15}{64}\frac{1}{\sqrt{2\pi}}\frac{1}{\lambda} + \ldots\right), && N=\infty, \lambda \gg 1
\end{aligned}$$

Now scalar and gauge propagators add up to a constant, summation of planar ladder diagrams give I_1 Bessel [Eriksson-Semenoff-Zarembo 99, Drukker-Gross 00, Pestun 07]

- $\mathcal{C}$: circle, $\vec{n}_{\theta_0}(s) = (0, 0, 0, \sin\theta_0 \cos s, \sin\theta_0 \sin s, \cos\theta_0)$ $\Rightarrow$ $\mathbf{\frac{1}{4}}$**-BPS** (8 susies)



The WL vev was conjectured to be given by the $\frac{1}{2}$- expression above by making the replacement $\lambda \to \lambda' = \lambda \cos^2\theta_0$ [Drukker]

$$\langle W_\Box(\text{circle}, \vec{n}_{\theta_0})\rangle = \frac{2}{\sqrt{\lambda}\cos\theta_0} I_1(\sqrt{\lambda}\cos\theta_0), \qquad N = \infty$$

$$= \exp\left(\sqrt{\lambda}\cos\theta_0 - \frac{3}{2}\ln\sqrt{\lambda} - \frac{3}{2}\ln\cos\theta_0 + \frac{1}{2}\ln\frac{2}{\pi} - \ldots\right), \quad \lambda \gg 1$$

This is the result we want to reproduce from string theory computations.

Equatorial loop on S^5: $\langle W_\Box(\text{circle}, \vec{n}_{\pi/2})\rangle = 1$

3 AdS/CFT parameters

$$\begin{array}{c} \mathcal{N} = 4 \text{ SYM in d} = 4 \\ G = U(N) \text{ gauge group} \end{array} \longleftrightarrow \begin{array}{c} IIB \text{ string theory} \\ \text{on } AdS_5 \times S^5 \end{array}$$

$(g_{YM}, N) \rightarrow \ \lambda = g_{YM}^2 N \ , \ \frac{1}{N}$ $\qquad$ $T_{eff} = \frac{L^2}{2\pi\alpha'} \ , \ g_s$

The parameters are related as

$$\lambda = \left(\frac{L^2}{\alpha'}\right)^2$$

$$\frac{\lambda}{4\pi N} = g_s$$

t' Hooft limit: keep λ fixed and take $N \rightarrow \infty$.

When doing so the perturbative expansion in terms of λ reorganizes as

planar and non-planar graphs $\leftrightarrow$ sphere/disk and higher genus (handles) string amplits

4 String Theory

The main importance of the AdS/CFT toolkit is that the gauge theory strong coupling $\lambda \gg 1$ regime is easily studied using perturbative string theory.

Why?

Wilson loop gauge observable in *fundamental representation* relates to the *string* partition function with the WL contour $\mathcal{C}$ being the boundary condition for the string

$$\langle W(\mathcal{C}, \vec{n}) \rangle = \int_{\partial X = \mathcal{C}, \vec{n}} [\mathcal{D}g \mathcal{D}X \mathcal{D}\Psi] \, e^{-S_{string}[g,X,\Psi]}$$

here

$$S_{string} = \frac{T_s}{2} \int d\tau d\sigma \sqrt{g} \left(g^{\alpha\beta} G_{MN}(X) \partial_\alpha X^M \partial_\beta X^N + \bar{\Psi} \mathrm{D} \Psi \right)$$

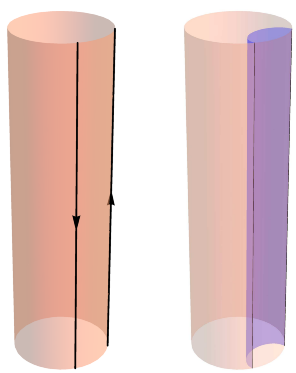
The crucial point is that the action becomes weighted by an effective string tension

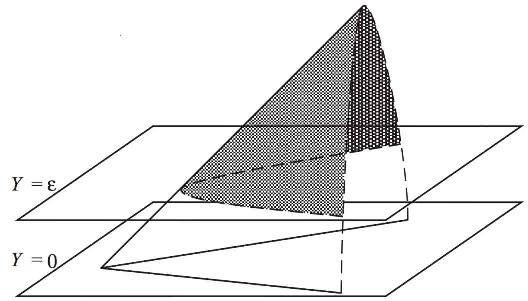
$$T_s = \frac{1}{2\pi\alpha'} \quad \to \quad \boxed{T_{eff} = \frac{L^2}{2\pi\alpha'} = \frac{\sqrt{\lambda}}{2\pi}} \quad \leftrightarrow \quad \frac{1}{\hbar} \qquad \leftarrow L^2 : \text{AdS radius}$$

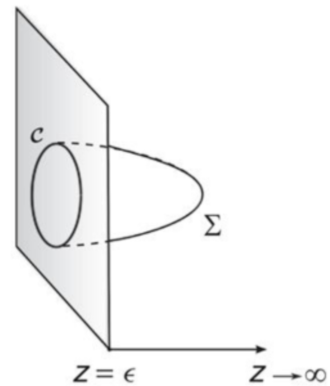
At strong coupling $\lambda \gg 1$ we perform a semiclassical expansion of the path integral.

Genus expansion is weighted by $g_s \sim \frac{1}{N}$.

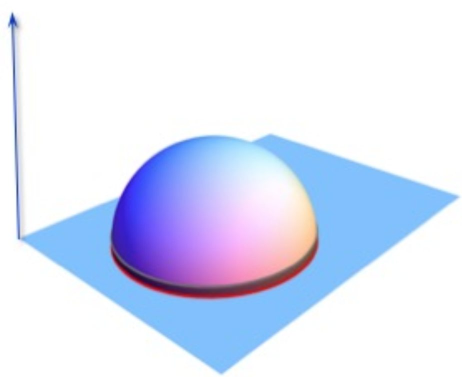
In t' Hooft limit the leading contribution comes from disk topology with no handles.

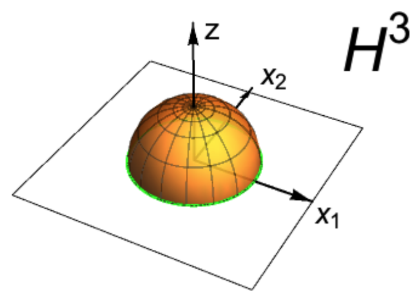


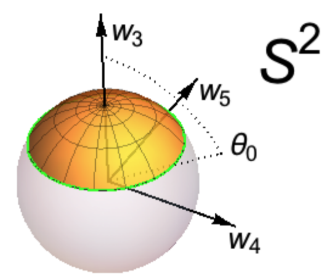




Parallel lines $\leftrightarrow$ $V_{q\bar{q}}$. Cusped line $\leftrightarrow$ Bremsstrahlung. Circular loop $\leftrightarrow$ AdS/CFT test







$\frac{1}{2}$-**BPS** loop: constant position in S^5. $\frac{1}{4}$-**BPS** loop: non trivial embedding in $S^2 \subset S^5$

$\mathcal{N} = 4$ Wilson loop at large N	$\leftrightarrow$	disk amplitude of IIB superstring with RR flux in $AdS_5 \times S^5$

Circular Loop in Fundamental Representation

Expanding the exact planar result for $\mathcal{N} = 4$ in the strong coupling $\lambda \gg 1$ regime

$$\ln\langle W^{1/2}(\text{circle})\rangle = \sqrt{\lambda} - \frac{3}{2}\ln\sqrt{\lambda} + \frac{1}{2}\ln\frac{2}{\pi} - \frac{3}{8}\frac{1}{\sqrt{\lambda}} + \dots$$

$$\ln\langle W^{1/4}(\text{circle}, \theta_0)\rangle = \sqrt{\lambda}\cos\theta_0 - \frac{3}{2}\ln\sqrt{\lambda} - \frac{3}{2}\ln\cos\theta_0 + \frac{1}{2}\ln\frac{2}{\pi} - \dots, \quad \lambda \gg 1, N = \infty$$

On general grounds we expect:

- $\sqrt{\lambda}$: should arise from classical worldsheet area, once properly renormalized S_{ren}.
- $\ln\sqrt{\lambda}$: correction is typical of zero modes. Drukker-Gross suggested its origin can be traced to the Fadeed-Popov diffeo fixing determinant. The FP determinant has 3 z.m. in disk topology hence $3 \times \log\lambda^{1/4}$ is found. Recall each zero mode contributes with $\hbar^{1/2}$.
- $\frac{1}{2}\ln\frac{2}{\pi}$ should come from measure factor of semiclassical partition function + fluctuation determinants over classical string solution.

To avoid the tricky (topological) issue related to FP ghosts the natural thing to do is to compare WL with same topology. The natural observable is then

$$\ln\frac{\langle W^{1/4}(\text{circle}, \theta_0)\rangle}{\langle W^{1/2}(\text{circle})\rangle} = \underbrace{\sqrt{\lambda}(\cos\theta_0 - 1)}_{\text{leading}} \underbrace{-\frac{3}{2}\ln\cos\theta_0}_{\text{1-loop}} + \mathcal{O}(\frac{1}{\sqrt{\lambda}})$$

STRING PARTITION FUNCTION TO 1-LOOP ORDER

About the expression

$$Z_{string} \;\approx\; \mathsf{C}\, e^{-\frac{\sqrt{\lambda}}{2\pi}\tilde{S}[X_{\mathsf{cl}}]}\,\frac{\det^{1/2}\mathcal{O}_F}{\det^{1/2}\mathcal{O}_B}\,\det\mathcal{O}_{FP}$$

C: Normalization factor of the path integral (measure) (eliminated when computing $\frac{W^{1/4}}{W^{1/2}}$)

X_{cl}: Classical string worldsheet above which we fluctuate (classical fermions $\Psi_{\mathsf{cl}} = 0$)

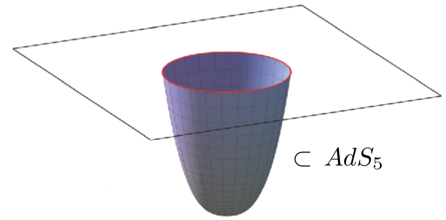
$\tilde{S}[X_{\mathsf{cl}}]$: Action evaluated on classical solution = Area.

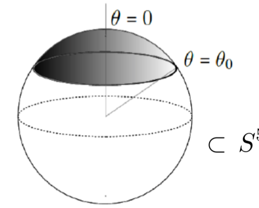
$\mathcal{O}_F$: fermionic fluctuations

$\mathcal{O}_B$: bosonic fluctuations.

$\mathcal{O}_{FP}$: FP diffeo fixing + κ- fixing. $\exists$ worldsheet CKV $\Longrightarrow$ zero modes (topological)

CLASSICAL $\frac{1}{4}$-BPS STRING SOLUTION





Background : $\quad ds^2 = L^2 \left(ds^2_{AdS_5} + d\Omega_5^2 \right)$

. AdS_5 in $\mathsf{H}_2 \times S^2$ foliation $\to$ $EAdS_5$ in $\mathsf{H}_2 \times \mathsf{S}^2$ foliation via $u \to iu$ and $\vartheta \to i\vartheta$

$$ds^2_{EAdS_5} = du^2 + \cosh^2 u \left(d\rho^2 + \sinh^2 \rho \, d\psi^2 \right) + \sinh^2 u \left(d\vartheta^2 + \sin^2 \vartheta \, d\varphi^2 \right)$$

. S^5 in $S^3 \times S^1$ foliation:

$$d\Omega_5^2 = d\theta^2 + \sin^2 \theta \, d\phi^2 + \cos^2 \theta \underbrace{\left(d\xi^2 + \cos^2 \xi \, d\alpha_1^2 + \sin^2 \xi \, d\alpha_2^2 \right)}_{\Omega_3},$$

String embedding: depends on latitude $\boldsymbol{\theta_0}$ and position $\boldsymbol{\Omega}^{(\mathbf{0})}$ in S^3

$$u = 0, \qquad \begin{matrix} \rho = \sigma, \\ \psi = \phi = \tau, \end{matrix} \qquad \Omega_3 = \boldsymbol{\Omega}^{(\mathbf{0})} = cte, \quad \sin \theta(\rho) = \frac{\sinh \rho \sin \boldsymbol{\theta_0}}{\cosh \rho + \cos \boldsymbol{\theta_0}}$$

Homogeneity of S^3 implies independence of $\boldsymbol{\Omega}^{(\mathbf{0})}$ in all physical quentities.

Induced geometry: is asymptotic to $EAdS_2$ $\forall$ $\boldsymbol{\theta_0}$ (from now on we set $L^2 = 1$)

$$ds^2 = M(\rho)(d\rho^2 + \sinh^2\rho\, d\phi^2), \qquad M(\rho) = 1 + \frac{\sin^2\boldsymbol{\theta_0}}{(\cosh\rho + \cos\boldsymbol{\theta_0})^2}$$

Worldsheets have disk topology $\Rightarrow$ 3 CKV $\forall$ slns.

On shell action $\equiv$ Worldsheet area is divergent

$$\tilde{S} = 2\pi \int_0^R \sinh\rho\, d\rho\, M(\rho) = 2\pi \frac{\sinh^2 R}{\cos\boldsymbol{\theta_0} + \cosh R} \approx 2\pi \left(e^R - \cos\boldsymbol{\theta_0} + O(e^{-R})\right)$$

Adding boundary Euler $\chi_b = \frac{1}{2\pi}\int ds\, \kappa_g$ (or performing Legendre transform), effectively eliminates the divergent piece leaving a negative regularized area:

$$\tilde{S}_{reg} = \tilde{S} - \chi_b = -2\pi\cos\boldsymbol{\theta_0}$$

This result successfully matches the leading order localization result

$$\boxed{\left.\frac{\langle W^{1/4}\rangle}{\langle W^{1/2}\rangle}\right|_{\text{leading}} \approx e^{-\frac{\sqrt{\lambda}}{2\pi}(\tilde{S}_{reg}[\theta_0] - \tilde{S}_{reg}[0])} = e^{\sqrt{\lambda}(\cos\theta_0 - 1)}}$$

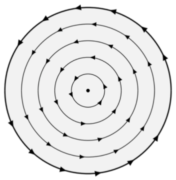
Important limits:

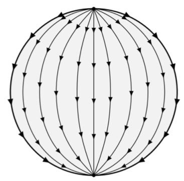
- $\frac{1}{2}$–BPS . $\boldsymbol{\theta_0 = 0}$: $\quad ds^2 = d\rho^2 + \sinh^2\!\rho\, d\phi^2, \quad M(\rho) = 1$

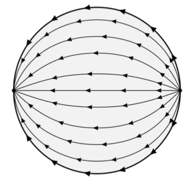
Induced geometry: EAdS$_2$. $SL(2,\mathbb{R}) \sim SO(2,1)$ isom $\rightarrow$ 3 KV leaving boundary invariant. Making $\rho = 2\,\text{arctanh}\, r \rightarrow ds^2 = \frac{4}{(1-r^2)^2}(dr^2 + r^2 d\phi^2)$, then CK equation is solved by

$$(P\xi)_{ab} = \nabla_a \xi_b + \nabla_b \xi_a - h_{ab} \nabla^c \xi_c = 0$$

$$\xi_{(1)} = \partial_\phi, \qquad \xi_{(2)} = (r^2-1)\sin\phi\,\partial_r - \frac{1}{r}(r^2+1)\cos\phi\,\partial_\phi, \qquad \xi_{(3)} = (r^2-1)\cos\phi\,\partial_r + \frac{1}{r}(r^2+1)\sin\phi\,\partial_\phi$$







CKV are non-zero at the boundary!

- $\frac{1}{4}$–BPS . $\boldsymbol{\theta_0 = \frac{\pi}{2}}$: $\quad ds^2 = M(\rho)(d\rho^2 + \sinh^2\!\rho\, d\phi^2), \quad M(\rho) = 1 + \frac{1}{\cosh^2\rho}$

Induced geometry: aEAdS$_2$, 1 KV + 2 CKV. Now an interesting thing happens:

The $\boldsymbol{\Omega}^{(\mathbf{0})}$ position at which the string is sitting in $S^3 \subset S^5$ is absent from the boundary point of view since S^3 collapses at $\theta = \frac{\pi}{2}$

$$d\Omega_5^2 = d\theta^2 + \sin^2\theta\, d\phi^2 + \cos^2\theta\, d\Omega_3^2\,, \qquad \sin\theta(\rho) = \tanh\rho \;\; \text{for} \;\; \boldsymbol{\theta_0} = \frac{\pi}{2}$$

But, S^3 inflates into the bulk due to non-trivial $\theta(\rho) \rightarrow \boldsymbol{\Omega}^{(\mathbf{0})}$ is relevant.

Outcome: a continuous set of solutions (moduli space) for given boundary conditions

$\leftrightarrow$ **expect zero modes in $\mathcal{O}_B$ for $\boldsymbol{\theta_0 = \frac{\pi}{2}}$** (a.k.a. Zarembo equatorial loops)

FLUCTUATIONS

$\delta X^M = E^M_{\underline{A}}(X)\,\xi^{\underline{A}}$: fluctuations are properly defined in tangent space.
Reason being SUSY, fermions only defined in tangent space.

$\delta\Psi$: Target space Green-Schwarz fermions (16-component) and Worldsheet scalars.
Reduced to 2×2 blocks by convenient choice of Dirac matrices.

GAUGE FIXING

Nambu-Goto: Static gauge. Trivial FP determinant.
8 physical transverse fluctuations + 8 two dimensional fermions

[Faraggi-Pando Zayas-GAS-Trancanelli 16]

Polyakov: Conformal gauge. FP and Longitudinal dets assumed to cancel each other.
8 physical transverse fluctuations + 8 two dimensional fermions

[Forini-Giangreco-Griguolo-Seminara-Vescovi 15]

κ-symmetry fixing gives trivial contribution.

FUNCTIONAL DETERMINANTS:

Quite generally computing a determinant involves:

1. Differential operator
2. Boundary conditions: for WL setup they are Dirichlet at worldsheet boundary.
3. An inner product:

$$S^{(2)} = \int d^2\sigma \sqrt{g}\, \phi \mathcal{O} \phi = \int d^2\sigma (\sqrt{g} M) \phi \left(\frac{1}{M} \mathcal{O} \right) \phi$$

Inner product / measure defines the operator to be diagonalized: $\mathcal{O}$ or $\mathcal{O}_M = M^{-1}\mathcal{O}$?

which crucially determines the set of normalizable fluctuations!

How does a change of measure affect a det? [Schwarz 79]

An important recurrent result in 2d is the relation between the dets of Weyl related operators: $\mathcal{O} = -\nabla_g^2 + X$ and $\tilde{\mathcal{O}} = \frac{1}{M}\mathcal{O}$,

$$A[\mathcal{O}, M] = \ln\det\tilde{\mathcal{O}} - \ln\det\mathcal{O} = \frac{1}{4\pi} \int d^2\sigma \sqrt{g} \ln M \left(\frac{1}{12} \nabla^2 \ln M + X - \frac{1}{6} R^{(2)} \right)$$

g is the 2d metric, $R^{(2)}[g]$ its scalar curvature, and X a space dependent mass term.

1-LOOP EFFECTIVE ACTION FOR $\frac{1}{4}$-BPS WILSON LOOP [FPzST 16]

Appropriate action is IIB Green-Schwarz, it incorporates spinor couplings to RR-fields.

1-loop effective action is a quotient between 8 fermions / 8 bosons

$$e^{-\Gamma^{1-\text{loop}}_{\text{effective}}(\theta_0)} = \frac{(\det\,\mathcal{O}_+(\theta_0))^{\frac{4}{2}}\;(\det\,\mathcal{O}_-(\theta_0))^{\frac{4}{2}}}{(\det\,\mathcal{O}_1(\theta_0))^{\frac{3}{2}}\;(\det\,\mathcal{O}_2(\theta_0))^{\frac{3}{2}}\;(\det\,\mathcal{O}_{3+}(\theta_0))^{\frac{1}{2}}\;(\det\,\mathcal{O}_{3-}(\theta_0))^{\frac{1}{2}}}$$

with measure being given by the induced metric $h_{\mu\nu} = M g_{\mu\nu}$

$$\mathcal{O}_1(\theta_0) = M^{-1}\left(-g^{\mu\nu}\nabla_\mu\nabla_\nu + 2\right), \qquad \mathcal{O}_2(\theta_0) = M^{-1}\left(-g^{\mu\nu}\nabla_\mu\nabla_\nu + V_2\right)$$

$$\mathcal{O}_{3\pm}(\theta_0) = M^{-1}\left(-g^{\mu\nu}D_\mu D_\nu + V_3\right), \qquad D_\mu = \nabla_\mu \pm i\mathcal{A}_\mu$$

$$\mathcal{O}_\pm(\theta_0) = M^{-\frac{1}{2}}\left(-i\left(\not{\mathcal{D}} + \frac{1}{4}\not{\partial}\ln M\right) - i\Gamma_{\underline{01}}\left(1+V\right) \pm W\right), \quad \mathcal{D}_\mu = \nabla_\mu \pm \frac{i}{2}\mathcal{A}_\mu$$

(coincide with those found by [FGGSV 15] for flat measure)

Here $g_{\mu\nu}$ and ∇_μ evaluated on AdS_2 metric. $\mathcal{A}_\rho = 0$, $\mathcal{A}_\tau = \mathcal{A}$ and

$$V_2(\rho) = -\frac{2\sin^2\boldsymbol{\theta_0}}{(\cosh\rho + \cos\boldsymbol{\theta_0})^2}, \quad V_3(\rho) = -\frac{\partial_\rho\mathcal{A}(\rho)}{\sinh\rho}, \quad V(\rho) = \frac{1}{\sqrt{M(\rho)}} - 1, \quad \cos\theta(\rho) = \frac{1+\cosh\rho\cos\boldsymbol{\theta_0}}{\cosh\rho + \cos\boldsymbol{\theta_0}}$$

$$W(\rho) = \frac{\sin^2\boldsymbol{\theta_0}}{\sqrt{M(\rho)}(\cosh\rho + \cos\boldsymbol{\theta_0})^2}, \quad \mathcal{A}(\rho) = 1 - \frac{1+\cosh\rho\cos\theta(\rho)}{\cosh\rho + \cos\theta(\rho)}, \quad M(\rho) = 1 + \frac{\sin^2\boldsymbol{\theta_0}}{(\cosh\rho + \cos\boldsymbol{\theta_0})^2}$$

. $\mathcal{A}_\mu$ smoothly collapses at the center of the disk, becomes non-zero at the boundary and vanishes in the $\frac{1}{2}$-BPS limit. Geometrically the components of the target space spin connection along the normal bundle: $\mathcal{A}^{\underline{ij}} = P[\Omega^{\underline{ij}}]$, P: pullback to the worldsheet.

. Fermions have chiral and non-chiral masses. Non-chiral mass vanishes in $1/2$-BPS lim.

The 1st correction to the Wilson loop ratio

$$\ln \frac{\langle W^{1/4} \rangle}{\langle W^{1/2} \rangle} = \sqrt{\lambda}(\cos\theta_0 - 1) \underbrace{-\frac{3}{2}\ln\cos\theta_0}_{\text{1-loop}} + \mathcal{O}(\frac{1}{\sqrt{\lambda}})$$

should be matched with

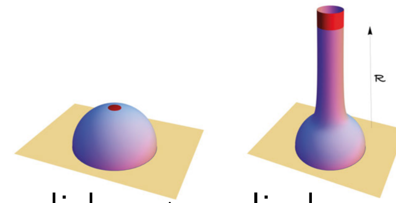
$$e^{-\Delta\Gamma^{\text{1-loop}}_{\text{effective}}(\theta_0)} = \left[\frac{\left(\frac{\det\, \mathcal{O}_+(\theta_0)}{\det\, \mathcal{O}_+(0)}\right)^4 \left(\frac{\det\, \mathcal{O}_-(\theta_0)}{\det\, \mathcal{O}_-(0)}\right)^4}{\left(\frac{\det\, \mathcal{O}_1(\theta_0)}{\det\, \mathcal{O}_1(0)}\right)^3 \left(\frac{\det\, \mathcal{O}_2(\theta_0)}{\det\, \mathcal{O}_2(0)}\right)^3 \left(\frac{\det\, \mathcal{O}_{3+}(\theta_0)}{\det\, \mathcal{O}_{3+}(0)}\right)^1 \left(\frac{\det\, \mathcal{O}_{3-}(\theta_0)}{\det\, \mathcal{O}_{3-}(0)}\right)^1} \right]^{\frac{1}{2}}$$

Perspectives to compute the determinants:

1. Scale the induced metric to flat space. Any 2d metric is conformally flat!
   [KT 08, FGGSV 15, FPzST 16, CMrZ 18]

2. Zeta function. Natural setup, honestly computed on the disk!
   [DGT 00, BT 14, FTV 17, AdFPzRS 18]

COMPUTING DETERMINANTS



Perspective 1: scale to flat space performs a change of topology: disk $\rightarrow$ cylinder

$$ds^2 = M(\rho)\sinh^2\rho\left(\frac{d\rho^2}{\sinh^2\rho} + d\phi^2\right) = \Omega(\sigma)\left(d\sigma^2 + d\phi^2\right) \quad \rightarrow \quad d\bar{s}^2 = \left(d\sigma^2 + d\phi^2\right)$$

Anomaly contributions of bosons and fermions cancel out: curved disk $\equiv$ flat cylinder.

Gelfand-Yaglom: developed for 1d second order operator dets in interval $(0, R)$.
It computes quotient of dets in terms of homogeneous solutions

$$\text{DD}: \quad \frac{\det\mathcal{O}}{\det\bar{\mathcal{O}}} = \frac{\psi(R)}{\bar{\psi}(R)} \quad \text{with} \quad \mathcal{O}\psi = 0,\ \bar{\mathcal{O}}\bar{\psi} = 0, \quad \psi(0) = 0,\ \psi'(0) = 1$$

Applied in 2d with circular symmetry by Fourier decomposing as

$$\ln\frac{\det\mathcal{O}(\theta_0)}{\det\mathcal{O}(0)} \equiv \sum_{l=-\infty}^{\infty} \ln\frac{\det\mathcal{O}_l(\theta_0)}{\det\mathcal{O}_l(0)} \qquad l = \mathbb{Z}, \mathbb{Z} + \frac{1}{2}.$$

For disk topology, solve in $\rho \in (\epsilon, R)$ and take appropriate limits at the end.
Result using this technique:

$$\Delta\Gamma^{\text{1-loop}}_{\text{effective}}(\theta_0) = \frac{3}{2}\ln\cos\theta_0 - \underbrace{\ln\cos\frac{\theta_0}{2}}_{\text{reminder} \leftrightarrow \text{mismatch}}$$

Remarks: potentially divergence in Fourier sum cancels due to susy multiplet structure.
GY method is insensitive to the measure!

[**FGGSV 15, FPzST 16** also KT 08, Drukker-Forini 11,Forini-Giangreco-Sax 12, Krjstiansen-Makeenko 12]

Perspective 2: Zeta function. Honest computation on the disk.

. [Camporesi-Higuchi 92]: Applied to hyperbolic space in '90s .

. [Drukker-Gross-Tseytlin 00]: showed exact cancellation of determinants for straight line WL. Agree with expectations from field thoery side: $\langle W^{1/2}(\text{line})\rangle_{\text{1-loop}} = 0$

. [Buchbinder-Tseytlin 14]: Circular $\frac{1}{2}$- dets via Heat Kernel/Zeta function on EAdS_2. Reobtain Kruczenski-Tirziu mismatch. Fermionic dets were transformed into 2nd order:

$$\text{strong coupling } \lambda \gg 1 : \qquad \langle W^{1/2}(\text{circle})\rangle_{\text{1-loop}} \neq \left(\frac{\det^{1/2}(-\not{D}^2+1)}{\det^3(-\nabla^2+2)\det^5(-\nabla^2)} \right)^{1/2}$$

. [Forini-Tseytlin-Vescovi 1702]: $\frac{1}{4}$-BPS latitude WL computed as a small deformation of the $1/2$-BPS WL (perturbatively in θ_0). Found a match for the quotient.
They profited from homogeneity of background EAdS_2 space. Heat Kernel in homogeneous space requires the volume of space. Common folklore takes $\text{Vol}(\text{EAdS}_2) = -2\pi$

$$\lambda \gg 1, (\theta_0)^2 \ll 1 : \qquad \frac{\langle W^{1/4}(\text{circle})\rangle}{\langle W^{1/2}(\text{circle})\rangle}\Big|_{\text{1-loop}, \theta_0^2} = \left[\frac{\frac{\det \mathcal{O}_F(\theta_0)}{\det \mathcal{O}_F(0)}}{\frac{\det \mathcal{O}_B(\theta_0)}{\det \mathcal{O}_B(0)}} \right]^{\frac{1}{2}}_{\theta_0^2}$$

$$\int d^2\sigma \sqrt{g} = \lim_{R\to\infty} \left[2\pi \int_0^R \sinh\rho \, d\rho = 2\pi(\cosh R - 1) \approx 2\pi \left(\cancel{\frac{e^R}{2}} - 1 + O(e^{-R}) \right) \right] = -2\pi$$

Motivated by [FTV 1702] we decided to attack the $\frac{1}{4}$-BPS in full generality using ζ-function.

. [Aguilera Damia-Faraggi-Pando Zayas-Rathee-Silva 1802]: Strategy was to scale the metric to EAdS_2

$$ds^2 = M(\rho)\left(d\rho^2 + \sinh^2\!\rho\, d\phi^2\right) \quad \to \quad d\bar{s}^2 = d\rho^2 + \sinh^2\!\rho\, d\phi^2$$

No change of topology!

Generically the $\frac{1}{4}$- operators take the form

$$\mathcal{O}_M = M^{-1}\bar{\mathcal{O}}, \qquad \bar{\mathcal{O}} = \underbrace{-\nabla^2}_{\text{AdS connection + gauge fields}} + m^2 + V$$

The determinants are therefore related according to

$$\ln\det\mathcal{O}_M = \ln\det\bar{\mathcal{O}} + A[\mathcal{O}, M]$$

Now a finite piece survives from the anomaly.

ζ-function: consider computing

$$\ln\det\frac{\bar{\mathcal{O}}(\theta_0)}{\bar{\mathcal{O}}(0)} \qquad \text{with} \quad \bar{\mathcal{O}}(0) \text{ a free operator in AdS } (V=0 \;\; \mathcal{A}=0)$$

Space is no longer homogeneous so we use the $U(1)$ circular isometry ∂_τ to Fourier decompose and re-express the 2d determinant in terms of 1d ones

$$\ln\frac{\det\bar{\mathcal{O}}(\theta_0)}{\det\bar{\mathcal{O}}(0)} \equiv \sum_{l=-\infty}^{\infty}\ln\frac{\det\bar{\mathcal{O}}_l(\theta_0)}{\det\bar{\mathcal{O}}_l(0)}$$

The problem with this expression is that the Fourier sum diverges!

In flat space this problem has been studied [Kirsten-Dunne 06]. We generalized their results to curved AdS and also we considered the presence of gauge fields.

Define

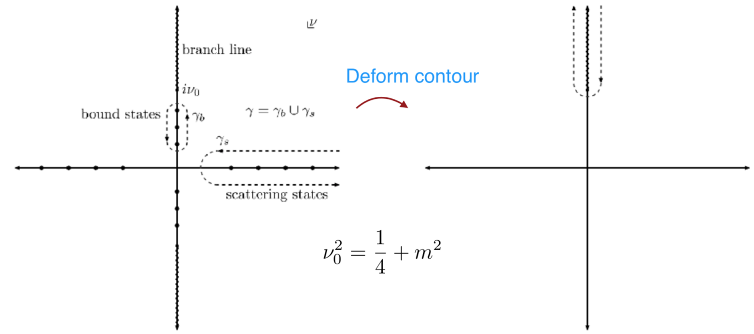
$$\ln\frac{\det\bar{\mathcal{O}}(\theta_0)}{\det\bar{\mathcal{O}}(0)} \equiv -\hat{\zeta}'_{\bar{\mathcal{O}}}(0) - \ln(\mu^2)\hat{\zeta}_{\bar{\mathcal{O}}}(0)\,, \quad \hat{\zeta}_{\bar{\mathcal{O}}}(s) \equiv \zeta_{\bar{\mathcal{O}}(\theta_0)}(s) - \zeta_{\bar{\mathcal{O}}(0)}(s)$$

with

$$\hat{\zeta}_{\bar{\mathcal{O}}}(s) = \sum_{l=-\infty}^{\infty}\hat{\zeta}_{\bar{\mathcal{O}}_l}(s)$$

Analytically continue to $s=0$ and evaluate. On general grounds: $\ln\frac{\det\bar{\mathcal{O}}(\theta_0)}{\det\bar{\mathcal{O}}(0)} \sim \frac{1}{l}$.

$$\lambda(k) = \nu_0^2 + k^2,\ \nu_0 = \sqrt{\tfrac{1}{4} + m^2}$$



By standard contour integration we reexpress $\zeta_{\bar{\mathcal{O}}}$

$$\zeta_{\bar{\mathcal{O}}_l}(s) \equiv \sum_i (\lambda_{(l),i})^{-s} = \oint_\gamma \frac{dk}{2\pi i} \left(k^2 + \nu_0^2\right)^{-s} \underbrace{\partial_k \ln f_l(k)}_{\text{simple poles at location of eigevalues}}$$

$$= \frac{\sin \pi s}{\pi} \int_{\nu_0}^{\infty} dk \left(k^2 - \nu_0^2\right)^{-s} \partial_k \ln f_l(ik)$$

With $f_l(k)$ the extension to the complex k-plane of the regular solution to

$$\bar{\mathcal{O}}_l f_{(l,k)}(\rho) = (\nu_0^2 + k^2) f_{(l,k)}(\rho) \ \text{ satisfying } \ f_{(l,k)}(R) = 0, \quad f_{(l,k)}(\rho) \approx \rho^{|l|}, \quad \rho \ll 1$$

which effectively puts the system in a box of radius R quantizing the spectrum k_i. After substracting the free operator one obtains

$$\hat{\zeta}_{\bar{\mathcal{O}}_l}(s) = \frac{\sin \pi s}{\pi} \int_{\nu_0}^{\infty} dk \ \left(k^2 - \nu_0^2\right)^{-s} \partial_\nu \ln g_l(ik) \ \text{ with } \ g_l(k) = \frac{f_{(l,k)}(R)}{f^{\text{free}}_{(l,k)}(R)}$$

$g_l(k)$ goes under the name of Jost function/phase shift. Note the resemblance with GY.

Substracting the $1/l$ divergence allows to regularize the sum. The divergence when added back can be easily be given meaning and analytically continued to $s=0$ obtaining finite expressions.

Bosons:

$$\ln \frac{\det \bar{\mathcal{O}}(\theta_0)}{\det \bar{\mathcal{O}}(0)} = \ln \frac{\det \bar{\mathcal{O}}_0}{\det \bar{\mathcal{O}}_0^{\text{free}}} + \sum_{l=1}^{\infty} \left(\ln \frac{\det \bar{\mathcal{O}}_l}{\det \bar{\mathcal{O}}_l^{\text{free}}} + \ln \frac{\det \bar{\mathcal{O}}_{-l}}{\det \bar{\mathcal{O}}_{-l}^{\text{free}}} + \frac{2}{l} \hat{\zeta}_{\bar{\mathcal{O}}}(0) \right) - 2 \left(\gamma + \ln \frac{\mu}{2} \right) \hat{\zeta}_{\bar{\mathcal{O}}}(0)$$
$$+ \int_0^\infty d\rho \, \sinh \rho \ln (\sinh \rho) \, V - q^2 \int_0^\infty d\rho \, \frac{\mathcal{A}^2}{\sinh \rho}$$
$$\hat{\zeta}_{\bar{\mathcal{O}}}(0) = -\frac{1}{2} \int_0^\infty d\rho \, \sinh \rho \, V$$

Fermions:

$$\ln \frac{\det \bar{\mathcal{O}}}{\det \bar{\mathcal{O}}^{\text{free}}} = \sum_{l=\frac{1}{2}}^{\infty} \left(\ln \frac{\det \bar{\mathcal{O}}_l}{\det \bar{\mathcal{O}}_l^{\text{free}}} + \ln \frac{\det \bar{\mathcal{O}}_{-l}}{\det \bar{\mathcal{O}}_{-l}^{\text{free}}} + \frac{2}{l+\frac{1}{2}} \hat{\zeta}_{\bar{\mathcal{O}}}(0) \right) - 2 \left(\gamma + \ln \frac{\mu}{2} \right) \hat{\zeta}_{\bar{\mathcal{O}}}(0)$$
$$+ \int_0^\infty d\rho \, \sinh \rho \ln (\sinh \rho) \left((m+V)^2 - W^2 - m^2 \right) - q^2 \int_0^\infty d\rho \, \frac{\mathcal{A}^2}{\sinh \rho}$$
$$- \int_0^\infty d\rho \, \sinh \rho \, W^2$$
$$\hat{\zeta}_{\bar{\mathcal{O}}}(0) = -\frac{1}{2} \int_0^\infty d\rho \, \sinh \rho \left((m+V)^2 - W^2 - m^2 \right)$$

The dependence on the renormalization scale $\ln(\mu^2)$ cancels in the final expression. This is reassuring in order to have an unambiguous result!

Putting all pieces together we find

$$\Delta\Gamma^{\text{1-loop}}_{\text{effective}}(\theta_0) = \underbrace{\frac{3}{2}\ln\cos\theta_0}_{\text{correct}} + \underbrace{2\left(4\sin^2\frac{\theta_0}{2} - \theta_0\sin\theta_0\right)}_{\text{reminder is } O(\theta_0^4)}$$

$$= -\frac{3}{4}\theta_0^2 + O\left(\theta_0^4\right)$$

The result leaves us with an uncomfortable feeling, coinciding with the Localization result at 1st order in θ_0-pertubation (coinciding with [FTV 1702]) but differing with the full result!

MATCHING OF $\frac{1}{4}$-BPS IN ANNULUS GEOMETRY

Introducing an auxiliary boundary at $\rho = \epsilon$ effectively changes the topology

disk$\rightarrow$ annulus!

Moreover, if we excise the center of the disk at $\rho = \epsilon$, we obtaining an annulus geometry.

$$S_\epsilon \simeq \frac{2\pi\epsilon^2}{(1+\cos\theta_0)} \quad \rightarrow \quad \epsilon = \left((1+\cos\theta_0)\,\frac{S_\epsilon}{2\pi}\right)^{1/2}$$

Small radial cutoff in diff invariant variables shows $\boldsymbol{\theta_0}$ dependence!

[Cagnazzo-Medina Rincon-Zarembo 1712] A delicate analysis of the determinants computed using phase shift method showed an anomaly due to the θ_0 dependence in the regulator. The anomaly exactly cancels the reminder.

[Medina Rincon-Tseytlin-Zarembo 1804]: fixed of normalization comparing to the equatorial WL (zero modes).

5 Summary and Open Questions

- Wilson loops are dual to string worldsheets with appropriate b.c.
- Leading order expansion at strong coupling matches between two sides.
- Fluctuation determinants for bosons and fermions as usual give ambiguous results. Many different techniques (GY, phase shift) with fermions computed at the 1st or 2nd order level give identical results for the cylinder.
- Match for $\frac{1}{4}$-BPS loop in $\mathcal{N} = 4$ was found by going to the cylinder and taking appropriate diff invariant reguator.

But..

- Role of susy is unclear in b.c.
- Cancellation of longitudinal and ghost dets was an hypothesis. b.c.? CKV are non-zero at the boundary, contrary to physical Dirichlet bc for WL. Krjstiansen-Makeenko?
- Zero modes and role of susy in equatorial loop? index?
- Drukker-Gross $\lambda^{-3/4}$ argument fails for IIA $\frac{1}{2}$-BPS WL! so?....

  In ABJM case massless modes appear. Zeta function technique becomes subtle.
- Zeta function had the benefit of not needing b.c, but mismatch? susy?
- Kappa-symmetry gauge fixing determinant gives a constant DGT. Never computed.
- Measure independence of induced worldsheet metric? But normalization constant as done by Zarembo precisely compares to equatorial loop were they exist!

Medina-Rincon PHD thesis June 2018: "This beautiful result highlights the importance and desperate need for a better understanding of the mathematical machinery required for these perturbative string theory calculations."

THANK YOU!

MOTIVATIONS: apologies for missing works

1. Maldacena 9803, Rey-Yee: Wilson loops in fundamental irrep and string worldsheets. Coupling to scalars. Parallel lines $\leftrightarrow$ gutter worldsheet. $V_{q\bar{q}}$ computation.
2. Drukker-Gross-Ooguri 9904: Coupling to scalars and gauge fields. Several issues related to cusp WL. Legendre transformation of action correctly regularizes.
3. Drukker-Gross-Tseytlin 0001: Proper foundation of semiclassical string partition funct. GS action, RR fields and fermions. Dets for straight line from zeta function and on-shell method. Wrote down circular WL dets. Longitudinal modes cancel FP det.
4. Erickson-Semenoff-Zarembo 0003: $\frac{1}{2}$- circular WL: perturbation theory. Summation of ladder diagrams at planar level give Bessel function. which relates to gaussian MM.
5. Drukker-Gross 0010: All N MM computation give Laguerre. Straight $\leftrightarrow$ circle WL are conformally related, but an anomaly $\Rightarrow \langle W(|)\rangle \neq \langle W(\bigcirc)\rangle$.
   3 CKV of disk are zero modes of FP determinant $\leftrightarrow \log\sqrt{\lambda}$
6. Zarembo 02: Classify susies of some WL according to ansatz relating $\dot{x}^\mu \leftrightarrow \vec{n}$. Stresses that moduli bring $\log\lambda^{1/4}$ contribution. Equatorial WL $\theta_0 = \frac{\pi}{2}$ solution has 3 moduli leading to a cancellation of subleading corrections, then $W = 1$.
7. Drukker 06: Based on perturbative computation, conjectures that $\frac{1}{4}$-BPS latitude WL. vev is given by as Bessell with $\lambda^2 \to \lambda^2\cos\theta_0$
8. Pestun 07: Localization of $\mathcal{N} = 4$ formulated on (curved) S^4 reduces to gaussian MM Circular WL in fundamental gives Laguerre proving ESZ and DG conjectures.

9. Drukker-Giombi-Ricci-Trancanelli 0711: Richer ansatz for susy WL. A subset of these contains latitude WL, the preserved susies allow to compute the vev via localization.
10. Kruczenski-Tirziu 08: Compute using Gelfand-Yaglom. Fermionic dets are 2nd order and square rooted. Compute Fermions/Bosons. Find mismatch.
11. Kristjansen-Makeenko 12: Circular dets using Gelfand-Dikii. b.c. for $\|$ modes
12. Buchbinder-Tseytlin 1404: Circular $\frac{1}{2}$- dets via Heat Kernel/Zeta function. Reobtain Kruczenski-Tirziu mismatch. Fermionic dets are 2nd order.
13. Forini-Giangreco Puletti-Griguolo-Seminara-Vescovi 1512: Write down $\frac{1}{4}$- dets. $W^{1/4}/W^{1/2}$- quotient estimated numerically. Mismatch with localization.
14. Faraggi-Pando Zayas-GAS-Trancanelli 1601: Analytic result for $\frac{1}{4}$- dets using GY. Organize susy multiplets. Reobtain FGPGSV mismatch. Fermionic dets are 1st order.
15. Forini-Tseytlin-Vescovi 1702: θ_0 perturbative Heat Kernel computation of latitude $\frac{1}{4}$-WL matches localization.
16. Cagnazzo-Medina Rincon-Zarembo 1712: Compute dets by spectral/phase shift methods reobtaining 13. & 14. Find an anomaly, which removes the discrepancy with localization, due to disk $\to$ cylinder topology change.
17. Aguilera Damia-Faraggi-Pando Zayas-Rathee-GAS 1802: Zeta function is θ_0-exact Mismatch with localization result. Fermionic dets are 2nd order and square rooted.
18. Medina Rincon-Tseytlin-Zarembo 1804: Fix the normalization of string path integral by relating CKV and configurational zero modes in equatorial loop.