Surface operators and duality relations in *N*=2 gauge theories

Alberto Lerda

Università del Piemonte Orientale and

INFN – Torino, Italy

UPO
UNIVERSITÀ DEL PIEMONTE ORIENTALE

INFN
Istituto Nazionale di Fisica Nucleare
SEZIONE DI TORINO

ICTP
The Abdus Salam
International Centre
for Theoretical Physics

Trieste, 10 July 2018

This talk is mainly based on:

- S.K. Ashok, M. Billò, E. Dell'Aquila, M. Frau, V. Gupta, R.R. John and A. L., *"Surface operators, dual quivers and residues"*, arXiv:1807.xxxxx
- S.K. Ashok, S. Ballav, M. Billò, E. Dell'Aquila, M. Frau, R.R. John and A. L., *"Surface operators in conformal SQCD"*, arXiv:180y.yyyyy

- S.K. Ashok, M. Billò, E. Dell'Aquila, M. Frau, V. Gupta, R.R. John and A. L., *"Surface operators in 5d gauge theories and duality relations"*, **JHEP 1805 (2018) 046**, arXiv:1712.06946
- S.K. Ashok, M. Billò, E. Dell'Aquila, M. Frau, V. Gupta, R.R. John and A. L., *"Surface operators, chiral rings and localization in N=2 theories"*, **JHEP 1711 (2017) 137**, arXiv:1707.08922
- S.K. Ashok, M. Billò, E. Dell'Aquila, M. Frau, R.R. John and A. L., *"Modular and duality properties of surface operators in N=2* gauge theories"*, **JHEP 1707 (2017) 068**, arXiv:1702.02833

but it builds on a very vast literature …

Plan of the talk

1. Introduction
2. Surface operators in N=2 gauge theories
3. Duality relations
4. Conclusions

Defects

- The study of how a QFT responds to the presence of **defects** has recently received a lot of attention.
- **Defects** are disturbances supported on sub-manifolds of the space-time that can be used to probe the theory in the bulk:

1. Local operators (*d=0*), *e.g.* chiral correlators

$$\left\langle \, \mathrm{Tr}\phi^{n_1}(x_1)\cdots\mathrm{Tr}\phi^{n_k}(x_k) \, \right\rangle$$

Defects

- The study of how a QFT responds to the presence of **defects** has recently received a lot of attention.

- **Defects** are disturbances supported on sub-manifolds of the space-time that can be used to probe the theory in the bulk:

1. Local operators (*d=0*), *e.g.* chiral correlators

$$\left\langle \mathrm{Tr}\phi^{n_1}(x_1)\cdots\mathrm{Tr}\phi^{n_k}(x_k) \right\rangle$$

2. Line operators (*d=1*), *e.g.* Wilson lines

$$\left\langle \mathrm{Tr}_R\, P\, \mathrm{e}^{\oint_C iA} \right\rangle$$

Defects

- The study of how a QFT responds to the presence of **defects** has recently received a lot of attention.
- **Defects** are disturbances supported on sub-manifolds of the space-time that can be used to probe the theory in the bulk:

1. Local operators (*d=0*), *e.g.* chiral correlators

$$\langle\, \mathrm{Tr}\phi^{n_1}(x_1)\cdots \mathrm{Tr}\phi^{n_k}(x_k)\,\rangle$$

2. Line operators (*d=1*), *e.g.* Wilson lines

$$\Big\langle \mathrm{Tr}_R\, P\, \mathrm{e}^{\oint_C iA} \Big\rangle$$

3. Surface operators (*d=2*)

Defects

- The study of how a QFT responds to the presence of **defects** has recently received a lot of attention.

- **Defects** are disturbances supported on sub-manifolds of the space-time that can be used to probe the theory in the bulk:

1. Local operators (*d=0*), *e.g.* chiral correlators

$$\langle\, \mathrm{Tr}\phi^{n_1}(x_1)\cdots \mathrm{Tr}\phi^{n_k}(x_k)\,\rangle$$

2. Line operators (*d=1*), *e.g.* Wilson lines

$$\Big\langle \mathrm{Tr}_R\, P\, \mathrm{e}^{\oint_C iA} \Big\rangle$$

3. Surface operators (*d=2*)

4. Domain walls (*d=3*)

Defects

- The study of how a QFT responds to the presence of **defects** has recently received a lot of attention.
- **Defects** are disturbances supported on sub-manifolds of the space-time that can be used to probe the theory in the bulk:

1. Local operators (*d=0*), *e.g.* chiral correlators

$$\left\langle \mathrm{Tr}\phi^{n_1}(x_1)\cdots\mathrm{Tr}\phi^{n_k}(x_k)\right\rangle$$

2. Line operators (*d=1*), *e.g.* Wilson lines

$$\left\langle \mathrm{Tr}_R\, P\, \mathrm{e}^{\oint_C iA}\right\rangle$$

3. Surface operators (*d=2*)

4. Domain walls (*d=3*)

Defects

- There are different ways of describing defects in a *4d* QFT:

  1. Modifying the QFT path-integral by imposing a singular behavior of the fields on the defect.

  2. Coupling the QFT to additional degrees of freedom localized on the defect.

  3. Using a String Theory realization of the QFT and introducing extra D-branes or M-branes to represent the defect.

  4. …

Defects

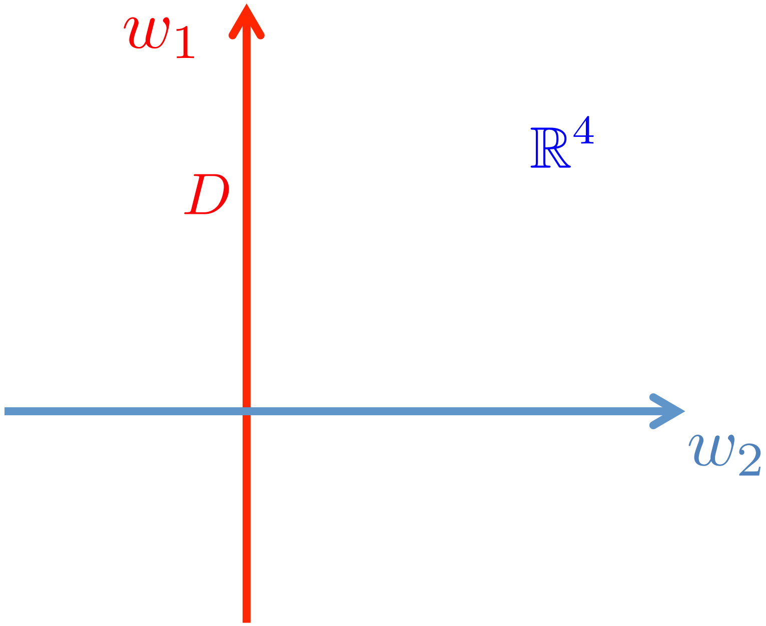
- There are different ways of describing defects in a *4d* QFT:

1. Modifying the QFT path-integral by imposing a singular behavior of the fields on the defect.
2. Coupling the QFT to additional degrees of freedom localized on the defect.
3. Using a String Theory realization of the QFT and introducing extra D-branes or M-branes to represent the defect.
4. …

Surface operators

Surface operators

- Surface operators in a *4d* theory are operators supported on a *2d* submanifold: $D \subset \mathbb{R}^4$
  - They are particular examples of non-local operators that play the role of “thermometers” for the QFT: when introduced in the path-integral, they provide us with valuable information (phases, non-perturbative features, ...) on the QFT. (Gukov+Kapustin ’13)



$$\mathbb{R}^4 \simeq \mathbb{C}^2 : (w_1, w_2)$$

$$D \simeq \mathbb{C} : (w_1, 0)$$

Surface operators

- Surface operators can be described:
  - as singularities (or monodromy defects) for the gauge field A_μ along the surface *D* in the *4d* space-time

monodromy defects

Surface operators

- Surface operators can be described:
  - as singularities (or monodromy defects) for the gauge field A_μ along the surface D in the $4d$ space-time
  - as coupled $2d$-$4d$ systems, namely as a $2d$ theory supported on D with a flavor symmetry group G that is gauged in $4d$.

monodromy defects

coupled $2d$-$4d$ systems

Surface operators

- Surface operators can be described:
  - as singularities (or monodromy defects) for the gauge field A_μ along the surface D in the $4d$ space-time
  - as coupled $2d$-$4d$ systems, namely as a $2d$ theory supported on D with a flavor symmetry group G that is gauged in $4d$.

Surface operators as monodromy defects

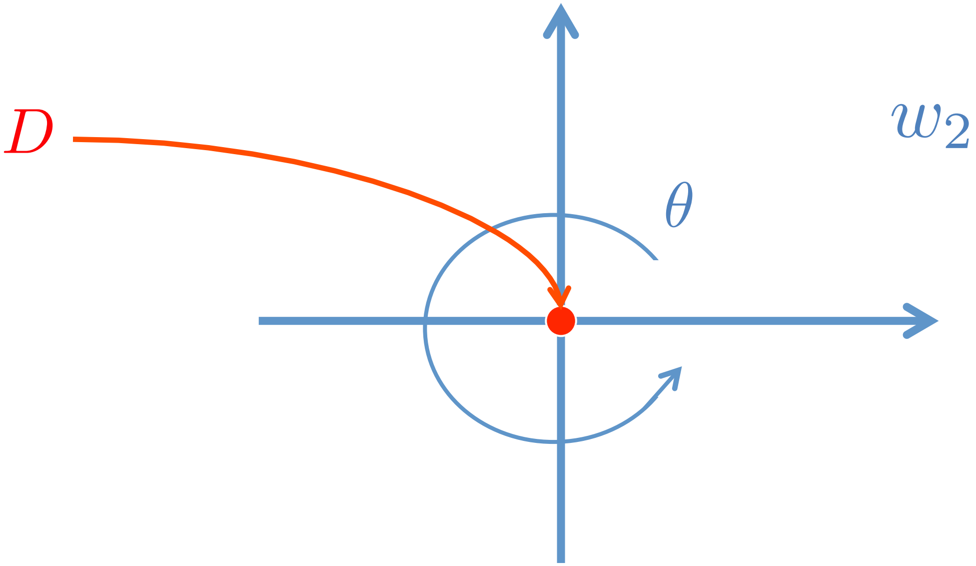
- The presence of the defect can be encoded in a singular behavior of the SU(*N*) gauge field on a loop around the defect:

$$A = -\,\mathrm{diag}\,(\underbrace{\gamma_1,\cdots,\gamma_1}_{n_1},\underbrace{\gamma_2,\cdots,\gamma_2}_{n_2},\cdots,\underbrace{\gamma_M,\cdots,\gamma_M}_{n_M})\;d\theta$$

where

$$\sum_i n_i = N \qquad , \qquad w_2 = \rho\,\mathrm{e}^{i\theta}$$

(Gukov+Witten ’06, ‘08)



Surface operators as monodromy defects

- The presence of the defect can be encoded in a singular behavior of the SU(*N*) gauge field on a loop around the defect:

$$A = -\,\mathrm{diag}\,(\underbrace{\gamma_1, \cdots, \gamma_1}_{n_1}, \underbrace{\gamma_2, \cdots, \gamma_2}_{n_2}, \cdots, \underbrace{\gamma_M, \cdots, \gamma_M}_{n_M})\; d\theta$$

where

$$\sum_i n_i = N \qquad , \qquad w_2 = \rho\, \mathrm{e}^{i\theta}$$

(Gukov+Witten '06, '08)

- This monodromy, *i.e.* the vector $\vec{n} = \{n_1, n_2, \ldots, n_M\}$, breaks the gauge symmetry SU(*N*) to a (Levi) subgroup $\mathbb{L}$, which characterizes the defect

$$\mathrm{SU}(N) \to \mathrm{S}\big[\mathrm{U}(n_1) \times \mathrm{U}(n_2) \times \cdots \times \mathrm{U}(n_M)\big]$$

Surface operators as monodromy defects

- In presence of the defect, quantized magnetic fluxes are allowed for each group $\mathrm{U}(n_i)$ factor:

$$\frac{1}{2\pi}\int_D \mathrm{Tr}\, F_{\mathrm{U}(n_i)} = m_i \ \in \mathbb{Z} \quad , \qquad \sum_i m_i = 0$$

- So besides the usual instanton factor

$$\exp\left[2\pi i\,\tau\Big(\frac{1}{8\pi^2}\int_{\mathbb{R}^4} \mathrm{Tr}\, F\wedge F\Big)\right]$$

complexified gauge coupling

in the path-integral we can insert also a new type of factor

$$\exp\left[2\pi i \sum_i \eta_i \Big(\frac{1}{2\pi}\int_D \mathrm{Tr}\, F_{\mathrm{U}(n_i)}\Big)\right]$$

constant “electric” parameters

Surface operators as monodromy defects

- In presence of the defect, quantized magnetic fluxes are allowed for each group $\mathrm{U}(n_i)$ factor:

$$\frac{1}{2\pi}\int_D \mathrm{Tr}\, F_{\mathrm{U}(n_i)} = m_i \;\in \mathbb{Z} \quad , \qquad \sum_i m_i = 0$$

- It can be shown that this implies that

$$\frac{1}{8\pi^2}\int_{\mathbb{R}^4} \mathrm{Tr}\, F \wedge F = k \;+ \sum_i \gamma_i\, m_i$$

instanton number

new contribution due to the defect

Surface operators as monodromy defects

- In presence of the defect, quantized magnetic fluxes are allowed for each group $\mathrm{U}(n_i)$ factor:

$$\frac{1}{2\pi}\int_D \mathrm{Tr}\, F_{\mathrm{U}(n_i)} = \boxed{m_i} \in \mathbb{Z} \quad , \qquad \sum_i m_i = 0$$

- It can be shown that this implies that

$$\frac{1}{8\pi^2}\int_{\mathbb{R}^4} \mathrm{Tr}\, F \wedge F = \boxed{k + \sum_i \gamma_i\, m_i}$$

- Thus

$$-S_{\mathrm{inst}} = 2\pi i\, \tau \Big(\frac{1}{8\pi^2}\int_{\mathbb{R}^4} \mathrm{Tr}\, F \wedge F \Big) + 2\pi i \sum_i \eta_i \Big(\frac{1}{2\pi}\int_D \mathrm{Tr}\, F_{\mathrm{U}(n_i)} \Big)$$

Surface operators as monodromy defects

- In presence of the defect, quantized magnetic fluxes are allowed for each group $\mathrm{U}(n_i)$ factor:

$$\frac{1}{2\pi}\int_D \mathrm{Tr}\, F_{\mathrm{U}(n_i)} = m_i \in \mathbb{Z} \quad , \qquad \sum_i m_i = 0$$

- It can be shown that this implies that

$$\frac{1}{8\pi^2}\int_{\mathbb{R}^4} \mathrm{Tr}\, F \wedge F = k + \sum_i \gamma_i\, m_i$$

- Thus

$$-S_{\mathrm{inst}} = 2\pi i\, \tau \Big(k + \sum_i \gamma_i\, m_i\Big) + 2\pi i \sum_i \eta_i\, m_i$$

$$= 2\pi i\, \tau\, k + 2\pi i \sum_i (\eta_i + \tau\, \gamma_i)\, m_i$$

Surface operators as monodromy defects

- The non-perturbative weight in the path-integral is then:

$$e^{2\pi i\,\tau\,k}\prod_i e^{2\pi i\,t_i\,m_i} \qquad \text{with} \quad \boxed{t_i = \eta_i + \tau\,\gamma_i}$$

- So that the instanton partition function becomes

$$Z_{\text{inst}} = \sum_{\{k,m_i\}} \left(e^{2\pi i\,\tau\,k} \prod_i e^{2\pi i\,t_i\,m_i} \right) Z_{\{k,m_i\}}$$

Surface operators as monodromy defects

- The non-perturbative weight in the path-integral is then:

$$e^{2\pi i\,\tau\,k}\prod_i e^{2\pi i\,t_i\,m_i} \qquad \text{with} \qquad \boxed{t_i = \eta_i + \tau\,\gamma_i}$$

- So that the instanton partition function becomes

$$Z_{\text{inst}} = \sum_{\{k,m_i\}} \Big(e^{2\pi i\,\tau\,k}\prod_i e^{2\pi i\,t_i\,m_i}\Big)\;Z_{\{k,m_i\}}$$

which, through a change of variables, takes the form

$$Z_{\text{inst}} = \sum_{\{d_i\}} q_i^{\,d_i}\;\,Z_{\{d_i\}}$$

where $d_i \in \mathbb{N}$ and

$$\left.\begin{aligned} &q_1 \sim e^{2\pi i\,t_1}\;, \\ &q_i \sim e^{2\pi i(t_i - t_{i-1})}\;, \quad i = 2, \cdots, M-1\;, \\ &q_M \sim e^{2\pi i\,\tau}\,e^{-2\pi i\,t_{M-1}}\;. \end{aligned}\right\} \;\Rightarrow\; q \sim e^{2\pi i\,\tau} = q_1 \cdots q_M$$

Surface operators as monodromy defects

- The non-perturbative weight in the path-integral is then:

$$e^{2\pi i\,\tau\,k}\prod_i e^{2\pi i\,t_i\,m_i} \qquad \text{with} \quad t_i = \eta_i + \tau\,\gamma_i$$

- So that the instanton partition function becomes

$$Z_{\text{inst}} = \sum_{\{k,m_i\}} \left(e^{2\pi i\,\tau\,k}\prod_i e^{2\pi i\,t_i\,m_i}\right) Z_{\{k,m_i\}}$$

which, through a change of variables, takes the form

$$Z_{\text{inst}} = \sum_{\{d_i\}} q_i^{d_i}\, Z_{\{d_i\}}$$

ramified instanton partition function

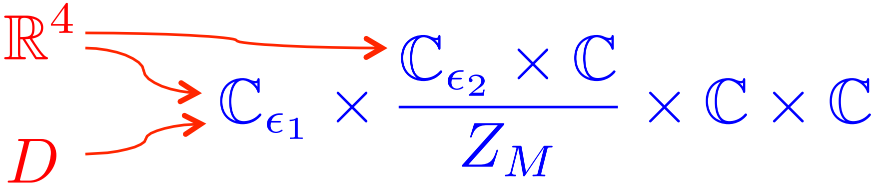
where $d_i \in \mathbb{N}$ and

$$\left.\begin{aligned} &q_1 \sim e^{2\pi i\,t_1}\ , \\ &q_i \sim e^{2\pi i(t_i - t_{i-1})}\ , \quad i = 2, \cdots, M-1\ , \\ &q_M \sim e^{2\pi i\,\tau}\, e^{-2\pi i\,t_{M-1}}\ . \end{aligned}\right\} \Rightarrow \quad q \sim e^{2\pi i\,\tau} = \ q_1 \cdots q_M$$

Ramified instanton partition function

- Like the usual instanton partition function, also the ramified instanton partition function can be computed **using localization**
  (Kanno, Tachikawa '11)

- The idea is to consider the orbifold



$$\mathbb{R}^4 \qquad D$$

$$\mathbb{C}_{\epsilon_1} \times \frac{\mathbb{C}_{\epsilon_2} \times \mathbb{C}}{Z_M} \times \mathbb{C} \times \mathbb{C}$$

in the presence of an Omega-background with parameters ϵ_1 and ϵ_2 to regulate the divergences and localize (Nekrasov '02,…)

- Introducing v.e.v's a for the adjoint scalars and going to a generic point in the Coulomb branch, we have

$$Z_{\text{inst}} = \sum_{\{d_i\}} q_i^{d_i} \boxed{Z_{\{d_i\}}(a, \epsilon_1, \epsilon_2)}$$

explicitly calculable

Ramified instanton partition function

- From $Z_{\rm inst}$ we obtain

prepotential **(twisted) superpotential**

$$\log(1+Z_{\rm inst}) = -\frac{F_{\rm inst}}{\epsilon_1\epsilon_2} + \frac{W_{\rm inst}}{\epsilon_1} + \cdots$$

- The prepotential *F* describes the effective *4d* dynamics

- The (twisted) superpotential *W* accounts for the effective dynamics on the *2d* defect

Ramified instanton partition function

- From Z_{inst} we obtain

prepotential **(twisted) superpotential**

$$\log(1+Z_{\text{inst}}) = -\frac{F_{\text{inst}}}{\epsilon_1\epsilon_2} + \frac{W_{\text{inst}}}{\epsilon_1} + \cdots$$

- The prepotential *F* describes the effective *4d* dynamics

- The (twisted) superpotential *W* accounts for the effective dynamics on the *2d* defect

- Indeed $$\frac{1}{\epsilon_1\epsilon_2} \sim \mathrm{Svol}(\mathbb{C}^2_{\epsilon_1\epsilon_2}) \sim \int d^4x\, d^2\theta_1\, d^2\theta_2$$

$$\frac{1}{\epsilon_1} \sim \mathrm{Svol}(\mathbb{C}_{\epsilon_1}) \sim \int d^2x\, d\theta_1\, d\bar{\theta}_1$$

Ramified instanton partition function

- From Z_{inst} we obtain

prepotential **(twisted) superpotential**

$$\log(1+Z_{\text{inst}}) = -\frac{F_{\text{inst}}}{\epsilon_1\epsilon_2} + \frac{W_{\text{inst}}}{\epsilon_1} + \cdots$$

- The prepotential *F* describes the effective *4d* dynamics

- The (twisted) superpotential *W* accounts for the effective dynamics on the *2d* defect

- For example, in the case $\mathrm{SU}(2) \to \mathrm{S}\big[\mathrm{U}(1)\times\mathrm{U}(1)\big]$, we have

$$Z_{\text{inst}} = 1 + \frac{q_1}{\epsilon_1(2a+\epsilon_1+\epsilon_2)} + \frac{q_2}{\epsilon_1(-2a+\epsilon_1+\epsilon_2)} + \cdots$$

$$\boxed{W_{\text{inst}} = \frac{q_1}{2a} - \frac{q_2}{2a} + \cdots} \qquad \langle\phi\rangle = \mathrm{diag}(a,-a)$$

Ramified instanton partition function

- From Z_{inst} we obtain

$$\log(1+Z_{\text{inst}}) = -\frac{\overset{\textbf{prepotential}}{F_{\text{inst}}}}{\epsilon_1\epsilon_2} + \frac{\overset{\textbf{(twisted) superpotential}}{W_{\text{inst}}}}{\epsilon_1} + \cdots$$

- The prepotential *F* describes the effective *4d* dynamics

- The (twisted) superpotential *W* accounts for the effective dynamics on the *2d* defect

Can we calculate *W* in a different way and obtain some physical intuition ?

Surface defect as a coupled *2d/4d* system

- One considers a *4d* gauge theory with group *G=SU(N)*

- One then couples it with a (2,2) *2d* GLSM with *G* as a global symmetry

(Gukov+Witten '06, …
Gadde+Gukov '13,
Gaiotto+Gukov+Seiberg '13, ...)

- Simplest example:

CP^1 *2d* σ – model with SU(2) flavour symmetry

Surface defect as a coupled *2d/4d* system

- One considers a *4d* gauge theory with group *G=SU(N)*
- One then couples it with a (2,2) *2d* GLSM with *G* as a global symmetry

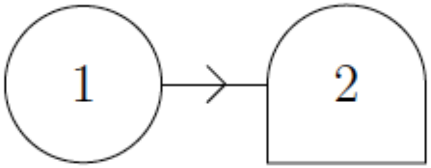
(Gukov+Witten ’06, …
Gadde+Gukov ‘13,
Gaiotto+Gukov+Seiberg ‘13, ...)

- Simplest example:

CP^1 *2d* σ – model with SU(2) flavour symmetry

⇕

2d U(1) gauge theory with a chiral field in the fundamental of SU(2)



Surface defect as a coupled *2d/4d* system

- One considers a *4d* gauge theory with group *G=SU(N)*

- One then couples it with a (2,2) *2d* GLSM with *G* as a global symmetry

(Gukov+Witten '06, …
Gadde+Gukov '13,
Gaiotto+Gukov+Seiberg '13, ...)

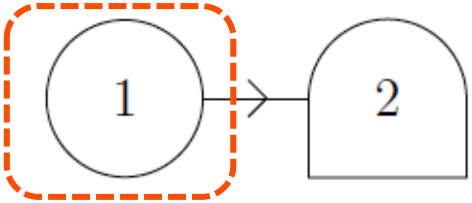
- Simplest example:

CP^1 *2d* σ – model with SU(2) flavour symmetry

⇕

2d U(1) gauge theory with a chiral field in the fundamental of SU(2)

2d U(1) gauge theory



Surface defect as a coupled *2d/4d* system

- One considers a *4d* gauge theory with group *G=SU(N)*

- One then couples it with a (2,2) *2d* GLSM with *G* as a global symmetry

(Gukov+Witten '06, …
Gadde+Gukov '13,
Gaiotto+Gukov+Seiberg '13, ...)

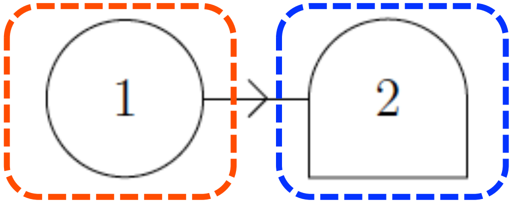
- Simplest example:

CP^1 *2d* σ – model with SU(2) flavour symmetry

↕

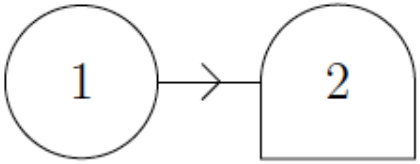
2d U(1) gauge theory with a chiral field in the fundamental of SU(2)

2d U(1) gauge theory



4d SU(2) gauge theory

The coupled *2d/4d* system

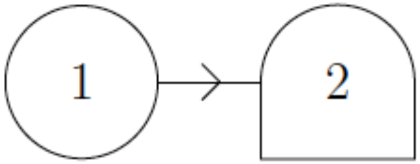


- Understanding the SU(2) node just as a flavour symmetry, we can write the effective twisted superpotential *W* for the CP^1 model as

(d'Adda+Di Vecchia+ …'82)

$$W = 2\pi i\, t\, \sigma - 2\Big[\sigma\Big(\log\frac{\sigma}{\mu} - 1\Big)\Big]$$

The coupled *2d/4d* system



- Understanding the SU(2) node just as a flavour symmetry, we can write the effective twisted superpotential *W* for the CP^1 model as

(d'Adda+Di Vecchia+ ...'82)

$$W = 2\pi i\, t\, \sigma - 2\Big[\sigma\Big(\log\frac{\sigma}{\mu} - 1\Big)\Big]$$

FI coupling

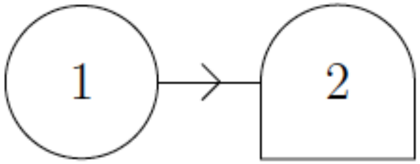
twisted U(1)
chiral superfield

2 because of the SU(2)
flavour symmetry

UV scale

This is a purely 2d point of view

The coupled *2d/4d* system



- Understanding the SU(2) node just as a flavour symmetry, we can write the effective twisted superpotential *W* for the CP^1 model as

(d'Adda+Di Vecchia+ ...'82)

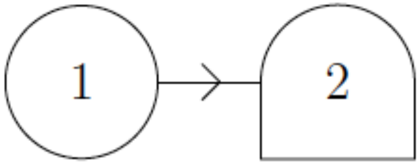
$$W = 2\pi i\, t\, \sigma - 2\left[\sigma\left(\log\frac{\sigma}{\mu} - 1\right)\right]$$

- Understanding the SU(2) node as a *4d* theory, and giving a vev to the adjoint scalar ϕ, we introduce a twisted mass for σ and *W* becomes

(Hanany+Hori '97)

$$W = 2\pi i\, t\, \sigma - \mathrm{Tr}\left[(\sigma - \langle\phi\rangle)\left(\log\frac{\sigma - \langle\phi\rangle}{\mu} - 1\right)\right]$$

The coupled *2d/4d* system



- Understanding the SU(2) node just as a flavour symmetry, we can write the effective twisted superpotential *W* for the CP^1 model as

(d'Adda+Di Vecchia+ ...'82)

$$W = 2\pi i\, t\, \sigma - 2\left[\sigma\left(\log\frac{\sigma}{\mu} - 1\right)\right]$$

- Understanding the SU(2) node as a *4d* theory, and giving a vev to the adjoint scalar ϕ, we introduce a twisted mass for σ and *W* becomes

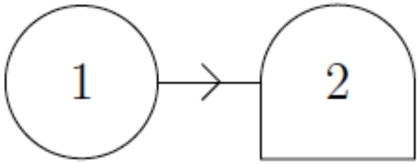
(Hanany+Hori '97)

$$W = 2\pi i\, t\, \sigma - \mathrm{Tr}\left[(\sigma - \langle\phi\rangle)\left(\log\frac{\sigma - \langle\phi\rangle}{\mu} - 1\right)\right]$$

4d vev's as twisted masses for σ

The *4d* theory is treated only classically

The coupled *2d/4d* system



- Understanding the SU(2) node just as a flavour symmetry, we can write the effective twisted superpotential *W* for the CP^1 model as

(d'Adda+Di Vecchia+ …'82)

$$W = 2\pi i\, t\, \sigma - 2\Big[\sigma\Big(\log\frac{\sigma}{\mu} - 1\Big)\Big]$$

- Understanding the SU(2) node as a *4d* theory, and giving a vev to the adjoint scalar ϕ, we introduce a twisted mass for σ and *W* becomes

(Hanany+Hori '97)

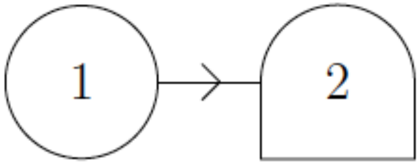
$$W = 2\pi i\, t\, \sigma - \mathrm{Tr}\Big[(\sigma - \langle\phi\rangle)\Big(\log\frac{\sigma - \langle\phi\rangle}{\mu} - 1\Big)\Big]$$

- Taking into account the quantum fluctuations of the *4d* SU(2) theory, we finally get

(Gaiotto+Gukov+Seiberg '13, ….)

$$\boxed{W = 2\pi i\, t\, \sigma - \Big\langle \mathrm{Tr}\Big[(\sigma - \phi)\Big(\log\frac{\sigma - \phi}{\mu} - 1\Big)\Big]\Big\rangle}$$

The coupled *2d/4d* system



- This 2d/4d quiver describes a *4d* SU(2) theory coupled with a *2d* U(1) theory inside.

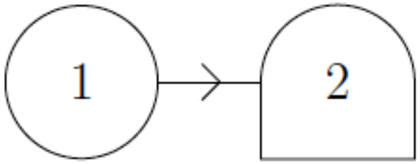
- Thus, it should describe the defect corresponding to

$$\mathrm{SU}(2) \to \mathrm{S}[\mathrm{U}(1) \times \mathrm{U}(1)]$$

and to the partition $\vec{n} = [1, 1]$.

Can we make a quantitative check ?

The coupled *2d/4d* system



- This 2d/4d quiver describes a *4d* SU(2) theory coupled with a *2d* U(1) theory inside.

- Thus, it should describe the defect corresponding to

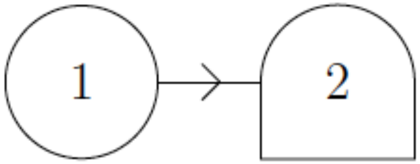
$$\mathrm{SU}(2) \to \mathrm{S}[\mathrm{U}(1) \times \mathrm{U}(1)]$$

and to the partition $\vec{n} = [1, 1]$.

Can we make a quantitative check ?

YES !

The coupled *2d/4d* system



- Consider the twisted superpotential

$$W = 2\pi i\, t\, \sigma - \left\langle \mathrm{Tr}\left[(\sigma-\phi)\left(\log\frac{\sigma-\phi}{\mu}-1\right)\right]\right\rangle$$

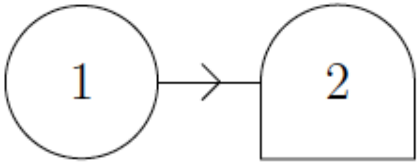
$$= -\left\langle \mathrm{Tr}\left[(\sigma-\phi)\left(\log\frac{\sigma-\phi}{\Lambda_1}-1\right)\right]\right\rangle$$

2d dynamically generated scale $\Lambda_1^2 = e^{2\pi i\, t}\,\mu^2$

- The *2d* vacuum is determined by

$$\exp\left(\frac{\partial W}{\partial\sigma}\right) = 1 \quad\Longleftrightarrow\quad \exp\left\langle \mathrm{Tr}\left[\log\frac{\sigma-\phi}{\Lambda_1}\right]\right\rangle = 1$$

The coupled *2d/4d* system



- Consider the twisted superpotential

$$W = 2\pi i\, t\, \sigma - \left\langle \mathrm{Tr}\left[(\sigma-\phi)\left(\log\frac{\sigma-\phi}{\mu} - 1\right)\right]\right\rangle$$

$$= -\left\langle \mathrm{Tr}\left[(\sigma-\phi)\left(\log\frac{\sigma-\phi}{\Lambda_1} - 1\right)\right]\right\rangle$$

2d dynamically generated scale $\Lambda_1^2 = e^{2\pi i\, t}\, \mu^2$

- The *2d* vacuum is determined by

$$\exp\left(\frac{\partial W}{\partial \sigma}\right) = 1 \quad \Longleftrightarrow \quad \exp\left\langle \mathrm{Tr}\left[\log\frac{\sigma-\phi}{\Lambda_1}\right]\right\rangle = 1$$

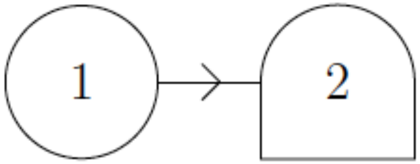
- This equation can be solved using the Seiberg-Witten curve of the *4d* SU(2) theory

$$y^2 = P_2(x)^2 - 4\Lambda^4$$

4d dynamically generated scale $\Lambda^4 = e^{2\pi i\, \tau}\, \mu^4$

characteristic polynomial of SW $P_2(x) = x^2 - \frac{1}{2}\langle \mathrm{Tr}\phi^2\rangle$

The coupled *2d/4d* system



- Consider the twisted superpotential

$$W = 2\pi i\, t\, \sigma - \left\langle \mathrm{Tr}\left[(\sigma-\phi)\left(\log\frac{\sigma-\phi}{\mu}-1\right)\right]\right\rangle$$

$$= -\left\langle \mathrm{Tr}\left[(\sigma-\phi)\left(\log\frac{\sigma-\phi}{\Lambda_1}-1\right)\right]\right\rangle$$

2d dynamically generated scale $\Lambda_1^2 = \boxed{e^{2\pi i\, t}}\mu^2$

- The *2d* vacuum is determined by

$$\exp\left(\frac{\partial W}{\partial\sigma}\right) = 1 \quad\Longleftrightarrow\quad \exp\left\langle \mathrm{Tr}\left[\log\frac{\sigma-\phi}{\Lambda_1}\right]\right\rangle = 1$$

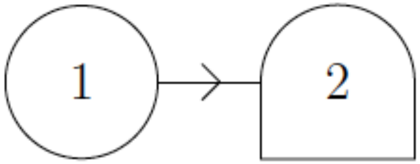
- This equation can be solved using the Seiberg-Witten curve of the *4d* SU(2) theory

$$y^2 = P_2(x)^2 - 4\Lambda^4$$

4d dynamically generated scale $\Lambda^4 = \boxed{e^{2\pi i\, \tau}}\mu^4$

characteristic polynomial of SW $P_2(x) = x^2 - \frac{1}{2}\langle\mathrm{Tr}\phi^2\rangle$

The coupled *2d/4d* system



- From the Seiberg-Witten relation $\left\langle \mathrm{Tr}\,\frac{1}{x-\phi}\right\rangle = \frac{P_2'(x)}{y}$, one gets

$$\left\langle \mathrm{Tr}\Big[\log\frac{\sigma-\phi}{\Lambda_1}\Big]\right\rangle = \int d\sigma \left\langle \mathrm{Tr}\Big[\frac{1}{\sigma-\phi}\Big]\right\rangle = \cdots$$

$$= \log\left(\frac{P_2(\sigma)+\sqrt{P_2(\sigma)^2-4\Lambda^4}}{2\Lambda_1^2}\right)$$

- With simple manipulations, the vacuum equation becomes

$$\exp\left(\frac{\partial W}{\partial\sigma}\right) = 1 \quad \Longleftrightarrow \quad P_2(\sigma) = \Lambda_1^2 + \frac{\Lambda^4}{\Lambda_1^2}$$

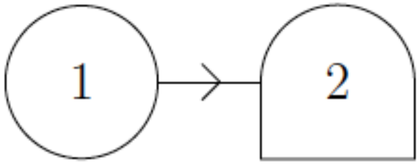
which is solved by

v.e.v. of φ

instanton terms

$$\sigma_\star = \sqrt{\frac{1}{2}\langle \mathrm{Tr}\,\phi^2\rangle + \Lambda_1^2 + \frac{\Lambda^4}{\Lambda_1^2}} = a + \frac{1}{2a}\left(\Lambda_1^2 + \frac{\Lambda^4}{\Lambda_1^2}\right) + \cdots$$

The coupled *2d/4d* system



- Evaluating *W* on the vacuum solution σ_* , we find

$$W(\sigma_\star)\big|_{\text{inst}} = \frac{1}{2a}\left(\Lambda_1^2 - \frac{\Lambda^4}{\Lambda_1^2}\right) + \cdots$$

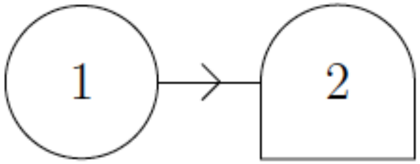
- Notice that

$$\Lambda_1^2 \sim e^{2\pi i\, t} \;\rightarrow\; q_1 \quad , \quad \frac{\Lambda^4}{\Lambda_1^2} \sim e^{2\pi i\, \tau}\; e^{-2\pi i\, t} \;\rightarrow\; \frac{q}{q_1} = q_2$$

and thus

$$\boxed{W(\sigma_\star)\big|_{\text{inst}} = \frac{q_1}{2a} - \frac{q_2}{2a} + \cdots}$$

The coupled *2d/4d* system



- Evaluating *W* on the vacuum solution σ_* , we find

$$W(\sigma_\star)\big|_{\text{inst}} = \frac{1}{2a}\left(\Lambda_1^2 - \frac{\Lambda^4}{\Lambda_1^2}\right) + \cdots$$

- Notice that

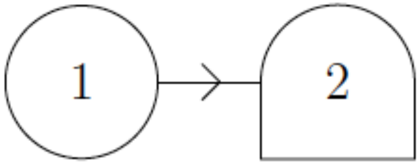
$$\Lambda_1^2 \sim e^{2\pi i\, t} \;\rightarrow\; q_1 \quad , \quad \frac{\Lambda^4}{\Lambda_1^2} \sim e^{2\pi i\, \tau}\; e^{-2\pi i\, t} \;\rightarrow\; \frac{q}{q_1} = q_2$$

and thus

$$\boxed{W(\sigma_\star)\big|_{\text{inst}} = \frac{q_1}{2a} - \frac{q_2}{2a} + \cdots}$$

- Furthermore, we see that $\Lambda_1 \gg \Lambda$

The coupled *2d/4d* system



- Evaluating *W* on the vacuum solution σ_* , we find

$$W(\sigma_\star)\big|_{\text{inst}} = \frac{1}{2a}\left(\Lambda_1^2 - \frac{\Lambda^4}{\Lambda_1^2}\right) + \cdots$$

- Notice that

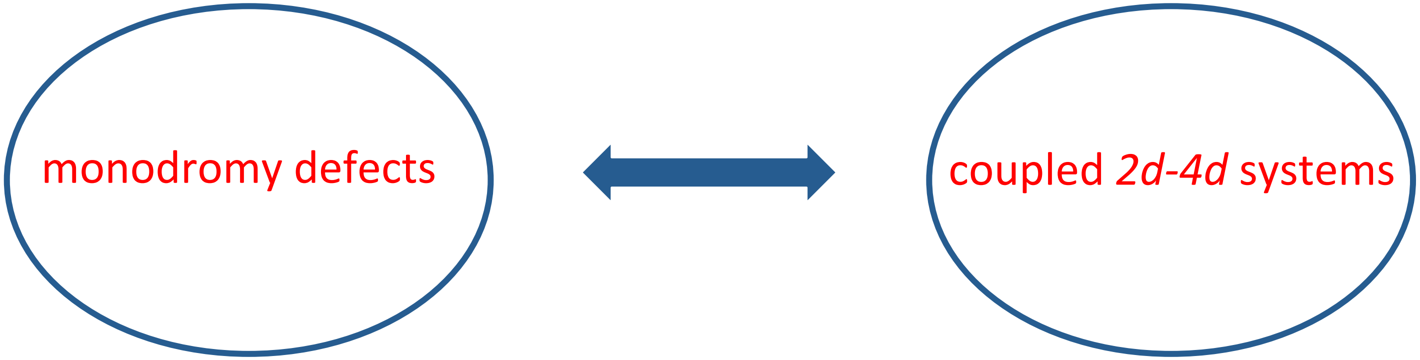
$$\Lambda_1^2 \sim e^{2\pi i\, t} \;\rightarrow\; q_1 \quad , \quad \frac{\Lambda^4}{\Lambda_1^2} \sim e^{2\pi i\, \tau}\; e^{-2\pi i\, t} \;\rightarrow\; \frac{q}{q_1} = q_2$$

and thus

$$W(\sigma_\star)\big|_{\text{inst}} = \frac{q_1}{2a} - \frac{q_2}{2a} + \cdots$$

This exactly matches the twisted superpotential due to ramified instantons !!

Recap



monodromy defects

coupled *2d-4d* systems

4d ⟶ *2d*

2d ⟶ *4d*

W_{inst} from localization

W_{inst} from vacuum equations

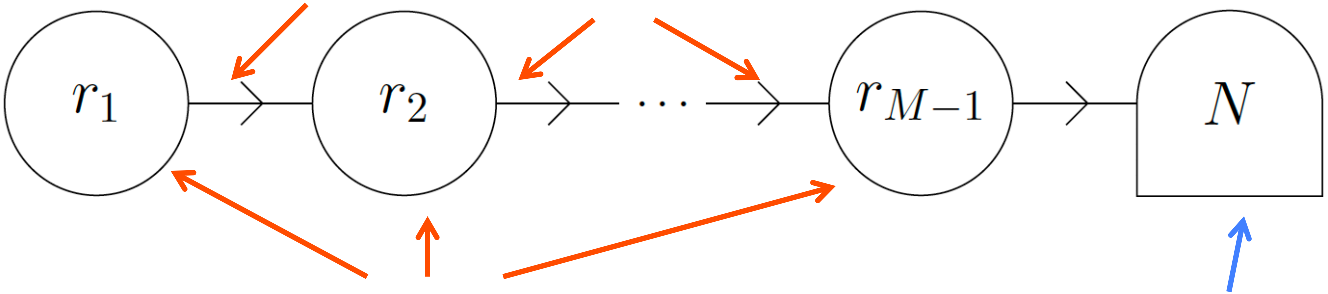
ramified instanton
counting parameters q_i

dynamically generated
scales Λ_i

$$\boxed{\left. W_{\text{inst}} \right|_{\text{loc}} = \left. W(\sigma) \right|_{\text{vacuum}}}$$

General case

- The quiver which describes the general defect in the SU(*N*) theory is



bi-fundamental *2d* chiral matter

2d U(r_i) gauge theories

4d SU(*N*) flavor theory

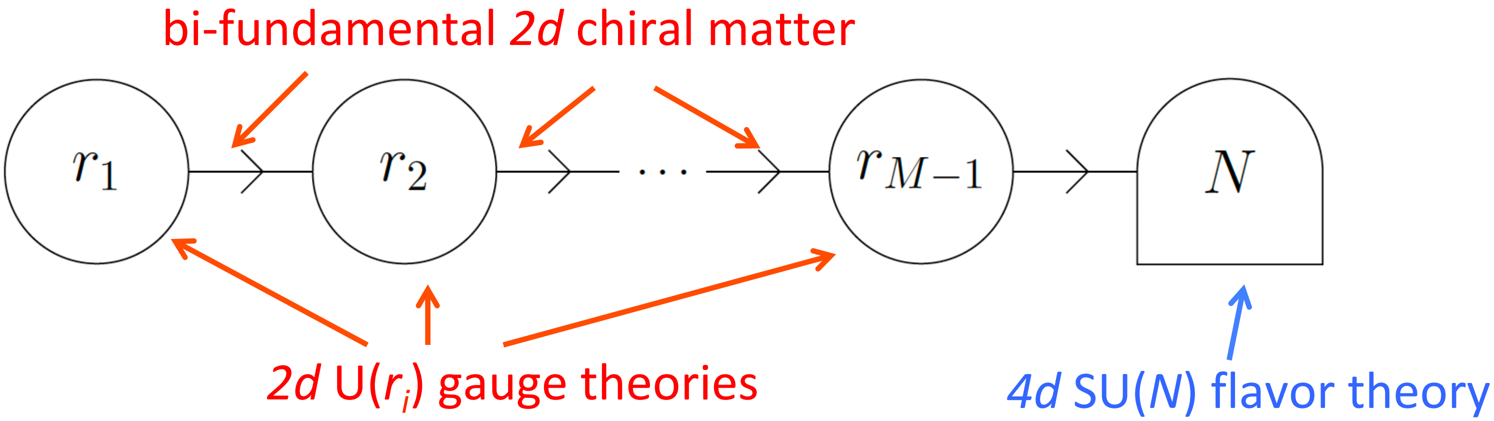
- This quiver corresponds to the monodromy defect with

$$\vec{n} = [n_1, \cdots, n_M] \qquad r_i = \sum_{j=1}^{i} n_j$$

$$\mathrm{SU}(N) \to \mathrm{S}\big[\mathrm{U}(n_1) \times \mathrm{U}(n_2) \times \cdots \times \mathrm{U}(n_M)\big]$$

General case

- The quiver which describes the general defect in the SU(*N*) theory is



- Given the structure of the quiver, the twisted superpotential is

$$W = 2\pi i \sum_{i=1}^{M-1} \sum_{s=1}^{r_i} t_i \, \sigma_s^{(i)} - \sum_{i=1}^{M-2} \sum_{s=1}^{r_i} \sum_{t=1}^{r_{i+1}} \varpi(\sigma_s^{(i)} - \sigma_t^{(i+1)})$$

$$- \sum_{s=1}^{r_{M-1}} \left\langle \mathrm{Tr} \, \varpi(\sigma_s^{(M-1)} - \phi) \right\rangle$$

$$\varpi(x) = x \log\left(\frac{x}{\mu} - 1\right)$$

General case

- The non-perturbative contributions to *W* are computed from the vacuum equations

$$\exp\left(\frac{\partial W}{\partial \sigma_s^{(i)}}\right) = 1$$

and **exactly match** those obtained from the localization approach!

General case

- The non-perturbative contributions to *W* are computed from the vacuum equations

$$\exp\left(\frac{\partial W}{\partial \sigma_s^{(i)}}\right) = 1$$

and **exactly match** those obtained from the localization approach!

Technical but important point

- The ramified instanton partition function is given in terms of an multiple integral; for example at 1-instanton we have

$$Z_1 = -\sum_{i=1}^{M} \frac{q_i}{\epsilon_1} \int \frac{d\chi_i}{2\pi i} \prod_{s=1}^{n_i} \frac{1}{\left(a_s - \chi_i + \frac{\epsilon_1+\epsilon_2}{2}\right)} \prod_{t=1}^{n_{i+1}} \frac{1}{\left(\chi_i - a_t + \frac{\epsilon_1+\epsilon_2}{2}\right)}$$

- Thus we have to specify the contour of integration, or equivalently specify which poles contribute to the integral

Integration contour

- The precise match is obtained by giving a positive imaginary part to the ε parameters and selecting the contribution from the following poles:

$$\chi_i = a_s + \frac{\epsilon_1 + \epsilon_2}{2} \quad s = 1 \cdots , n_i \ , \quad i = 1, \cdots , M - 1$$

$$\chi_M = a_t - \frac{\epsilon_1 + \epsilon_2}{2} \quad t = 1 \cdots , n_1$$

(Gorsky+LeFloch+… ‘17
Ashok+Billo+… ‘17,’ 18)

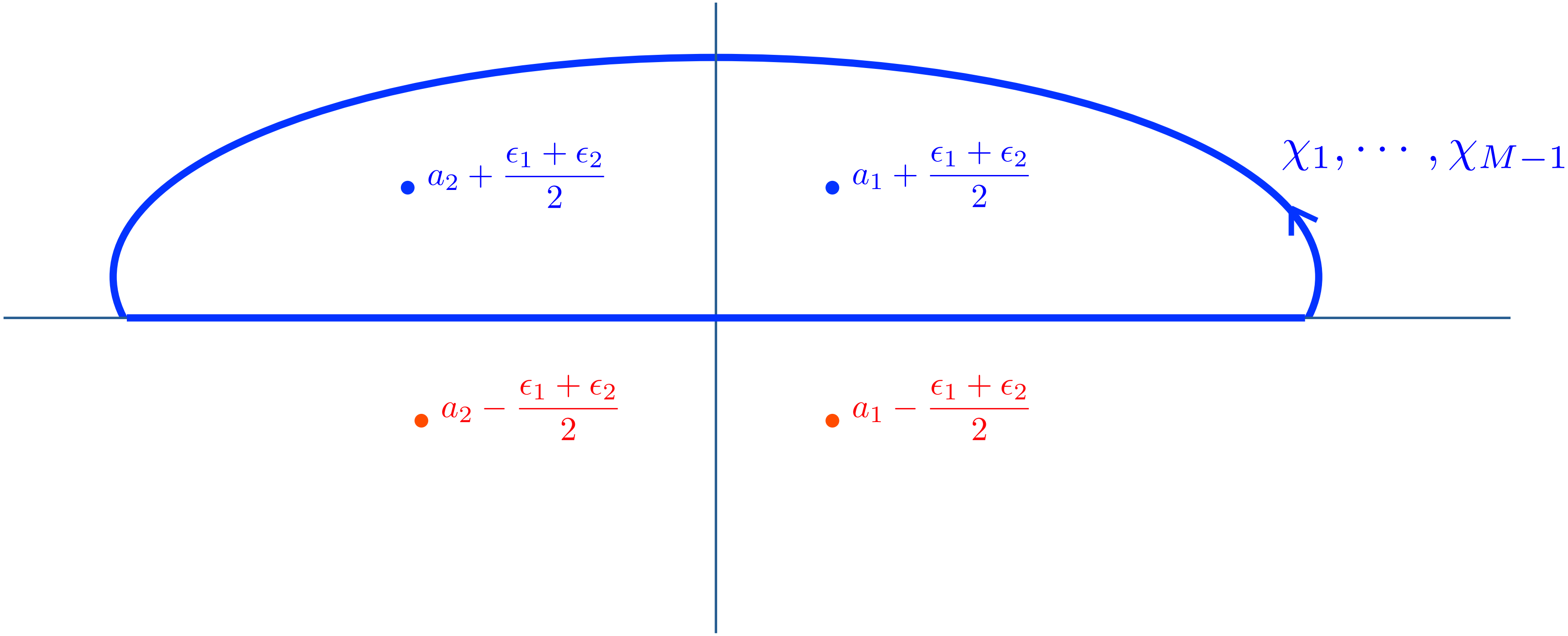
Integration contour

- The precise match is obtained by giving a positive imaginary part to the ε parameters and selecting the contribution from the following poles:

$$\chi_i = a_s + \frac{\epsilon_1+\epsilon_2}{2} \quad s = 1\cdots, n_i \ , \quad i = 1, \cdots, M-1$$

$$\chi_M = a_t - \frac{\epsilon_1+\epsilon_2}{2} \quad t = 1\cdots, n_1$$

(Gorsky+LeFloch+... '17 Ashok+Billo+... '17,' 18)



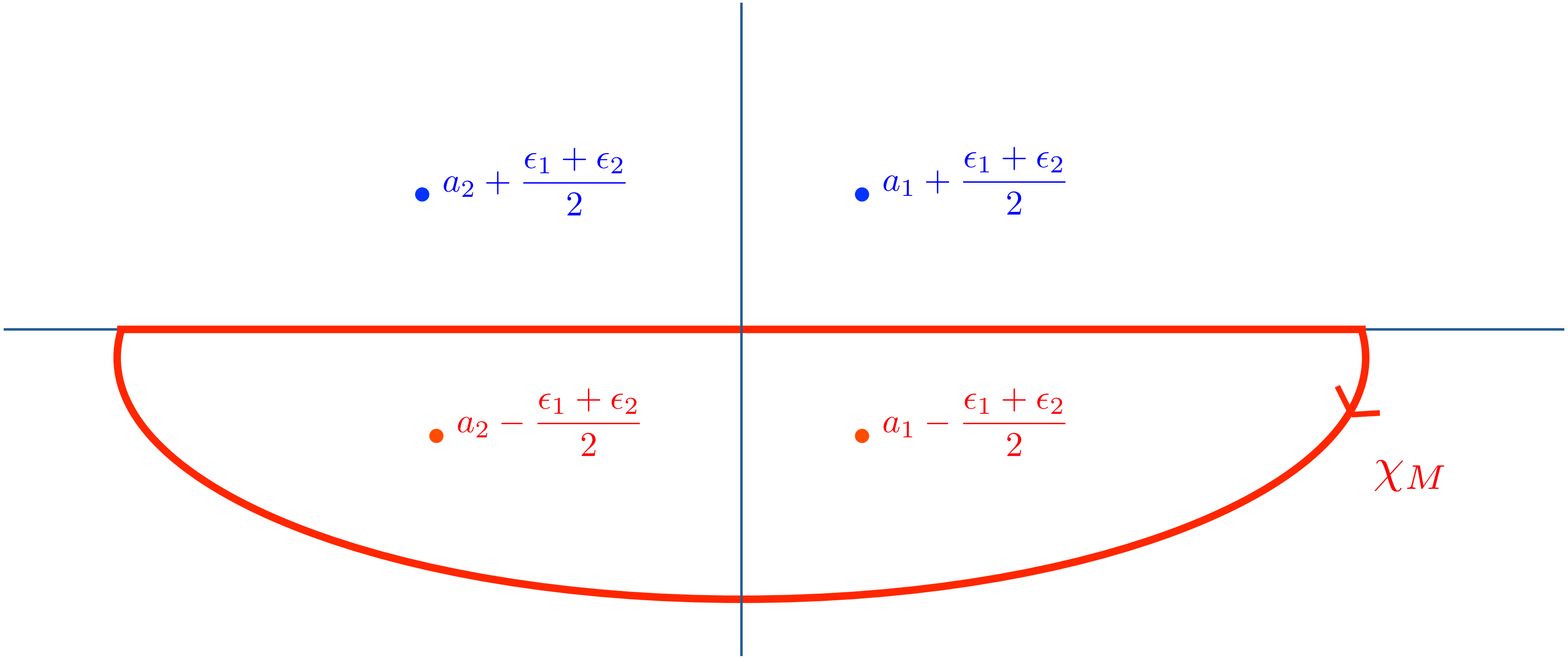
Integration contour

- The precise match is obtained by giving a positive imaginary part to the ε parameters and selecting the contribution from the following poles:

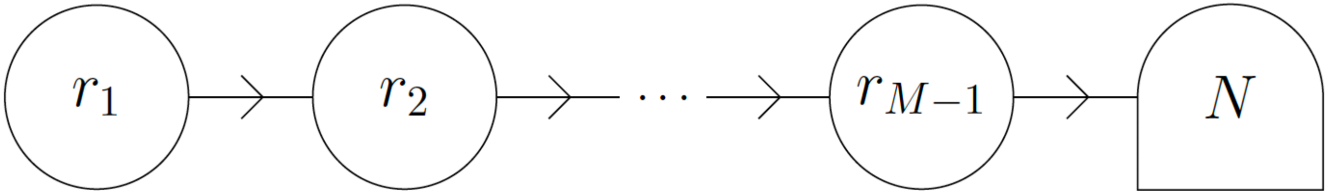
$$\chi_i = a_s + \frac{\epsilon_1 + \epsilon_2}{2} \quad s = 1 \cdots , n_i \ , \quad i = 1, \cdots , M - 1$$

$$\chi_M = a_t - \frac{\epsilon_1 + \epsilon_2}{2} \quad t = 1 \cdots , n_1$$

(Gorsky+LeFloch+… ‘17
Ashok+Billo+… ‘17,’ 18)



Integration contour



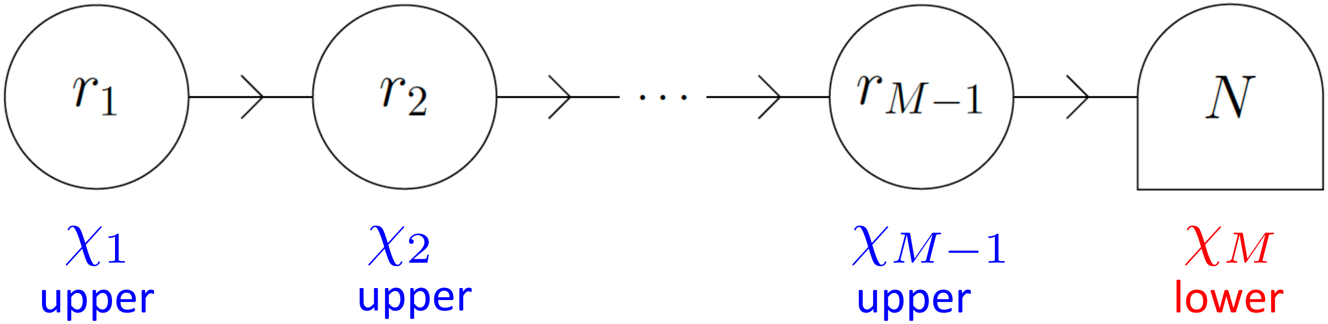
χ_1
upper

χ_2
upper

χ_{M-1}
upper

χ_M
lower

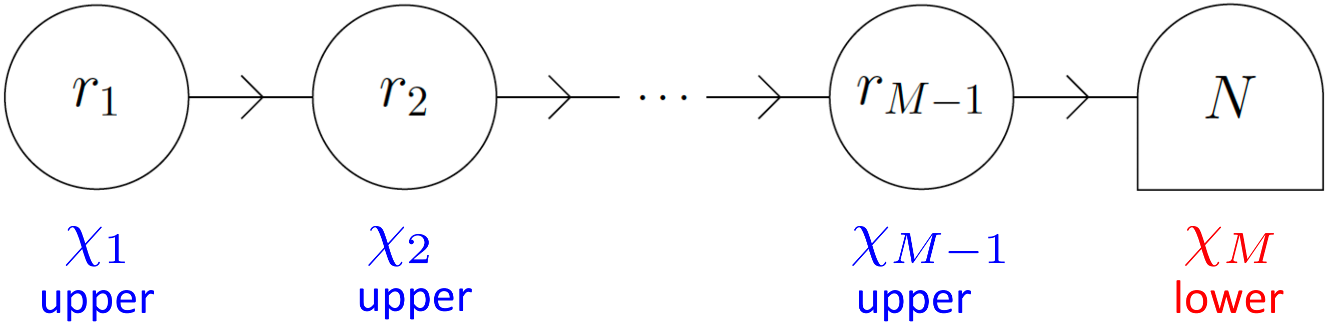
Integration contour



- This integration prescription can be elegantly specified with **a Jeffrey-Kirwan vector η** given by

$$\eta = -\chi_1 - \chi_2 \cdots - \chi_{M-1} + \xi_M \chi_M \qquad (\xi_M \gg 1)$$

Integration contour

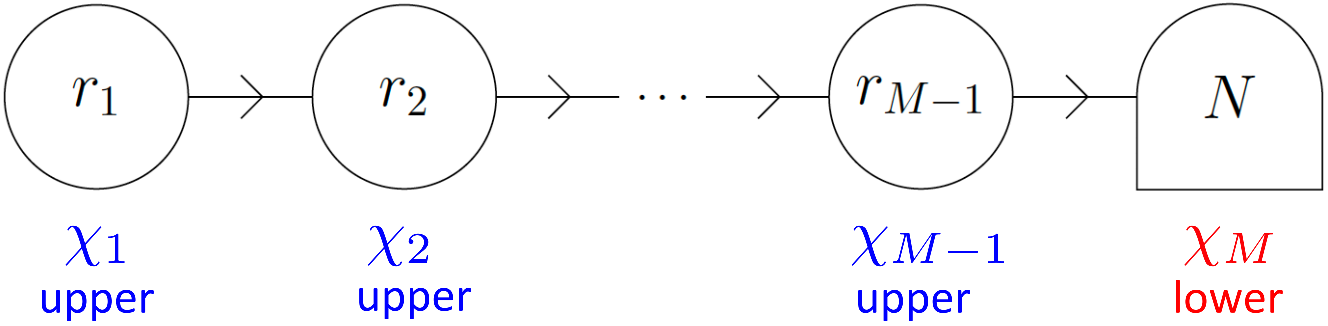


- This integration prescription can be elegantly specified with **a Jeffrey-Kirwan vector η** given by

- sign of the beta-function for the FI couplings of the U(r_i)

$$\eta = -\chi_1 - \chi_2 \cdots - \chi_{M-1} + \xi_M \chi_M \qquad (\xi_M \gg 1)$$

Integration contour



- This integration prescription can be elegantly specified with **a Jeffrey-Kirwan vector η** given by

- sign of the beta-function for the FI couplings of the U(r_i)

$$\eta = -\xi_1\,\chi_1 - \xi_2\,\chi_2 \cdots - \xi_{M-1}\,\chi_{M-1} + \xi_M\,\chi_M$$

FI couplings of the U(r_i) nodes of the quiver

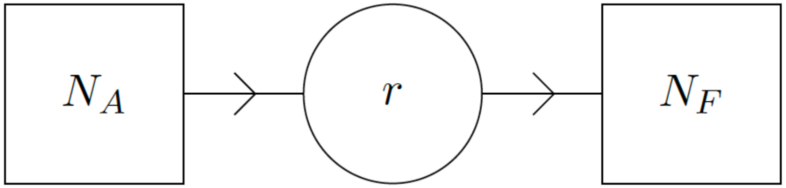
$$\xi_1 \ll \xi_2 \ll \cdots \ll \xi_{M-1} \ll \xi_M \quad \leftrightarrow \quad \Lambda_1 \gg \Lambda_2 \gg \cdots \gg \Lambda_{M-1} \gg \Lambda$$

Duality

Duality

- Consider a *2d* U(*r*) theory with N_F fundamentals and N_A ($< N_F$) anti-fundamentals

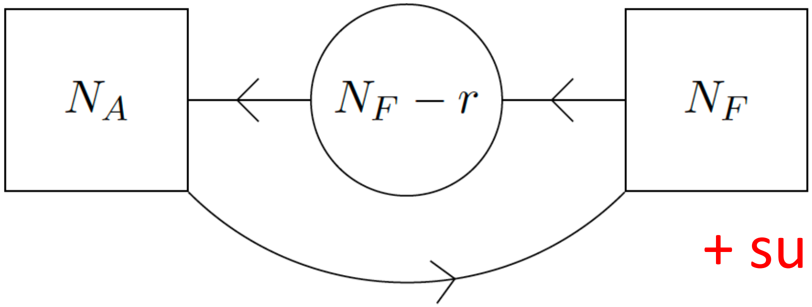
A)



- This theory is dual to a *2d* U($N_F - r$) theory with the roles of fundamentals and anti-fundamentals reversed

(Seiberg '94)

B)



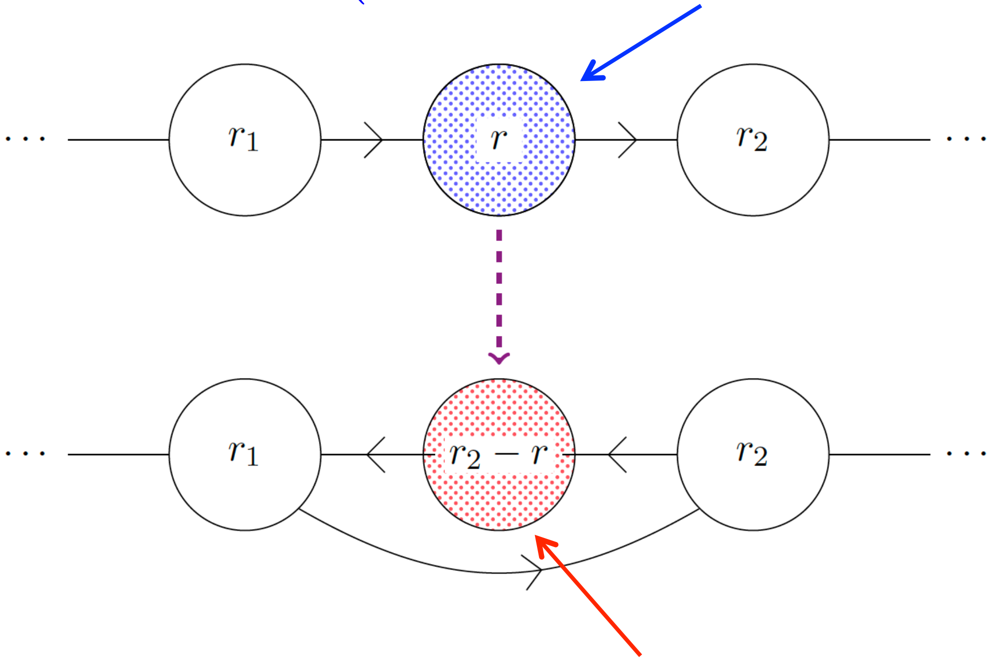
+ superpotential term

$$W^A_{\text{class}} = 2\pi i\, t\, \mathrm{Tr}\sigma \quad \to \quad W^B_{\text{class}} = -2\pi i\, t\, \mathrm{Tr}\sigma + 2\pi i\, t \sum_{f=1}^{N_F} m_f$$

Duality

- We can apply this duality rule to the quiver representing a surface defect

$$W_{\text{class}} = \cdots + 2\pi i\big(t_1\,\mathrm{Tr}\,\sigma^1 + t\,\mathrm{Tr}\,\sigma + t_2\,\mathrm{Tr}\,\sigma^2\big) + \cdots$$

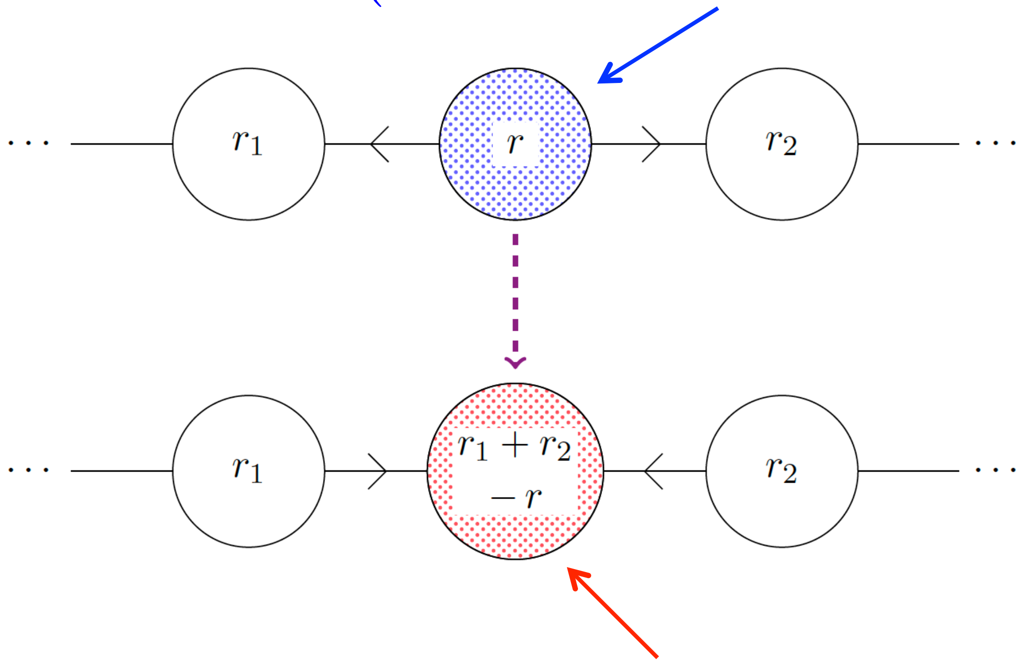


$$W_{\text{class}} = \cdots + 2\pi i\big(t_1\,\mathrm{Tr}\,\sigma^1 - t\,\mathrm{Tr}\,\sigma + (t_2 + t)\,\mathrm{Tr}\,\sigma^2\big) + \cdots$$

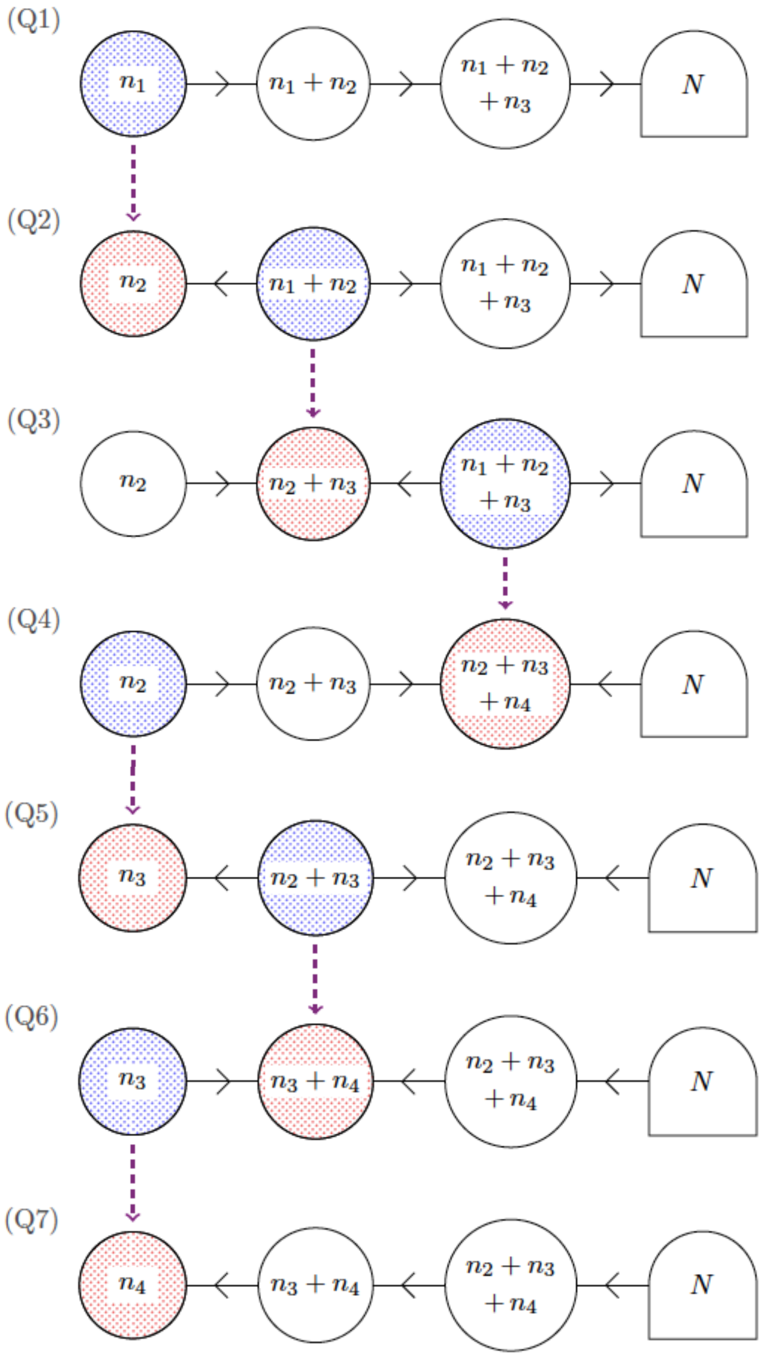
Duality

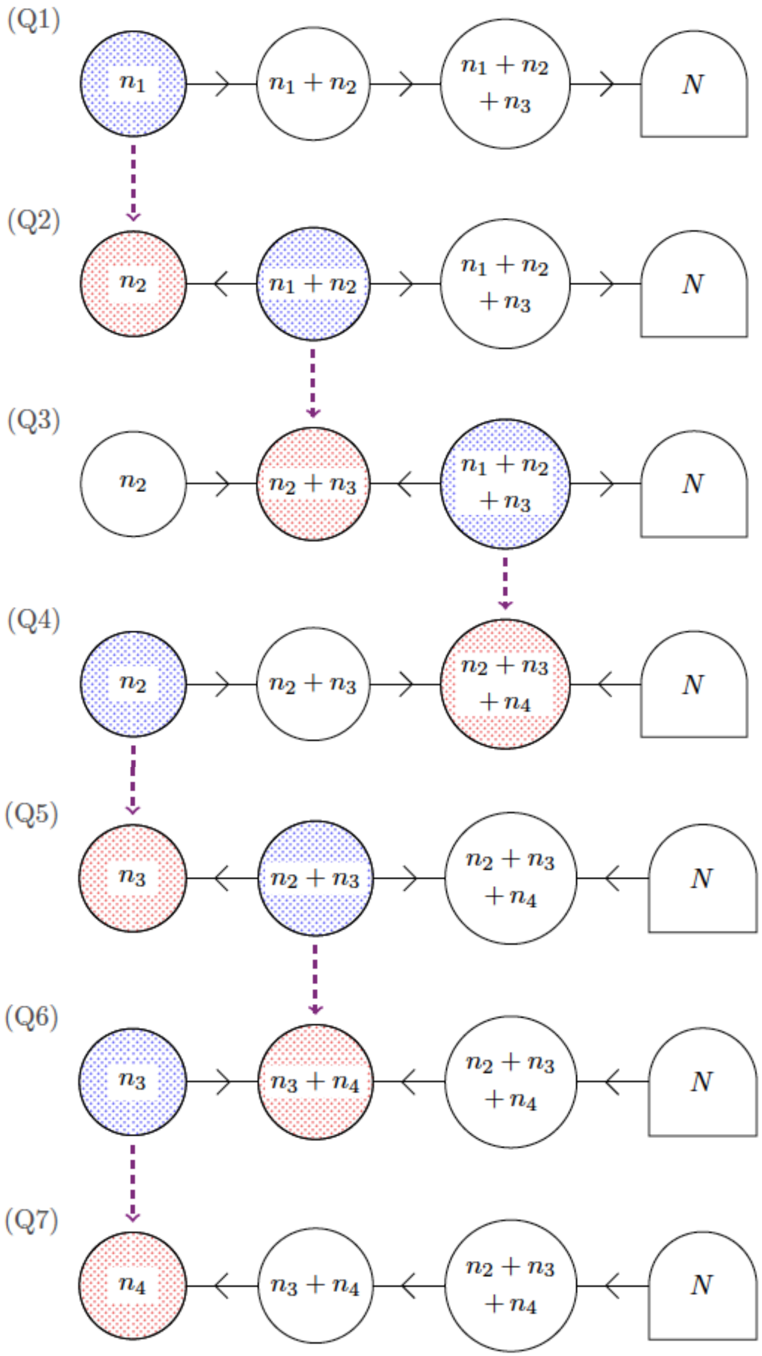
- We can apply this duality rule to the quiver representing a surface defect

$$W_{\text{class}} = \cdots + 2\pi i\big(t_1 \operatorname{Tr}\sigma^1 + t \operatorname{Tr}\sigma + t_2 \operatorname{Tr}\sigma^2\big) + \cdots$$



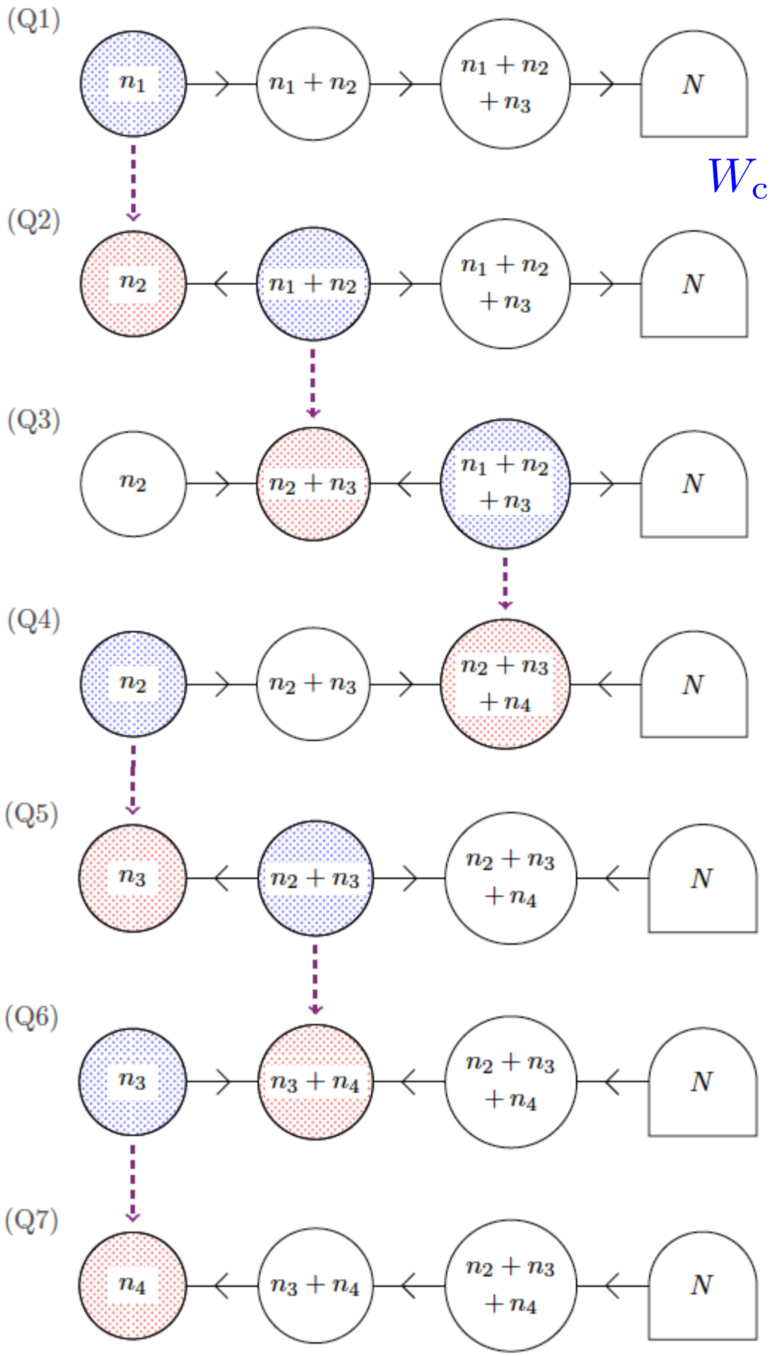
$$W_{\text{class}} = \cdots + 2\pi i\big((t_1 + t) \operatorname{Tr}\sigma^1 - t \operatorname{Tr}\sigma + (t_2 + t) \operatorname{Tr}\sigma^2\big) + \cdots$$


(Q1)
n_1
$n_1 + n_2$
$n_1 + n_2 + n_3$
N
(Q2)
n_2
$n_1 + n_2$
$n_1 + n_2 + n_3$
N
(Q3)
n_2
$n_2 + n_3$
$n_1 + n_2 + n_3$
N
(Q4)
n_2
$n_2 + n_3$
$n_2 + n_3 + n_4$
N
(Q5)
n_3
$n_2 + n_3$
$n_2 + n_3 + n_4$
N
(Q6)
n_3
$n_3 + n_4$
$n_2 + n_3 + n_4$
N
(Q7)
n_4
$n_3 + n_4$
$n_2 + n_3 + n_4$
N

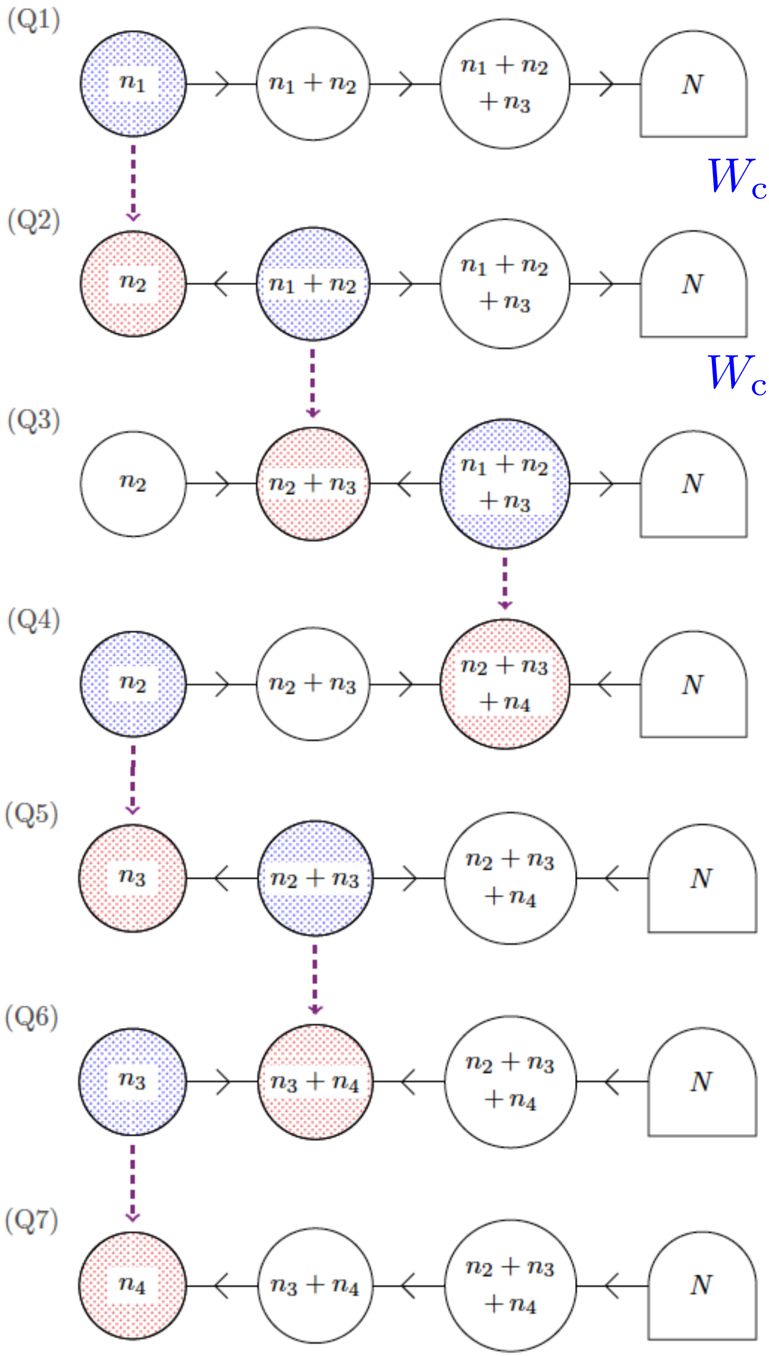


All these different quiver theories provide alternative descriptions of the **same surface operator** corresponding to

$$\mathrm{SU}(N)$$
$$\downarrow$$
$$\mathrm{S}\big[\mathrm{U}(n_1)\times\mathrm{U}(n_2)\times\mathrm{U}(n_3)\times\mathrm{U}(n_4)\big]$$

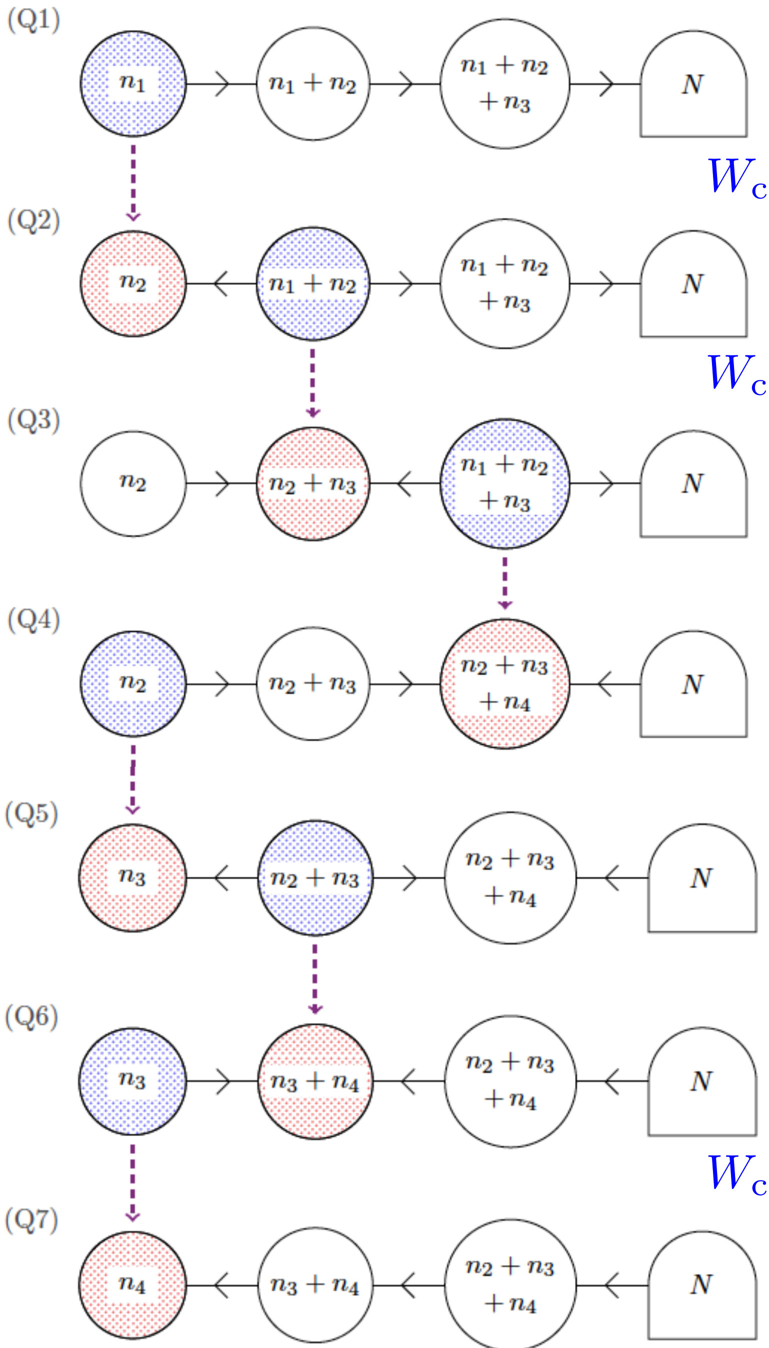


$$W_{\text{class}} = 2\pi i\big(t_1 \operatorname{Tr} \sigma^1 + t_2 \operatorname{Tr} \sigma^2 + t_3 \operatorname{Tr} \sigma^3\big)$$



$$W_{\text{class}} = 2\pi i\big(t_1 \operatorname{Tr}\sigma^1 + t_2 \operatorname{Tr}\sigma^2 + t_3 \operatorname{Tr}\sigma^3\big)$$

$$W_{\text{class}} = 2\pi i\big(-t_1 \operatorname{Tr}\sigma^1 + (t_1 + t_2) \operatorname{Tr}\sigma^2 + t_3 \operatorname{Tr}\sigma^3\big)$$



$$W_{\text{class}} = 2\pi i\big(t_1 \operatorname{Tr}\sigma^1 + t_2 \operatorname{Tr}\sigma^2 + t_3 \operatorname{Tr}\sigma^3\big)$$

$$W_{\text{class}} = 2\pi i\big(-t_1 \operatorname{Tr}\sigma^1 + (t_1+t_2) \operatorname{Tr}\sigma^2 + t_3 \operatorname{Tr}\sigma^3\big)$$

$$\vdots$$

$$W_{\text{class}} = 2\pi i\big(t_3 \operatorname{Tr}\sigma^1 - (t_2+t_3) \operatorname{Tr}\sigma^2 - t_1 \operatorname{Tr}\sigma^3\big)$$

$$W_{\text{class}} = 2\pi i\big(-t_3 \operatorname{Tr}\sigma^1 - t_2 \operatorname{Tr}\sigma^2 - t_1 \operatorname{Tr}\sigma^3\big)$$

Duality

- For any of these quiver theories we can work out the *q* vs Λ map just as we did for the simplest quiver

- For example, after one duality step we find

$$q_1 \sim \Lambda_1^{n_1+n_2} \quad , \quad q_2 \sim \frac{\Lambda_2^{n_1+2n_2+n_3}}{\Lambda_1^{n1+n2}} \quad , \quad q_3 \sim \Lambda_3^{n_3+n_4} \quad , \quad q_4 \sim \frac{\Lambda^{2N}}{q_1 q_2 q_3}$$

Note that the exponents are the (absolute values of the) beta-functions for the FI couplings ξ_i, and *2N* is the beta-function of the *4d* SU(*N*) theory.

Duality

- For any of these quiver theories we can work out the *q* vs Λ map just as we did for the simplest quiver

- For example, after one duality step we find

$$q_1 \sim \Lambda_1^{n_1+n_2} \ , \quad q_2 \sim \frac{\Lambda_2^{n_1+2n_2+n_3}}{\Lambda_1^{n1+n2}} \ , \quad q_3 \sim \Lambda_3^{n_3+n_4} \ , \quad q_4 \sim \frac{\Lambda^{2N}}{q_1 q_2 q_3}$$

- Moreover, the sign change in the FI term of the first node

$$W_{\text{class}} = 2\pi i\big(-t_1 \operatorname{Tr}\sigma^1 + (t_1+t_2)\operatorname{Tr}\sigma^2 + t_3 \operatorname{Tr}\sigma^3\big)$$

leads to consider a new Jeffrey-Kirwan vector

$$\eta = +\xi_1\,\chi_1 - \xi_2\,\chi_2 \cdots - \xi_{M-1}\,\chi_{M-1} \ + \xi_M\,\chi_M$$

χ_1 is now integrated in the lower plane !

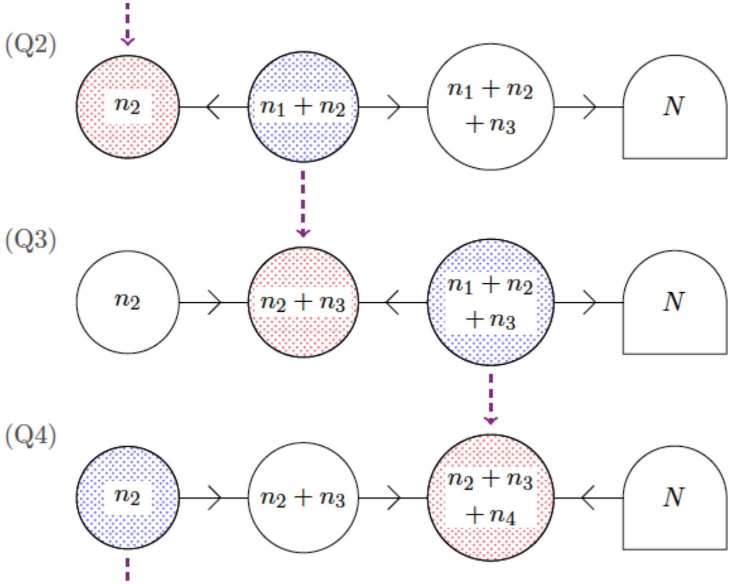
Duality

- Solving the vacuum equations

$$\exp\left(\frac{\partial W}{\partial \sigma}\right) = 1$$

leads again to a perfect match (term by term) with the superpotential of ramified instantons with the new JK parameter!

- The same is true for any other quiver diagram of the sequence we considered



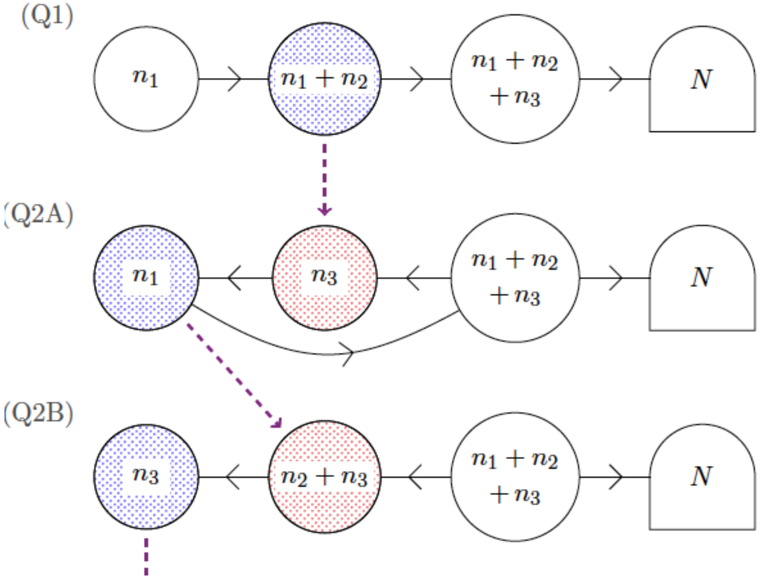
Duality

- Solving the vacuum equations

$$\exp\left(\frac{\partial W}{\partial \sigma}\right) = 1$$

leads again to a perfect match (term by term) with the superpotential of ramified instantons with the new JK parameter!

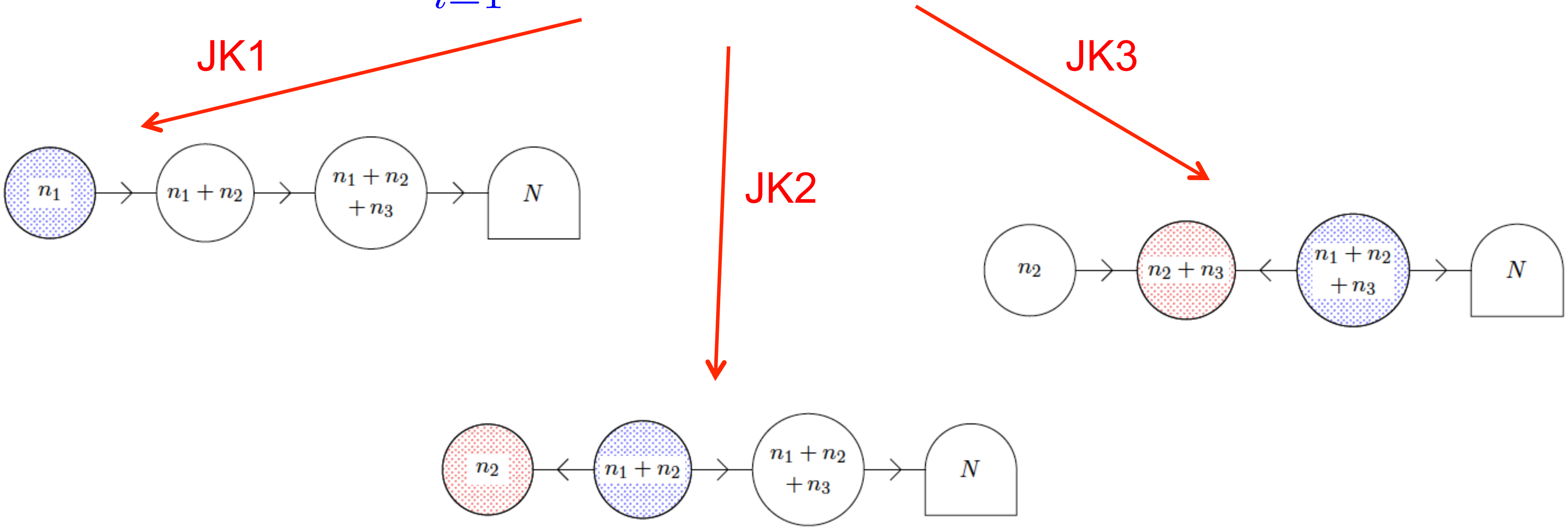
- The same is true for any other quiver diagram of the sequence we considered, and also in sequences with non-linear quivers



Duality

- Starting from the same localization integrand and choosing different integration contours (or different Jeffrey-Kirwan vectors) we obtain a perfect match with different quiver theories related to each other by duality transformations

$$\prod_{i=1}^{M} \oint \frac{d\chi_i}{2\pi i} \frac{q_i^{d_i}}{d_i!} \; Z_{d_i}(a, \epsilon_1, \epsilon_2)$$



Conclusions

Summary of results

monodromy defects	coupled *2d*-*4d* systems
$[n_1, n_2, \ldots, n_M]$	Ranks of gauge groups
4d v.e.v’s of SU(N)	*2d* twisted masses
Ramified instanton counting parameters q_i	Dynamically generates scales Λ_i
W_{inst} from localization	W_{inst} from vacuum eqs.
Integration contour	Duality frame

Conclusions

- Similar analysis can be done for other type of theories
  - *N=2** theories
  - SQCD's (with matter in fundamental representation)
  - Chern-Simons like theories in *3d/5d*
  - Theories with other groups (orthogonal, symplectic,…)
- Explore the connection with integrable models
- …

Thanks a lot for your attention!