Twisting and localization in Supergravity: equivariant cohomology of BPS black holes

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Workshop on Supersymmetric Localization and Holography: Black Hole Entropy and Wilson Loops

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Based on arXiv:1806.04479 with Sameer Murthy (King's College)

Motivation

✓ **Supersymmetric localization** [Duistermatt-Heckman, Witten, Schwarz-Zaboronsky]

- a very powerful tool for exact computation

- many applications for field theories in various backgrounds

[Nekrasov, Pestun….]

✓ **Application to supergravities ?**

It could also be applied to the SUGRA because the supersymmetric localization principle is very general.

Need an off-shell formulation of supergravity

We have off-shell formulations for SUGRA up to N=2 (4d), called "Superconformal formulation".

[de Wit, van Proeyen, van Holten…]

[de Wit, V. Reys '17] for Euclidean SUGRA

Will provide exact computation of supergravity.
We can see quantum/exact holography

✓ Our interest is in **BPS black hole entropy** for AdS2/CFT1

Black hole entropy formula

- For large charge limit (thermodynamic limit) in BPS black hole

$$\frac{A_H(p,q)}{4G_N} = \ln d_{micro}(p,q)$$

[Bekenstein-Hawking] [Strominger, Vafa '96]

✓ Our interest is in **BPS black hole entropy** for AdS2/CFT1

Black hole entropy formula

- For finite charge

$$\frac{A_H(p,q)}{4G_N} + O(\frac{1}{Q}) \;=\; \ln d_{micro}(p,q)$$

[Strominger, Vafa ’96]

✓ Our interest is in **BPS black hole entropy** for AdS2/CFT1

Black hole entropy formula

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$$\frac{A_H(p,q)}{4G_N} + O(\frac{1}{Q}) = \ln d_{micro}(p,q)$$

[Strominger, Vafa '96]

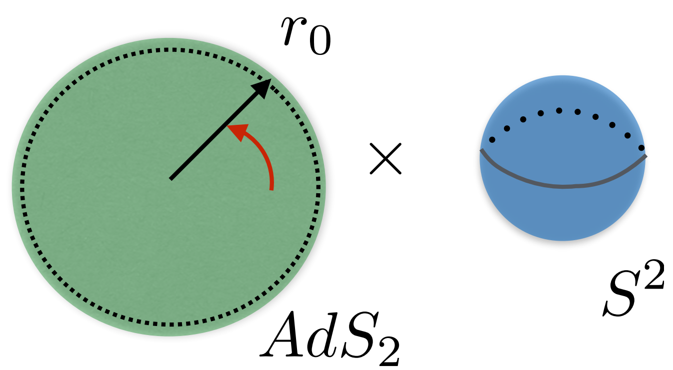
→ **Quantum entropy function**

✓ For extremal black hole, the entropy formula has been generalized to

Quantum entropy function **[Sen '08]**

$$\exp\left(S^{\text{qu}}_{\text{BH}}(q,p)\right) \;\equiv\; W(q,p) \;=\; \int_{\text{AdS}_2} [D\phi_{\text{sugra}}]\, \exp\Big(-i\, q_I \oint_\tau A^I - S_{\text{sugra}}(\phi_{\text{sugra}})\Big)$$

✓ It is a partition function in supergravity with Wilson loop and AdS2 boundary condition.



✓ Many tests for perturbative quantum correction

[Sen, Banerjee, Gupta, Mandal, Lal, Thakur, '10-'14, Larsen, Keeler, Lisbão '14,'15]

✓ We want to apply the supersymmetric localization.

Supersymmetric Localization for QEF

Modify the action

$$S \longrightarrow S + \lambda\, Q_{\rm eq} \mathcal{V}$$

$$\mathcal{V} \;=\; \int d^4x \, \sqrt{\mathring{g}} \, \sum_{\psi} \, \overline{\psi} \, Q_{\rm eq} \psi$$

- We choose a canonical choice for $\mathcal{V}$:

  The summation is over all the physical fermions of the theory.

- The algebra of our fermionic symmetry closes to compact bosonic symmetry:

$$Q_{\rm eq}^2 = H \; (= L_0 - J_0)$$

- At $\lambda \to \infty$ the saddle point approximation is exact, and new saddle point appears which satisfies $Q_{\rm eq} \mathcal{V} = 0$ i.e. "**Localization saddle point**"

$$Q_{\rm eq}\, \psi \;=\; 0 \,, \qquad \text{for all physical fermions } \psi$$

Supersymmetric Localization for QEF

Localization saddle point solutions

- The Weyl multiplet is localized to $AdS_2 \times S^2$ configuration. [Gupta, Murthy '12]

$$ds^2 = \ell^2\left(d\eta^2 + \sinh^2\eta\, d\tau^2\right) + \ell^2\left(d\psi^2 + \sin^2\psi\, d\phi^2\right)$$

where ℓ is scale parameter fixed to arbitrary constant by the Weyl scaling symmetry.

- The off-shell contribution of gravity comes in the physical metric $G_{\mu\nu}$ from the scalar in vector multiplet through the Kahler potential, for its relation to metric in Weyl multiplet

$$G_{\mu\nu} = e^{\mathcal{K}(X,\bar{X})} g_{\mu\nu}$$

- In vector multiplets sector, the solution is labeled by one parameter for each multiplets.

[Dabholkar, Gomes, Murthy '10]

$$X^I = X^I_* + \frac{C^I}{\ell \cosh\eta}\,, \qquad Y^I_{12} = \frac{2C^I}{\ell^2 \cosh^2\eta}\,.$$

Supersymmetric Localization for QEF

- [Dabholkar, Gomes, Murthy '11]

$$W^{\text{pert}}(q,p) = \int_{\mathcal{M}_Q} \prod_{I=0}^{n_{\text{v}}} d\phi^I \, \exp\Big(-\pi\, q_I\, \phi^I + 4\pi\, \text{Im}F\big((\phi^I + \text{i}p^I)/2\big)\Big)\, Z^{Q_{\text{eq}}\mathcal{V}}_{\text{1-loop}}(\phi^I)$$

where $\phi^I = e_*^I + 2C^I$

- Considered some N=2 truncation of N=8 SUGRA with an assumption of $Z_{1-\text{loop}}$ and considered the microstate counting of $\frac{1}{8}$ BPS black hole in type II on T^6 , and showed that the integration over the saddle point would give precise agreement.

- The measure should be given through the 1-loop determinant.

Supersymmetric Localization for QEF

- [IJ, R. Gupta, Y. Ito; S. Murthy, V. Rey '15]

  1-loop determinate has the following universal form,

$$Z_{\text{1-loop}}(\phi^I) \;=\; \exp\big(-a_0\,\mathcal{K}(\phi^I + \mathrm{i}p^I)\big)\,.$$

$$a_0^{\text{vec}} \;=\; -a_0^{\text{hyp}} \;=\; -1/12\,.$$

  agree with the on-shell perturbative computation by [Sen]

- **1-loop for gravity multiplets ?** $a_0^{\text{Weyl}} = ?$

Supersymmetric Localization for QEF

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  agree with the on-shell perturbative computation by [Sen]

- **1-loop for gravity multiplets ?** $\quad a_0^{\text{Weyl}} = \dfrac{23}{12}$

  How and whether it reproduces the consistent result ?

We address two questions.

1. What is the global supercharge $Q_{\mathbf{eq}}$ in supergravity?

[de Wit's talk]

- "There is no global SUSY in a theory of gravity."
  In SUGRA, the supersymmetry is gauged. Not a symmetry of functional integral.
- Fix a background through boundary condition, $AdS_2 \times S^2$
  then, the global supercharge $Q_{\mathbf{eq}}$ is inherited from the symmetry of the background.
- Split the gravitational fields into background and quantum part.
- Need to define action of the global supercharge on the quantum fluctuation.

1. What is the global supercharge $Q_{\mathbf{eq}}$ in supergravity?

- For BRST quantization of SUSY gauge theory, we use equivariant charge

$$Q_{\rm eq} = Q + Q_{\rm brst}$$

such that

$$Q_{\rm eq}^2 = H$$

- Example: U(1) gauge theory

$$Q^2 A_\mu = \mathcal{L}_v A_\mu - \partial_\mu (v^\nu A_\nu)\,, \qquad Qc = v^\nu A_\nu$$

$$\begin{aligned}(Q + Q_{\rm brst})^2 A_\mu &= (Q^2 + QQ_{\rm brst} + Q_{\rm brst}Q + Q_{\rm brst}^2) A_\mu \\ &= Q^2 A_\mu + Q\partial_\mu c \\ &= \mathcal{L}_v A_\mu\end{aligned}$$

- Finding Q transformation for all the ghost in SUGRA can be demanding problem. The difficulty comes from that the algebra in SUGRA is Not Lie algebra but “soft algebra” : field dependent structure constant.

2. What are the twisted variables in supergravity and $Q_{\mathbf{eq}}$ - cohomology?

- For 1-loop determinant we will use index theory.

$$\begin{array}{ccc} \Phi & \longrightarrow & Q_{\text{eq}}\Phi \\ {\scriptstyle D_{10}}\downarrow & & \downarrow \\ \Psi & \longrightarrow & Q_{\text{eq}}\Psi \end{array}$$

Φ "elementary" boson

Ψ "elementary" fermion

- Once we organize all the fields in this representation, then 1-loop determinant reduces to

$$Z_{\text{1-loop}} = \sqrt{\frac{\det_{\Psi} H}{\det_{\Phi} H}}$$

and this can be reproduce by computing the equivariant index.

2. What are the twisted variables in supergravity and $Q_{\mathbf{eq}}$ - cohomology?

- and this can be reproduce by computing the equivariant index.

$$\mathrm{ind}(D_{10})(t) := \mathrm{Tr}_{\mathrm{Ker}D_{10}}\, e^{-iHt} - \mathrm{Tr}_{\mathrm{Coker}D_{10}}\, e^{-iHt}$$
$$= \mathrm{Tr}_{\Phi} e^{-iHt} - \mathrm{Tr}_{\Psi} e^{-iHt}$$

$$\mathrm{ind}(D_{10})(t) = \sum_n a(n)\, e^{-i\lambda_n t} \longrightarrow Z_{\text{1-loop}} = \prod_n \lambda_n^{-a(n)}$$

- Thus the information of the cohomological variable is essential in this computation.
  - ✓ cf. [Bae, Imbimbo, Rey '15] [Imbimbo, Rosa '18] for use of twisting for supersymmetric solutions. Here all fluctuations.

$$
\begin{array}{ccc}
\Phi & \longrightarrow & Q_{\mathrm{eq}}\Phi \\
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\end{array}
$$

1. What is Q_{eq} for SUGRA ?

2. What are the elementary variables Φ , Ψ ?

$$\begin{array}{ccc} \Phi & \longrightarrow & Q_{\rm eq}\Phi \\ D_{10}\downarrow & & \downarrow \\ \Psi & \longrightarrow & Q_{\rm eq}\Psi \end{array}$$

1. **What is** $Q_{\rm eq}$ **for SUGRA ?**

   Background field method of BRST and its modification

   [de Wit, S. Murthy, V. Reys '18]

2. **What are the elementary variables** $\Phi\,,\Psi$ **?**

   Find a twisting of spinor variables

Modified BRST

[de Wit, S. Murthy, V. Reys '18]

Background field method of BRST

Split fields into background + quantum

$$\phi^i = \mathring{\phi}^i + \tilde{\phi}^i \,, \qquad c^\alpha \to \mathring{c}^\alpha + c^\alpha$$

Then the usual BRST transformation for full fields are

$$\begin{aligned} \delta_{\rm brst}\, \phi^i &= \Lambda\, (\mathring{c} + c)^\alpha \, R(\phi)_\alpha{}^i \,, \\ \delta_{\rm brst}\, (\mathring{c} + c)^\alpha &= -\frac{1}{2}\, (\mathring{c} + c)^\gamma \Lambda\, (\mathring{c} + c)^\beta \, f(\phi)_{\beta\gamma}{}^\alpha \;. \end{aligned}$$

$$\boxed{\delta_{\rm brst}{}^2 = 0}$$

Modified BRST

[de Wit, S. Murthy, V. Reys ’18]

Background field method of BRST

Split fields into background + quantum

$$\phi^i = \mathring{\phi}^i + \widetilde{\phi}^i \,, \qquad c^\alpha \to \mathring{c}^\alpha + c^\alpha$$

It is natural to to read off the transformation of quantum fields

$$\begin{aligned}
\delta_{\rm brst}\, \mathring{\phi}^i &= \Lambda\, \mathring{c}^\alpha\, R(\mathring{\phi})_\alpha{}^i \\
\delta_{\rm brst}\, \widetilde{\phi}^i &= \Lambda\, (c + \mathring{c})^\alpha\, R(\phi)_\alpha{}^i - \Lambda\, \mathring{c}^\alpha\, R(\mathring{\phi})_\alpha{}^i \\
\delta_{\rm brst}\, \mathring{c}^\alpha &= -\frac{1}{2}\, \mathring{c}^\gamma \Lambda\, \mathring{c}^\beta\, f(\mathring{\phi})_{\beta\gamma}{}^\alpha \\
\delta_{\rm brst}\, c^\alpha &= -\frac{1}{2}\, (c + \mathring{c})^\gamma \Lambda\, (c + \mathring{c})^\beta\, f(\phi)_{\beta\gamma}{}^\alpha + \frac{1}{2}\, \mathring{c}^\gamma \Lambda\, \mathring{c}^\beta\, f(\mathring{\phi})_{\beta\gamma}{}^\alpha
\end{aligned}$$

$$\boxed{{\delta_{\rm brst}}^2 = 0}$$

Modified BRST

[de Wit, S. Murthy, V. Reys '18]

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$$\delta_{\rm brst}\, \mathring{\phi}^i \;=\; \Lambda\, \mathring{c}^\alpha\, R(\mathring{\phi})_\alpha{}^i$$

$$\delta_{\rm brst}\, \widetilde{\phi}^i \;=\; \Lambda\, (c + \mathring{c})^\alpha\, R(\phi)_\alpha{}^i - \Lambda\, \mathring{c}^\alpha\, R(\mathring{\phi})_\alpha{}^i$$

$$\delta_{\rm brst}\, \mathring{c}^\alpha \;=\; -\frac{1}{2}\, \mathring{c}^\gamma \Lambda\, \mathring{c}^\beta\, f(\mathring{\phi})_{\beta\gamma}{}^\alpha$$

$$\boxed{\delta_{\rm brst}\, c^\alpha \;=\; -\frac{1}{2}\, (c + \mathring{c})^\gamma \Lambda\, (c + \mathring{c})^\beta\, f(\phi)_{\beta\gamma}{}^\alpha + \frac{1}{2}\, \mathring{c}^\gamma \Lambda\, \mathring{c}^\beta\, f(\mathring{\phi})_{\beta\gamma}{}^\alpha}$$

Simple!   If the algebra is Lie algebra, then the transformation rule will reproduce the one of usual background field method in field theory.

Modified BRST

[de Wit, S. Murthy, V. Reys '18]

Fix the background.

- There is no background value of ghost, except the isometry

Choose $\mathring{c}^{\alpha}$ **to be an isometry parameter (Killing spinor)**

Isometry

$$\delta_{\text{brst}}\, \mathring{\phi}^i = \Lambda\, \mathring{c}^{\alpha}\, R(\mathring{\phi})_{\alpha}{}^{i}$$

$$\delta_{\text{brst}}\, \widetilde{\phi}^i = \Lambda\, (c + \mathring{c})^{\alpha}\, R(\phi)_{\alpha}{}^{i} - \Lambda\, \mathring{c}^{\alpha}\, R(\mathring{\phi})_{\alpha}{}^{i}$$

$$\delta_{\text{brst}}\, \mathring{c}^{\alpha} = -\frac{1}{2}\, \mathring{c}^{\gamma} \Lambda\, \mathring{c}^{\beta}\, f(\mathring{\phi})_{\beta\gamma}{}^{\alpha}$$

$$\delta_{\text{brst}}\, c^{\alpha} = -\frac{1}{2}\, (c + \mathring{c})^{\gamma} \Lambda\, (c + \mathring{c})^{\beta}\, f(\phi)_{\beta\gamma}{}^{\alpha} + \frac{1}{2}\, \mathring{c}^{\gamma} \Lambda\, \mathring{c}^{\beta}\, f(\mathring{\phi})_{\beta\gamma}{}^{\alpha}$$

$$\delta_{\text{brst}}{}^2 = 0$$

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[de Wit, S. Murthy, V. Reys '18]

Fix the background.

- There is no background value of ghost, except the isometry

Choose $\mathring{c}^{\alpha}$ to be an isometry parameter (Killing spinor)

- For non-compact space, this isometry parameter is not normalizable, and is no longer gauge symmetry.

- We do not need to introduce additional ghost of ghost.

Modified BRST

[de Wit, S. Murthy, V. Reys '18]

Fix the background.

- There is no background value of ghost, except the isometry

Choose $\mathring{c}^{\alpha}$ **to be an isometry parameter (Killing spinor)**

Isometry

$$\delta_{\text{eq}}\, \mathring{\phi}^i = \Lambda\, \mathring{c}^{\alpha}\, R(\mathring{\phi})_{\alpha}{}^{i}$$

$$\delta_{\text{eq}}\, \widetilde{\phi}^i = \Lambda\, (\mathring{c} + c)^{\alpha}\, R(\phi)_{\alpha}{}^{i} - \Lambda\, \mathring{c}^{\alpha}\, R(\mathring{\phi})_{\alpha}{}^{i}$$

$$\delta_{\text{eq}}\, \mathring{c}^{\alpha} = 0$$

Deformation

$$\delta_{\text{eq}}\, c^{\alpha} = -\frac{1}{2}(\mathring{c} + c)^{\gamma}\Lambda(\mathring{c} + c)^{\beta}\, f(\phi)_{\beta\gamma}{}^{\alpha} + \frac{1}{2}\mathring{c}^{\gamma}\Lambda\, \mathring{c}^{\beta}\, f(\phi)_{\beta\gamma}{}^{\alpha}$$

- Then the algebra equivariantly closes to bosonic symmetry with rigid parameter.

$$\delta_{\text{eq}}^2 = \delta_{\mathring{\xi}}\,, \qquad \mathring{\xi}^{\alpha} = \frac{1}{2}\Lambda_2\, \mathring{c}^{\gamma}\Lambda_1\, \mathring{c}^{\beta}\, f(\mathring{\phi})_{\beta\gamma}{}^{\alpha}$$

Modified BRST

Anti-ghost and auxiliary field

$$\delta_{\rm eq}\, b_\alpha = \Lambda\, B_\alpha\,,$$
$$\delta_{\rm eq}\, B_\alpha = \frac{1}{2}\,\mathring{c}^{\,\sigma}\Lambda\,\mathring{c}^{\,\delta}\, f(\mathring{\phi})_{\delta\sigma}{}^{\beta}\, f(\phi)_{\beta\alpha}{}^{\gamma}\, b_\gamma\,.$$

$$\delta_{\rm eq}^2\, b_\alpha = \mathring{\xi}^{\beta} f(\phi)_{\beta\alpha}{}^{\gamma}\, b_\gamma\,,$$

$$\delta_{\rm eq}^2\, B_\alpha = \mathring{\xi}^{\beta} f(\phi)_{\beta\alpha}{}^{\gamma} B_\gamma + \mathring{\xi}^{\beta}(\mathring{c}+c)^{\kappa} R(\phi)_\kappa{}^i \partial_i f(\phi)_{\beta\alpha}{}^{\gamma} b_\gamma$$

- If $f(\phi)_{\beta\alpha}{}^{\gamma}$ is constant, then the algebra is closed.
- It is possible by the observation that the index β is for bosonic symmetry.

$$\delta_{\rm eq}^2 = \mathring{\delta}_{\mathring{\xi}}$$

Application to supergravity

- Generically, for supergravity softness of $f(\phi)_{\beta\alpha}{}^{\gamma}$ appears only from anti commutator of supersymmetries.

- For the case of supergravity, (D=4 N=2 superconformal gavity)

  the “modified BRST” gives the equivariant symmetry

$$Q_{\rm eq}^2 = \mathcal{L}_{\mathring{v}} + \sum_{I,{\rm bos}} \delta_I(\mathring{\varepsilon}_3^I)$$

where

$$\mathring{v}^{\mu} := \frac{1}{2}\mathring{c}^J \mathring{c}^I f_{IJ}{}^{\mu}(\mathring{\phi}) \,,$$
$$\mathring{\varepsilon}_3^I := \frac{1}{2}\mathring{c}^K \mathring{c}^J f_{JK}{}^{I}(\mathring{\phi}) \,,$$

Matter coupled to supergravity

- General formulation can be applied in the same manner when matter coupled to supergravity.
- This formalism systemize the construction the equivariant charge that was constructed in SUSY gauge theories. :

For rigid limit of SUGRA coupled to YM theory recovers the field theory

cf. Pestun '07, Hama-Hosomichi '12, David-Gava-Gupta-Narain '16

$$
\begin{aligned}
Q_{\rm eq}\widetilde{\phi}^i_m &= \mathring{c}^A {R_A}^i(\mathring{\phi}+\widetilde{\phi}_m) + c^I {R_I}^i(\mathring{\phi}+\widetilde{\phi}_m) \\
Q_{\rm eq}c^I &= -\frac{1}{2}\mathring{c}^C\mathring{c}^B({f_{BC}}^I(\mathring{\phi}+\widetilde{\phi}_m) - {f_{BC}}^I(\mathring{\phi})) + \frac{1}{2}c^K c^J {f_{JK}}^I \\
Q_{\rm eq}b_I &= B_I \\
Q_{\rm eq}B_I &= \mathcal{L}_{\frac{1}{2}\mathring{c}^B\mathring{c}^A {f_{AB}}^\mu(\mathring{\phi})} b_I + \frac{1}{2}\mathring{c}^B\mathring{c}^A {f_{AB}}^J(\mathring{\phi}) {f_{JI}}^K b_K \,,
\end{aligned}
$$

$$
\begin{aligned}
Q_{\rm eq}^2 &= \mathcal{L}_{\mathring{v}} + \sum_{A\in {\rm bos}} \delta_A(\mathring{\varepsilon}^A_3) + \delta_G(\mathring{a}) \,, \\
\mathring{a}^I &= \frac{1}{2}\mathring{c}^B\mathring{c}^A {f_{AB}}^I(\mathring{\phi}) \,.
\end{aligned}
$$

Twisting and cohomological classification

- Reorganize the fields into the **representation of cohomology complex**.

$$
\begin{array}{ccc}
\Phi & \longrightarrow & Q_{\mathrm{eq}}\Phi \\
{\scriptstyle D_{10}}\big\downarrow & & \big\downarrow \\
\Psi & \longrightarrow & Q_{\mathrm{eq}}\Psi
\end{array}
$$

.

- This reorganization is a change of variable: **local and invertible**

- Find an appropriate choice of **twisting** of spinors such that we can find the cohomological variables and the change of variables is non-singular.

Twisted field and algebra

- 1. Choose a way of twisting and make sure that it is invertible.
- 2. Start with a given component ϕ_R of boson (or fermion) in some representation R of gauge group. Lorentz , R-symmetry etc..
- 3. Consider its variation $Q_{\rm eq}\phi_R$ which may be a composite combination of bosons and fermions with some coefficient made of Killing spinor and background value . Find a fermion ψ_R of the same representation with ϕ_R which linearly appears.
- 4. ψ_R should not involve derivative, also the coefficient of this term should be regular. Otherwise the invertibility will not be guaranteed.
- 5. If we can find such ψ_R then we classify the ϕ_R as the elementary bosonic variable in Φ and may exclude ψ_R from the elementary fermionic variable Ψ .
- 6. Keep the process until the end. If we fail, reconsider the twisting.

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- 6. Keep the process until the end. If we fail, reconsider the twisting.

This procedure guarantees the invertibility if considering small fluctuation.
We assume it holds even for large fluctuation.

Exercise: N=2 U(1) gauge multiple

- vector multiplet $\big(A_\mu, X, \lambda^i, Y_{ij}\big)$ 9 B+8 F d.o.f.

  and U(1) ghost multiplet (b, c, B) 1b + 2F d.o.f.

- Choose a twisted variable using the production by $(\gamma_5\varepsilon^i\,, \gamma^\mu\varepsilon^i\,, \varepsilon_{ij}\varepsilon^j)$

$$\lambda \;=\; \overline{\varepsilon}_i\gamma_5\lambda^i\,, \qquad \lambda_\mu \;=\; \overline{\varepsilon}_i\,\gamma_\mu\,\lambda^i\,, \qquad \lambda^{ij} \;=\; -2\,\varepsilon^{(i}\,C\lambda^{j)}$$

  inverse relation is

$$\lambda^i \;=\; (\overline{\varepsilon}_j\,\varepsilon^j)^{-1}\big(\gamma_5\varepsilon^i\lambda + \gamma^\mu\varepsilon^i\,\lambda_\mu + \epsilon_{jk}\,\varepsilon^k\,\lambda^{ij}\big)$$

- cf. We could have used another twisting using $(\varepsilon^i\,, \gamma^\mu\gamma_5\varepsilon^i\,, \varepsilon_{ij}\gamma_5\varepsilon^j)$

Exercise: N=2 U(1) gauge multiple

- Investigate the variation and follow the procedure.

10B + 10F d.o.f. fall into a rep of this equivariant algebra

Φ	Ψ
$\widetilde{A}_\mu\,,\widetilde{X}_2$	$\lambda^{ij}\,,b\,,c$

elementary bosons

$$Q_{\rm eq}\,\widetilde{A}_\mu \;=\; \lambda_\mu + \partial_\mu c$$
$$Q_{\rm eq}\,\widetilde{X}_2 \;=\; \lambda$$

elementary fermions

$$Q_{\rm eq} c = v^\mu \widetilde{A}_\mu + {\rm i}\widetilde{X}_1(\bar{\varepsilon}_i\varepsilon^i) + \widetilde{X}_2(\bar{\varepsilon}_i\gamma_5\varepsilon^i)$$
$$Q_{\rm eq} b \;=\; B$$
$$Q_{\rm eq}\,\lambda^{ij} \;=\; \bar{\varepsilon}_k\varepsilon^k\,Y^{ij} + 2\varepsilon^{(i}C\gamma^\mu\varepsilon^{j)}\partial_\mu X_2$$
$$+\,\varepsilon_+^{(i}C\gamma^{ab}\varepsilon_+^{j)}\left[F_{ab}^- - \frac{1}{8}(X_1 - {\rm i}X_2)T_{ab}^-\right]$$
$$+\,\varepsilon_-^{(i}C\gamma^{ab}\varepsilon_-^{j)}\left[F_{ab}^+ - \frac{1}{8}(X_1 + {\rm i}X_2)T_{ab}^+\right]$$

Weyl multiplet

- Weyl multiplet $\left(e_\mu^a\,, \psi_\mu^i\,, A_\mu^D\,, A_\mu^R\,, \mathcal{V}_{\mu\,j}^{\,i}\,; T_{ab}^{ij}\,, \chi^i\,, D\right)$
- 24 B+ 24 F d.o.f. after removing gauge redundancies.
- 43 B + 40 F d.o.f. if we keep all degree of freedom.
- Add 51 B +54 F ghost degree of freedom.
- Similar classification of 94B +94F fields as a representation of the equivariant algebra

Weyl multiplet

Local symmetry	Gauge fields	Degrees of freedom
g.c.t	e_μ^a	16B
Dilatation D	A_μ^D	4B
Sp. conf. K^a	f_μ^a	composite
Lorentz M_{ab}	ω_μ^{ab}	composite
$SO(1,1)_R$	A_μ^R	4B
$SU(2)_R$	$\mathcal{V}_{\mu\,j}^{\,i}$	12B
Q-susy	ψ_μ^i	32F
S-susy	ϕ_μ^i	composite

	Auxiliary fields	Degrees of freedom
	$T_{ab}^{\pm}$	6B
	D	1B
	χ^i	8F

Ghost multiplets

Local symmetry	Ghosts	Degrees of freedom
g.c.t	(c_μ, b_μ, B_μ)	8F 4B
Dilatation D	(c_D, b_D, B_D)	2F B
Sp. conf. K^a	$(c^a_K\,, b^a_K\,, B^a_K)$	8F 4B
Lorentz M_{ab}	$(c^{ab}_M\,, b^{ab}_M\,, B^{ab}_M)$	12F 6B
$U(1)_R$	(c_R, b_R, B_R)	2F 1B
$SU(2)_R$	$(c^{\;i}_{R\,j}\,, b^{\;i}_{R\,j}\,, B^{\;i}_{R\,j})$	6F 3B
Q-susy	$(c^i_Q\,, b^i_Q\,, B^i_Q)$	16B 8F
S-susy	$(c^i_S\,, b^i_S\,, B^i_S)$	16B 8F

Twisted variables

- By projection of Killing spinors, $(\gamma_5\varepsilon^i\,, \gamma^\mu\varepsilon^i\,, \varepsilon_{ij}\varepsilon^j)$ or $(\varepsilon^i\,, \gamma^\mu\gamma_5\varepsilon^i\,, \varepsilon_{ij}\gamma_5\varepsilon^j)$

  we found a choice

$$
\begin{aligned}
\psi_\mu = \varepsilon_i\gamma_5\psi_\mu^i\,, \quad {\psi_\mu}^a &= \varepsilon_i\gamma^a\psi_\mu^i\,, \quad \psi_\mu^{ij} = -2\varepsilon^{(i}C\psi_\mu^{j)}\,, \\
\chi = \varepsilon_i\chi^i\,, \quad \chi^a &= \varepsilon_i\gamma_5\gamma^a\chi^i\,, \quad \chi^{ij} = -2\varepsilon^{(i}C\gamma_5\chi^{j)}\,, \\
c_S = \varepsilon_i\gamma_5 c_S^i\,, \quad c_S^a &= \varepsilon_i\gamma^a c_S^i\,, \quad c_S^{ij} = -2\varepsilon^{(i}Cc_S^{j)}\,, \\
c_Q = \varepsilon_i\gamma_5 c_Q^i\,, \quad {c_Q}^a &= \varepsilon_i\gamma^a c_Q^i\,, \quad c_Q^{ij} = -2\varepsilon^{(i}Cc_Q^{j)}\,.
\end{aligned}
$$

- Inverse relation

$$
\begin{aligned}
\psi_\mu^i &= (\varepsilon_i\varepsilon^i)^{-1}\left(\psi_\mu\gamma_5\varepsilon^i + {\psi_\mu}^a\gamma_a\varepsilon^i + \psi_\mu^{ij}\varepsilon_{jk}\varepsilon^k\right)\,, \\
\chi^i &= (\varepsilon_i\varepsilon^i)^{-1}\left(\chi\varepsilon^i + \chi^a\gamma_a\gamma_5\varepsilon^i + \chi^{ij}\varepsilon_{jk}\gamma_5\varepsilon^k\right)\,, \\
c_S^i &= (\varepsilon_i\varepsilon^i)^{-1}\left(c_S\gamma_5\varepsilon^i + c_S^a\gamma_a\varepsilon^i + c_S^{ij}\varepsilon_{jk}\varepsilon^k\right)\,, \\
c_Q^i &= (\varepsilon_i\varepsilon^i)^{-1}\left(c_Q\gamma_5\varepsilon^i + c_Q^a\gamma_a\varepsilon^i + c_Q^{ij}\varepsilon_{jk}\varepsilon^k\right)\,.
\end{aligned}
$$

Variation of fields

•

$$
\begin{aligned}
Q_{\rm eq}\widetilde{e}_\mu{}^a &= \bar{\varepsilon}_i\gamma^a\psi_\mu{}^i + c^\nu\partial_\nu e_\mu^a + \partial_\mu c^\nu e_\nu^a + c^{ab}e_{\mu b} - c_D e_\mu^a + \overline{c_Q}_i\gamma^a\psi_\mu{}^i\,,\\
Q_{\rm eq}\psi_\mu{}^i &= 2\mathcal{D}_\mu(\varepsilon+c_Q)^i + c^\nu\partial_\nu\psi_\mu{}^i + \partial_\mu c^\nu\psi_\nu{}^i + \tfrac14 c^{ab}\gamma_{ab}\psi_\mu{}^i - \tfrac12 c_D\psi_\mu{}^i - \tfrac12 c_R\gamma_5\psi_\mu{}^i\\
&\quad + c^i{}_j\psi_\mu{}^i + \mathrm{i}\tfrac{1}{16}T^{ab}\gamma_{ab}\gamma_\mu(\varepsilon+c_Q)^i + \gamma_\mu\gamma_5(\eta+c_S)^i\,,\\
&= 2\widetilde{\mathcal{D}}_\mu\varepsilon^i + \mathring{\gamma}_\mu\gamma_5 c_S^i + \mathrm{i}\tfrac{1}{16}\gamma_{ab}(T^{ab}\gamma_\mu - \mathring{T}^{ab}\mathring{\gamma}_\mu)\varepsilon^i + 2\mathcal{D}_\mu c_Q^i + c^\nu\partial_\nu\psi_\mu{}^i + \partial_\mu c^\nu\psi_\nu{}^i\\
&\quad + \tfrac14 c^{ab}\gamma_{ab}\psi_\mu{}^i - \tfrac12 c_D\psi_\mu{}^i - \tfrac12 c_R\gamma_5\psi_\mu{}^i + c^i{}_j\psi_\mu{}^i + \mathrm{i}\tfrac{1}{16}\gamma_{ab}T^{ab}\gamma_\mu c_Q^i + \widetilde{\gamma}_\mu\gamma_5 c_S^i + \widetilde{\gamma}_\mu\gamma_5\eta^i
\end{aligned}
$$

etc…

• Write them in terms of twisted variables and try to the cohomological classicfication.

Cohomological classification

Φ	Ψ
$\widetilde{e}_\mu^a\,,\widetilde{A}_\mu^R\,,\widetilde{A}_\mu^D\,,\widetilde{T}_{ab}^{+/-}$	$\psi_\mu\,,\psi_\mu^{ij}\,,\chi\,,$
$c_Q\,,\,c_Q^{ij}\,,$	$c^\mu\,,c_M^{ab}\,,c_D\,,$
$b_Q\,,\,b_{Qa}\,,\,b_Q^{ij}\,,$	$b_\mu\,,b_M^{ab}\,,b_D\,,$
$b_S\,,\,b_{Sa}\,,\,b_S^{ij}$	$b_K^a\,,b_R\,,b_{Rj}^i$

$$Q_{\rm eq}\tilde{e}_\mu{}^a = \psi_\mu{}^a + \cdots$$

$$\begin{aligned} Q_{\rm eq}\,\psi_\mu &= -c_{S\mu} + \widetilde{A}_\mu^R\bar{\varepsilon}_i\varepsilon^i + \cdots\,, \\ Q_{\rm eq}\,\psi_\mu^{ij} &= \widetilde{\mathcal{V}}_\mu{}^{(i}{}_k\epsilon^{j)k} + \cdots\,, \end{aligned}$$

Cohomological classification

Φ	Ψ
$\widetilde{e}_\mu^a\,,\widetilde{A}_\mu^R\,,\widetilde{A}_\mu^D\,,\ \widetilde{T}_{ab}^{+/-}$	$\psi_\mu\,,\psi_\mu^{ij}\,,\chi\,,$
$c_Q\ ,\ c_Q^{ij}\,,$	$c^\mu\,,c_M^{ab}\,,c_D\,,$
$b_Q\ ,\ b_{Qa}\ ,\ b_Q^{ij}\ ,$	$b_\mu\,,b_M^{ab}\,,b_D\,,$
$b_S\ ,\ b_{Sa}\ ,\ b_S^{ij}$	$b_K^a\,,b_R\,,b_{Rj}^i$

$$Q_{\rm eq}\,\widetilde{T}_{ab}^{\pm}\ =\ 4{\rm i}\,(\bar\varepsilon_i\varepsilon^i)^{-1}\,\varepsilon_{\mp}^{(i}C\,\gamma_{ab}\,\gamma_5\,\varepsilon_{\mp}^{j)}\,\chi_{ij}\ +\cdots\ ,$$

- In terms of $SU(2)_+\times SU(2)_-\times SU(2)_R$

  $\tilde{T}_{ab}^+$ and $\tilde{T}_{ab}^-$ (1,3,1) and (3,1,1) but χ^{ij} (1,1,3)

- Twisting procedure maps χ^{ij} to $\tilde{T}_{ab}^-$ or $\tilde{T}_{ab}^+$ depending on a point of manifold.

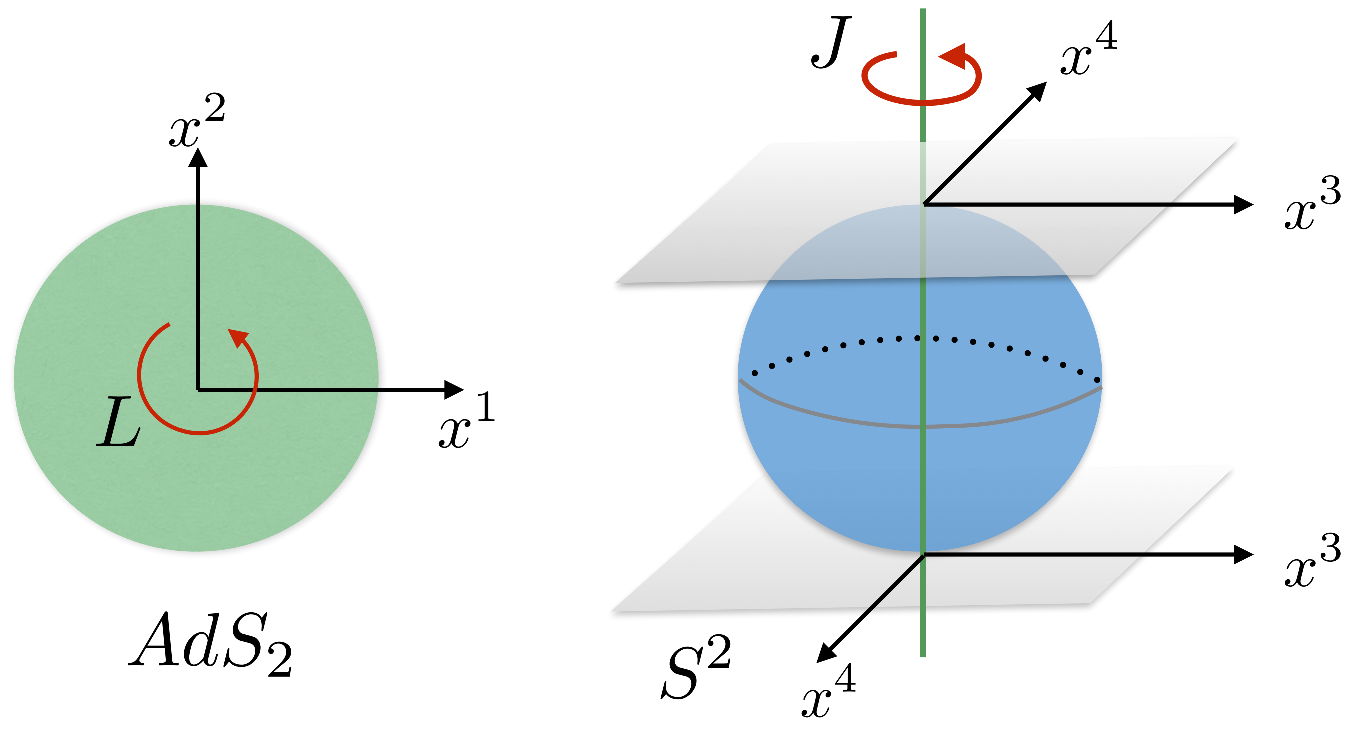
Index and 1-loop

Atiyah-Bott fixed point formula

- We apply Atiyah-Bott fixed point formula to compute the index

$$\mathrm{ind}(D_{10}) = \sum_{\{x|\widetilde{x}=x\}} \frac{\mathrm{Tr}_\Phi\, e^{-itH} - \mathrm{Tr}_\Psi\, e^{-itH}}{\det(1-\partial\widetilde{x}/\partial x)}$$

- There are two fixed point under H= L-J: One is the center of AdS2 with the north pole of S2, the other is the center of AdS2 with the south pole of S2



Topological twisting

- At the fixed point, the twisting between $SU(2)_R$ symmetry and one of SU(2) in Lorentz group $SU(2)_+ \times SU(2)_-$ happen.

- At the fixed points, the chiral and anti chiral part of Killing spinor is reduced.

- At north pole : $\text{at } \eta = 0 \text{ and } \psi = 0$

$$\varepsilon^i_{\ +\alpha} = 0 \,, \qquad \varepsilon^i_{\ -\dot{\alpha}} \propto (\sigma_3)^i_{\ \dot{\alpha}}$$

- At south pole: $\text{at } \eta = 0 \text{ and } \psi = \pi$

$$\varepsilon^i_{\ +\alpha} \propto (\sigma_3)^i_{\ \alpha} \,, \qquad \varepsilon^i_{\ -\dot{\alpha}} = 0$$

- Therefore, $SU(2)_R$ symmetry is identified with the inverse of $SU(2)_-$ at north pole, the inverse of $SU(2)_+$ at south pole.

Twisted representation

- At the fixed point, the twisting between SU(2) R symmetry and one of SU(2) in Lorentz group SU(2) x SU(2) happen.

Φ	NP: $SU(2)_+ \times SU(2)_{-R}$ rep SP: $SU(2)_{+R} \times SU(2)_-$ rep	Ψ	NP: $SU(2)_+ \times SU(2)_{-R}$ rep SP: $SU(2)_{+R} \times SU(2)_{-R}$ rep
$\widetilde{e}^a_\mu$	$(3,3)+(1,3)+(3,1)+(1,1)$	ψ_μ	$(2,2)$
$\widetilde{A}^R_\mu$	$(2,2)$	ψ^{ij}_μ	$(2,4)$ at NP/$(4,2)$ at SP + $(2,2)$
$\widetilde{A}^D_\mu$	$(2,2)$	χ	$(1,1)$
$T^{+/-}_{ab}$ at NP/SP	$(1,3)$ at NP/$(3,1)$ at SP	c_μ	$(2,2)$
c_Q	$(1,1)$	c^{ab}_M	$(1,3)+(3,1)$
c^{ij}_Q	$(1,3)$ at NP/$(3,1)$ at SP	c_D	$(1,1)$
b_Q	$(1,1)$	b_R	$(1,1)$
$b_{Q\mu}$	$(2,2)$	${b_R}^i_j$	$(1,3)$ at NP/$(3,1)$ at SP
${b_Q}^{ij}$	$(1,3)$ at NP/$(3,1)$ at SP	b_μ	$(2,2)$
b_S	$(1,1)$	b_D	$(1,1)$
$b_{S\mu}$	$(2,2)$	b^a_K	$(2,2)$
${b_S}^{ij}$	$(1,3)$ at NP/$(3,1)$ at SP	b^{ab}_M	$(1,3)+(3,1)$

1-loop determinant

- Using the Atiyah-Bott fixed point formula,

$$\mathrm{ind}(D_{10}) = \frac{2(q^2+q^{-2}) - 6(q+q^{-1}) + 8}{(1-q^{-1})^2(1-q)^2} \times 2$$

- From the index we read off

$$Z_{1\text{-loop}}(\phi^I) = \exp(-a_0 \mathcal{K}(\phi^I))$$

$$a_0^{\text{Weyl, bulk}} = \frac{11}{12}$$

- Recall that the zero mode contribution of Weyl multiplet is [Sen]

$$a_0^{\text{Weyl, bdry}} = 1 \ (-3 \text{ from bosons and } +4 \text{ from fermions})$$

Adding this, we obtain

$$a_0^{\text{Weyl}} = 23/12$$

Which is consistent with on-shell computation.

Summary and outlook

- ✓ We have constructed the equivariant supercharge for N=2 SUGRA, and classified the cohomological variables with appropriate twisting of variables.
- ✓ The index computation gives the 1-loop for Weyl multiplets, which agrees with on-shell perturbative computation.
- ✓ We hope that this work brings some clarity to the idea of twisting and localization in supergravity.
- ✓ It may be useful in other directions.
- ✓ Other systems can be interesting AdS_{d+1}/CFT_d
- ✓ Relation of Twisting of supergravity to topological gravity?

  [Witten '88] [Baulieu, Bellon, Reys '12] [Bae, Imbimbo, Rey, Rosa]

Thank you!

Remark 1

[IJ, Yuto Ito, Rajesh Gupta arXiv:1504.01700]

• The eigen value of H is $\frac{n}{\ell}$.   How do we get 1-loop which depends on localization saddle through the Kahler potential $e^{\mathcal{K}(C)}$?

Integration measure

• Definition of the functional integration means

$$\mathcal{D}X^I = \prod_{x,I} \mathrm{d}X^I(x)\mathcal{J}(X)$$

• The Jacobian can be determined by using "ultra locality argument",

[Fujikawa, Yasuda '84 ; Bern, Blau, Mottola '91
Moore, Nelson '86]

Integration measure

- Consider the kinetic terms for graviton and scalars in the action

$$\int \mathrm{d}x^4 \sqrt{g} e^{-K} [R_g + N_{IJ} \partial_\mu X^I \partial_\nu \bar{X}^J g^{\mu\nu}]$$

- The metric $g_{\mu\nu}$ is not physical metric, which is related to the physical metric in Einstein frame.

$$G_{\mu\nu} = g_{\mu\nu} e^{-K} \,, \qquad e^{-K} = \frac{\ell_P^2}{\ell^2}$$

- Then we get standard E-H action, and the kinetic terms of the scalars are

$$\int \mathrm{d}x^4 \sqrt{G} e^{K} N_{IJ} \partial_\mu X^I \partial_\nu \bar{X}^J G^{\mu\nu}$$

- Looking at the factors in front of each kinetic term, the definition of norm is dictated as

$$||\delta X||^2 := \int \mathrm{d}^4 x \sqrt{G} e^{K} N_{IJ} \delta X^I \delta \bar{X}^J = \int \mathrm{d}^4 x \sqrt{g_0} \ell^2 \ell_P^2 N_{IJ} \delta X^I \delta \bar{X}^J$$

Integration measure

- By the following normalization condition

$$1 = \int \mathcal{D}X\mathcal{D}\bar{X} e^{-||\delta X||^2}$$

the integration measure is defined as

$$\mathcal{D}X\mathcal{D}\bar{X} = \prod_{x,I} \mathrm{d}X^I(x)\mathrm{d}\bar{X}^I(x) \det(\ell_P^2 \ell^2 N_{IJ})$$

- Similarly, for all the fields we can define the measure.

- Note that the physical radius factor becomes, by localization, $\ell_P(X, \bar{X}) = \ell_P(\vec{C})$

- The problem essentially becomes computation of the regularized power $\ell_P(\vec{C})$

Integration measure

- In order to compute the regularized number, we first use the field redefinition

$$\tilde{X}^I := X^I \ell_P(\vec{C})\ell\,, \quad \tilde{W}^I_\mu := W^I_\mu \ell_P(\vec{C})\,, \quad etc..$$

- and compute 1-loop partition function with these variables.

✓ To relate the redefined bosonic and fermionic field, it also becomes natural to consider

redefinition of the equivariant operator $\tilde{Q}_{\rm eq} := \ell^{1/2}\ell_P^{-1/2} Q_{\rm eq}$

so that the eigenvalue of the square of the operator is in terms of physical length

and so the result of the 1-loop is in terms of the physical length $\tilde{Q}^2_{\rm eq} \to 1/\ell_P$.

Remark 2 : Boundary modes (Example for 1-form)

- Boundary modes(Discrete zero mode): [Camporesi, Higuchi]

  Since AdS2 is non-compact geometry, it forces us to consider

$$W^l = d\Phi^l \,, \qquad \Phi^l = \frac{1}{\sqrt{2\pi|l|}} \left[\frac{\sinh \eta}{1+\cosh \eta} \right]^{|l|} e^{il\theta} \,, \qquad l = \pm 1, \pm 2, \pm 3, \cdots$$

  which do not vanish at the boundary of AdS2, but still normalizable.

- These modes not only make QV=0, but also the original action vanishing since the field strength is zero.

  .

- Nevertheless, these are not pure gauge, because Φ^l are not normalizable.

  Thus this mode cannot be gauged away, and we have to separately take into account it in the path integral . The regularized result is well understood. [Sen, Gupta]