The topological structure of supergravity

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JHEP 1603 (2016) 169, J. Bae, C.I. S.J Rey, D. Rosa
JHEP 1805 (2018) 112, C. I. and D. Rosa,
C. I., V. Pedemonte and D. Rosa, in progress

The topological sectors of supergravity

- Today I am going to describe two different topological structures which sit inside supergravity.
- The first one is very generic: it exists in any dimensions and in any supergravity.
- The second structure exists for a certain class of supergravity theories, which includes $N = (2, 2)$ and $N = (4, 4)$ in $d = 2$ and $N = 2$ in $d = 4$.
- In $d = 2$ case, I will describe the relationship between these two structures which emerges in their application to localization.

The BRST formulation of supergravity

- I will start by revisiting the BRST formulation of supergravity, for the purpose of setting the notation.
- This formulation requires introducing:
  - anti-commuting ghosts for bosonic symmetries;
  - commuting ghosts for fermionic symmetries;
  - a nilpotent operator s acting on ghost and other matter fields.

Ghosts and superghosts

- (Poincaré) Supergravity bosonic symmetries include:
  - Diffeos with ghost ξ^μ
  - YM symmetries like local Lorentz and local R-symmetries with ghost c, living in the total = Lorentz+YM Lie algebra
- Fermionic symmetries:
  - Local supersymmetries with ghosts ζ^i, with $i = 1, \ldots N$, which are Majorana spinors.

The BRST transformations of the supersymmetric ghost

- The action of the BRST s on the supersymmetric ghost is

$$s\,\zeta^i = i_\gamma(\psi^i) + \text{diffeos} + \text{gauge}$$
$$\gamma^\mu \equiv -\frac{1}{2}\,\bar{\zeta}_i\,\Gamma^A\,\zeta^i\,e^\mu_A$$

  where i_γ is the contraction of a form by the vector ghost bilinear γ^μ and $\psi^i = \psi^i_\mu\,dx^\mu$ are the gravitinos.
- The BRST transformation of the vierbein is

$$s\,e^A = \bar{\zeta}_i\,\Gamma^A\psi^i + \text{diffeos} + \text{gauge}$$

The BRST algebra

- One can show that the BRST algebra of any supergravity theory takes the form

$$S^2 = \mathcal{L}_\gamma + \delta_{i_\gamma(A)+\phi}$$

- S is obtained from s by subtracting the transformations associated to the bosonic gauge symmetries

$$S = s + \delta_c + \mathcal{L}_\xi$$

- $\mathcal{L}_\xi$ is the Lie derivative along the vector field ξ^μ.
- δ_c is the gauge transformation with parameter c.

The γ^μ and ϕ ghost bilinears

We see that the BRST algebra is fully characterized by two bilinears of the commuting ghosts ζ^i

$$\boxed{\gamma^\mu \quad : \text{a commuting vector fields}}$$

$$\boxed{\phi \quad : \text{scalars in the total gauge Lie algebra}}$$

The γ^μ ghost bilinears

- The vector bilinear γ^μ has an universal expression [Baulieu & Bellon, 1986]

$$\gamma^\mu \equiv -\frac{1}{2}\,\bar{\zeta}_i\,\Gamma^A\,\zeta^i\,e^\mu_A$$

- The scalar ghost bilinear

$$\phi = \phi^{AB}\,\frac{1}{2}\,\sigma_{AB} + \phi^I\,T^I$$

valued in the total = Lorentz +R-symmetry gauge Lie algebra is model dependent: it characterises the specific supergravity one is considering.

The N=(2,2) d=2 ϕ ghost bilinears

For $N = (2,2)$ in $d = 2$ supergravity, ϕ has Lorentz and gauge $U(1)_R$ components

$$\phi_{\mathrm{Lorentz}} = \eta^{ab} F_a N_b \qquad \phi_{gauge} = \frac{1}{2} \epsilon^{ab} F_a N_b$$

where

- ζ is the Dirac superghost;
- $F_1 \equiv \bar{\zeta}\, \zeta \qquad F_2 \equiv \bar{\zeta}\, \Gamma_3\, \zeta$
- $N_a = \star N_a^{(2)}$ are the duals of the graviphoton field strengths;
- η^{ab} is a $O(1,1)$ Lorentzian metric $\eta^{11} = -\eta^{22} = 1$;
- ϵ^{ab} is the Levi Civita tensor in 2 dimensions.

The N=(4,4) Lorentz ϕ ghost bilinears

$\phi_{Lorentz}$ of $N = (4,4)$ in $d = 2$ is abelian as well

$$\phi_{\text{Lorentz}} = \eta^{ab}\, N_a\, F_b$$

where

- ζ^i are 2 Dirac superghosts in the fundamental of $SU(2)_R$;
- $F_1 \equiv \bar{\zeta}_i\, \zeta^i \qquad F_2 \equiv \bar{\zeta}_i\, \Gamma_3\, \zeta^i \qquad F_3 + i\, F_4 \equiv \bar{\zeta}^c_i\, \Gamma_3\, \zeta^i$
- $N_a = \star N_a^{(2)},\ a = 0, 1, 2, 3$ are scalars duals to 2-forms that we will call (in analogy to the $N = 2$ sugra) graviphoton field strengths.
- η^{ab} is a $O(1,3)$ Lorentzian metric (when space-time signature is Euclidean).

The N=(4,4) d=2 gauge ϕ^I ghost bilinear

ϕ_{gauge} of $N=(4,4)$ in $d=2$ supergravity takes values in the $SU(2)_R$ algebra

$$\phi_{gauge} = \phi^I_{gauge}\,\tau^I = (N_0\,\sigma^I + N_1\,\tilde{\sigma}^I + ((N_3 + i\,N_4)\,\hat{\sigma}^I + h.c.))\,\tau^I$$

where the σ's are the non-gauge invariant ghost bilinears

- $\sigma^I \equiv \bar{\zeta}_i\,(\tau^I)^i{}_j\zeta^j$
- $\tilde{\sigma}^I \equiv \bar{\zeta}_i\,(\tau^I)^i{}_j\,\Gamma_3\zeta^j$
- $\hat{\sigma}^I = \bar{\zeta}^c_i\,(\tau^I)^i{}_j\,\Gamma_3\,\zeta^j \qquad \zeta^c_i \equiv C\,\zeta^{*i}$

Invariants of ϕ_{gauge} for $N = (4,4)$ in $d = 2$

Thanks to the Fierz identities one can write the gauge invariants combinations of the gauge ghost bilinears in a manifestly $O(1,3)$ invariant form

$$\mathrm{tr}\,\phi^2_{gauge} = (N_a F^a)^2 - F_a^2 N_a^2$$

which only involves the gauge invariant bilinears F_a.

The BRST transformations of ghost bilinears

- The basic observation is that ghost bilinears γ^μ and ϕ have remarkable and universal BRST transformation properties:

$$
\begin{aligned}
S\,\gamma^\mu &= 0 \\
S\,\phi &= i_\gamma(\lambda)
\end{aligned}
$$

Topological gravity inside supergravity

- The BRST transformation rule of γ^μ is the one of the superghost of topological gravity.
- One finds also

$$
\begin{aligned}
s\,\xi^\mu &= -\frac{1}{2}\,\mathcal{L}_\xi\,\xi^\mu + \gamma^\mu \\
S\,g_{\mu\nu} &= \bar{\zeta}_i\,\Gamma_{(\mu}\,\psi^i_{\nu)} \equiv \psi_{\mu\nu} \equiv \text{Topological gravitino} \\
S\,\psi_{\mu\nu} &= \mathcal{L}_\gamma\,g_{\mu\nu}
\end{aligned}
$$

which are precisely the BRST transformations of topological gravity

Topological YM inside supergravity

- The BRST transformation rules of ϕ are those of the superghost of topological YM coupled to topological gravity

$$S\,\phi = i_\gamma(\lambda)$$
$$S\,\lambda = i_\gamma(F) - D\,\phi$$
$$S\,F = -D\,\lambda$$

- $F^{(2)} = dA + A^2 = R^{(2)} + \mathcal{F}_{gauge}$
- λ is the topological gaugino

$$S\,A = \lambda \equiv \text{Topological gaugino}$$

Topological YM coupled to topological gravity

- Summarizing, there exists a universal subsector of composites of supergravity fields transforming under BRST precisely as the fields of topological YM coupled to topological gravity.
- This topological structure is emergent, and, therefore it is not obvious, yet, what is its fate at quantum level.
- Later I will discuss its relevance to localization, for which supergravity is a classical background.

Topological YM coupled to topological gravity

- Topological YM coupled to topological gravity (C.I. 2010, C.I & D. Rosa 2015) can also be defined as a microscopic theory. In this theory topological gravitinos and gauginos are independent elementary fields, unlike in supergravity.
- This theory computes the De Rham cohomology on the product space $\mathrm{Met}(M) \times \mathcal{A}(M)$ of metrics and connections on a manifold M, equivariant with respect to the action of Diffeos and gauge transformations.

Topological YM coupled to topological gravity

- The coupling of topological gravity to topological Yang-Mills has not been explored yet, as far as I know.
- It should provide a field theoretical way to study the metric dependence of Donaldson invariants, wall-crossing phenomena, quantum topological anomalies etc.

The second topological structure of supergravity

- More topological multiplets emerge whenever gauge invariant scalar bilinears F_a of the commuting ghosts ζ^i — not depending on other bosonic fields — exist.
- For lack of a better name, I will refer to supergravities with this property as “twistable”.

Gauge invariant scalar ghost bilinears

- $d = 2 \; N = 2$

$$F_1 = \bar{\zeta}\,\zeta \qquad F_2 = \bar{\zeta}\,\Gamma_3\,\zeta$$

- $d = 2 \; N = 4$

$$F_1 = \bar{\zeta}_i\,\zeta^i \quad F_2 = \bar{\zeta}_i\,\Gamma_3\,\zeta^i \quad F_3 + i\,F_4 = \bar{\zeta}_i^c\,\epsilon^{ij}\,\zeta_j$$

- $d = 4 \; N = 2$

$$F = \epsilon^{\alpha\beta}\,\epsilon_{ij}\,\zeta^i_\alpha\,\zeta^j_\beta \qquad \overline{F} = \epsilon_{\dot{\alpha}\dot{\beta}}\,\epsilon^{ij}\,\bar{\zeta}^{\dot{\alpha}}_i\,\bar{\zeta}^{\dot{\beta}}_j$$

The BRST transformations of the F_a

- Since,

$$S\,\zeta^i = i_\gamma(\psi^i)$$

one obtains

$$S\,F_a = i_\gamma(\chi_a^{(1)})$$

where $\chi_a^{(1)}$ is a fermionic one-form of ghost number 1

$$\chi_a^{(1)} = \bar{\zeta}\,X_a\,\psi + \bar{\psi}\,X_a\,\zeta$$

and, schematically,

$$F_a = \bar{\zeta}\,X_a\,\zeta$$

The BRST multiplet of gauge invariant ghost bilinears

- BRST descent equations ensue from the supergravity BRST algebra

$$
\begin{aligned}
S\,F_a &= i_\gamma(\chi_a^{(1)}) \\
S\,\chi_a^{(1)} &= -d\,F_a + i_\gamma(N_a^{(2)}) \\
S\,(N_a^{(2)}) &= -d\,\chi_a^{(1)}
\end{aligned}
$$

where $N_a^{(2)}$ is a bosonic two-form of ghost number 0.

Superfields

- In short, when the supergravity is "twistable", topological scalar multiplets exist

$$\mathbb{H}_a = F_a + \chi_a^{(1)} + N_a^{(2)}$$
$$(S + d - i_\gamma)\,\mathbb{H}_a = 0$$

with a which labels the invariant ghost bilinears.

Fierz identities

The invariants ghost bilinears satisfy Fierz identities, involving the composite superghost of topological gravity γ^μ:

- For $N = 2$ $d = 2$

$$F_0^2 - F_1^2 = \gamma^2$$

- For $N = 4$ $d = 2$

$$F_0^2 - F_1^2 - F_2^2 - F_3^2 = \gamma^2$$

Duality symmetry

- Thus in $d = 2$

$$\eta^{ab} F_a F_b = \gamma^2$$

  the F_a's sit in the vector representation of a global duality group which is $SO(1,1)$ for $N = 2, d = 2$ and $SO(1,3)$ for $N = 4, d = 2$.
- Since γ^μ belongs to the topological gravity multiplet, the Fierz identities establish a connection between the curvature topological multiplets and the $\mathbb{H}_a$ topological multiplets.

Application to Localization

- Localization is a long-known property of both supersymmetric (SQFT) and topological (TQFT) theories, by virtue of which semi-classical approximation becomes, in certain cases, exact. [Witten '88, Pestun '07,...]

Localization and supergravity

- Conserved currents of the SQFT that one would like to probe couple to gauge fields which must sit in supergravity multiplets.
- Therefore to identify localizable backgrounds of SQFT one couples supersymmetric matter field theories to classical supergravity: setting the supersymmetry variations of the fermionic supergravity fields — both gravitinos and gauginos — to zero, one obtains equations for the local supersymmetry spinorial parameters.

Generalized Killing Spinor equations

- These differential equations, that are often named generalized Killing spinor (GKS) equations, admit non-trivial solutions only for special configurations of the bosonic fields of the supergravity multiplet.
- The relevant supergravity and the particular GSK equations depend on the global symmetries of the specific SQFT one is interested in.

GKS equations for $N = 2$ $d = 2$

The $N = (2, 2)$ $d = 2$ GKS equations write

$$S\,\psi_\mu = (\partial_\mu + \frac{i}{2}\,\omega_\mu - i\mathcal{A}_\mu)\,\zeta - \frac{i}{2}\,N_1\,\Gamma_\mu\zeta - \frac{i}{2}\,N_2\,\Gamma_\mu\Gamma_3\zeta = 0$$

where

- $\mathcal{A}_\mu$ is the $U(1)_R$ gauge field
- N_1 and N_2 are scalars duals of the graviphoton backgrounds.

GKS equations for $N = 4$ $d = 2$

The $N = (4,4)$ $d = 2$ GKS equations write

$$
\begin{aligned}
& S\,\psi_\mu = \partial_\mu\,\zeta^i + i\,\mathcal{A}^I_\mu\,(\tau^I)^i{}_j\,\zeta^j + \frac{1}{2}\,i\,\omega_\mu\,\Gamma_3\,\zeta^i + \\
& +2\,i\,\big[N_1\,\Gamma_\mu\,\zeta^i + N_0\,\Gamma_3\,\Gamma_\mu\,\zeta^i + \\
& \qquad\qquad -(N_2 + i\,N_3)\,\Gamma_\mu\,\Gamma_3\,\epsilon^{ij}\,\zeta^c_j\big] = 0
\end{aligned}
$$

where:

- $\mathcal{A}^I_\mu$, $I = 1,2,3$ are the $SU(2)_R$ gauge fields;
- N_a, $a = 0,1,2,3$ are scalars backgrounds.

The old topologically twisted solution

- It has been known for a long time that the $N = (2,2)$ $d = 2$ GKS equations admit, for generic space-time topologies, the topologically twisted solution

$$\mathcal{A}_\mu = \frac{1}{2}\,\omega_\mu \qquad N_1 = N_2 = 0$$

Modern genus zero solutions

- More solutions of the $d = 2$ GKS eqs were found for spheric world-sheet topology [Benini& Cremonesi: '12, Doroud et al.: '12, Closset&Cremonesi: '14, Closset, Cremonesi, Park: '15].
- GKS equations have also been studied in higher dimensions and a host of new solutions have been found as well [Hama et al. '12, Klare et al. '12...]

GKS equations and topological structures

- There is no general strategy to construct solutions of GKS equations.
- One application of the topological structures of supergravity that we illustrated earlier is to provide a systematic way to find and classify solutions of GKS equations.
- I am going to describe how to obtain the general solutions $d = 2\ N = (4, 4)$ GKS equations (which include as a specific case the $d = 2\ N = (2, 2)$ GKS equations).

Supersymmetric supergravity backgrounds

- Supersymmetric bosonic backgrounds are obtained by setting to zero the supergravity BRST variations of all the fermionic supergravity fields. We will refer to the set of such backgrounds as the localization locus.

The topological localization equations

- On the localization locus also the BRST variation of the composite topological fermions must vanish as well

$$
\begin{aligned}
S\,\psi_{\mu\nu} = 0 &\Leftrightarrow \text{The topological gravitino eq.}\\
S\,\lambda = 0 &\Leftrightarrow \text{The topological gaugino eq.}\\
S\,\chi_a = 0 &\Leftrightarrow \text{The topological scalar eq.}
\end{aligned}
$$

The topological gravitino equations

- The first equation

$$S\,\psi_{\mu\nu} = D_\mu\,\gamma_\nu + D_\nu\,\gamma_\mu = 0$$

  states that the vector bilinear γ^μ is an isometry of the space-time metric $g_{\mu\nu}$
- This is a well-known result which was obtained quite early in the GKS literature.

The topological gaugino equations

- The topological gaugino equation $S\,\lambda = 0$

$$D\,\phi - i_\gamma(F) = 0$$

appears to be a novel equation which has not been yet explored in either supergravity or topological field theory literature.

The topological gaugino equations

- In the context of supergravity, the topological gaugino equation splits into equations valued in the Lorentz local algebra and in the R-symmetry YM symmetry algebra.

$$D\,\phi_{Lorentz} - i_\gamma(R^{(2)}) = 0$$
$$D\,\phi_{gauge} - i_\gamma(\mathcal{F}^{(2)}_{gauge}) = 0$$

- When either one of these algebras is non-abelian, these equations are non-linear.

The equivariant Chern classes

- To extract the gauge invariant content of the topological gaugino equation, define the generalized field strength

$$\mathbb{F} = F^{(2)} + \phi$$

and the generalized Chern classes

$$\mathrm{Tr}\,\mathbb{F}^n = \mathrm{Tr}(F + \phi)^n = \mathrm{Tr}\,F^n + n\,\mathrm{Tr}\,F^{n-1}\,\phi + \cdots + \mathrm{Tr}\,\phi^n$$

which are gauge invariant polyforms.

De Rham γ-equivariant cohomology

- The topological gaugino equation implies that the generalized Chern classes $\mathrm{Tr}\,\mathbb{F}^n$ are equivariant extensions of the ordinary Chern classes:

$$(d - i_\gamma)\,\mathrm{Tr}\,\mathbb{F}^n = 0$$

- The differential

$$\mathcal{D}_\gamma \equiv d - i_\gamma \qquad \mathcal{D}_\gamma^2 = -\mathcal{L}_\gamma$$

is the coboundary operator defining the de Rham cohomology of polyforms on space-time, equivariant with respect to the action associated to the Killing vector γ^μ.

Integral invariants of the localization locus

- The ordinary Chern classes are integer-valued.
- In the examples we computed so far also their γ-equivariant extensions $\mathrm{Tr}\,\mathbb{F}^n$ are also integer.
- Different values of the γ-equivariant classes label different branches of the localization locus. On each of these branches moduli spaces of inequivalent localizing backgrounds may exist— all with the same (integral) values of the γ-equivariant Chern classes.

- It should be stressed that the topological gravitino and gaugino equations do not, in general, completely characterize the localization locus.
- Additional, independent, equations are obtained by setting to zero the variations of other, independent, gauge invariant composite fermions.
- When the ghost bilinears F_a exists one gets precisely these extra topological equations.

The scalar topological equations

- For each F_a, the $S\,\chi_a^{(1)} = 0$ equation leads to the topological scalar equation

$$
\begin{aligned}
& d\,F_a - i_\gamma(N_a^{(2)}) = 0 \\
& \Rightarrow \mathcal{D}_\gamma\,\mathbb{H}_a = 0 \qquad \text{with } \mathbb{H}_a \equiv F_a + N_a^{(2)} = 0
\end{aligned}
$$

  showing that the polyforms $\mathbb{H}_a$ are γ-equivariantly closed.
- For $d = 2$, $N = 2, 4$ supergravities, the 2-forms $N_a^{(2)}$ are the graviphoton field strengths.

Solving the scalar topological equations in d=2

- The scalar topological equations determine the backgrounds $N_a^{(2)}$ given the ghost bilinears F_a:

$$N_a = \frac{\epsilon^{\mu\nu}}{\sqrt{g}\,\gamma^2}\,\gamma_\mu\,\partial_\nu\,F_a$$

- The ghost bilinears are not independent, but must satisfy Fierz identity

$$\eta^{ab}\,F_a\,F_b = \gamma^2$$

The supersymmetric backgrounds

- The F_a do not obey any more constraints: given the metric and its associated Killing structure γ^μ, and F_a's satisfying

$$\eta^{ab} F_a F_b = \gamma^2,$$

  i.e. 3 independent γ-invariant functions, one obtains, up to gauge transformations, a covariantly constant spinor $\zeta^i(F_a)$ and the corresponding graviphoton and gauge backgrounds $N_a^{(2)}(F_a), A_\mu^I(F_a)$.

The gauge backgrounds

Just to give an idea of how the general solution (in a given gauge) looks

$$
\begin{aligned}
A^1_\theta = & -\frac{16\left(F_0^2 F_2(N_0+N_1)\right)}{128(F_0-F_1)\left(F_2^2+F_3^2\right)} + \\
& \frac{\left(-2F_0(F_1F_2(N_0+N_1)+F_3(F_2n_3-F_3n_2))\right)}{128(F_0-F_1)\left(F_2^2+F_3^2\right)} + \\
& +\frac{16\ \left(F_1^2F_2(N_0+N_1)+2F_1F_3(F_2N_3-F_3N_2)\right)}{128(F_0-F_1)\left(F_2^2+F_3^2\right)} + \\
& +\frac{\left(-F_2\left(F_2^2+F_3^2\right)(N_0-N_1)\right)}{128(F_0-F_1)\left(F_2^2+F_3^2\right)} + \\
& -\frac{16F_2(F_0-F_1)\cos(\theta)+F_2\sin^2(\theta)(N_0-N_1)}{128(F_0-F_1)\left(F_2^2+F_3^2\right)}
\end{aligned}
$$

etc. etc.

Cohomological invariance

- Given a solution $\mathbb{H}_a$ of the scalar topological equations, cohomologically equivalent solutions are associated to every $\mathcal{L}_\gamma$ invariant 1-form $\omega_a^{(1)}$

$$
\begin{aligned}
& \mathbb{H}'_a = \mathbb{H}_a + \mathcal{D}_\gamma\, \omega_a^{(1)} \Leftrightarrow \\
\Leftrightarrow \qquad & F'_a = F_a + i_\gamma(\omega_a^{(1)}) \quad N_a^{(2)\,\prime} = N_a^{(2)} + d\, \omega_a^{(1)} \\
& \mathcal{L}_\gamma\, \omega_a^{(1)} = 0
\end{aligned}
$$

- Since the $\mathcal{D}_\gamma$ cohomological symmetry is inherited by the original supergravity BRST symmetry, it is natural to conjecture that localizing backgrounds corresponding to cohomologically equivalent solutions give rise to the same partition function.

Finite dimensional moduli space?

- If this is true, then the moduli space of supersymmetric backgrounds of $d = 2 \quad N = (4, 4)$ is parametrized by the three independent $\mathcal{D}_\gamma$-cohomology classes.
- For the round metric, the representatives can be chosen to be

$$F_a = A_a - N_a \cos\theta \qquad a = 1, 2, 3$$

$$F_0^2 = \gamma^2 + \sum_{a=1}^{3} F_a^2$$

where A_a and N_a are constants.

The equivariant Fierz identity

- The Fierz identity $\eta^{ab} F_a F_b = \gamma^2$ extends to the identity of equivariantly closed polyforms

$$\eta^{ab} \mathbb{H}_a \mathbb{H}_b = \gamma^2 + \star\phi_{Lorentz} \qquad \phi_{Lorentz} = \sqrt{g}\, \epsilon_{\mu\nu} D^\mu \gamma^\nu$$

- This equation allows to express the curvature polyform $\mathbb{R}$ in terms of the scalar $\mathbb{H}_a$ ones.
- Other Fierz identities express the gauge polyform $\mathbb{F}^I$ also in terms of the scalar $\mathbb{H}_a$ ones.

The dual of γ-equivariant polyforms

- To obtain these relations one introduces a derivation L which maps equivariantly closed polyforms to equivariantly closed polyforms:

$$L(\mathbb{H}_a) \equiv N_a + \star\Delta_\gamma F_a$$

where

$$\Delta_\gamma F_a \equiv d^\dagger \frac{1}{\gamma^2} \star d F_a$$

$$\mathcal{D}_\gamma L(\mathbb{H}_a) = 0$$

Curvature backgrounds in terms of the scalar closed polyforms

- $N = 2$ $d = 2$

$$\mathbb{R} = \eta^{ab}\,\mathbb{H}_a\,L(\mathbb{H}_b)$$
$$\mathbb{F} = \epsilon^{ab}\,\mathbb{H}_a\,L\,(\mathbb{H}_b)$$

- $N = 4$ $d = 2$

$$\mathbb{R} = \eta^{ab}\,\mathbb{H}_a\,L(\mathbb{H}_b)$$
$$\mathbb{D}^{ab} \equiv \epsilon^{abcd}\,\mathbb{H}_c\,L\,(\mathbb{H}_d)$$
$$\mathrm{Tr}\,\mathbb{F}^2 = \mathbb{D}^{ab}\,\mathbb{D}_{ab}$$

The integrability of GKS equations

- These relations between the curvature polyforms and the $\mathbb{H}_a$ polyforms are the topological counterpart of the integrability equations of the GKS equations.
- These relations are manifestly invariant under the $SO(1,3)$ global duality symmetry, which shows that this duality group acts on the space of the supersymmetric backgrounds.

A host of new localizing backgrounds

- In $N = 4$ $d = 2$ one obtains a huge amount of new localizable backgrounds, which include all the previously known $N = 2$ $d = 2$ backgrounds and many both with more and with less supersymmetry.
- It would be interesting to compute matter partition function as functions of the data $\{g_{\mu\nu}, \gamma^{\mu}, F_a\}$ which determine the supersymmetric backgrounds, to verify, among other things, its cohomological properties.

Conclusions

- Supergravity contains an emergent composite universal topological subsector, described by topological gravity coupled to topological YM.
- Certain “twistable” extended supergravities contain a second topological structure which consists of scalar topological multiplets $\mathbb{H}_a = F_a + \chi_a^{(1)} + N_a^{(2)}$ coupled to topological gravity.
- The two structures are related, “on shell”, by certain topological “integrability” conditions that we worked out explicitly in $d = 2$ and $N = 2$ and $N = 4$

Open problems and outlook

- Explore the new host of localizing backgrounds of $N = (4, 4)$ in $d = 2$ that we found.
- Find the relation between the two structures for $N = 2$ $d = 4$ supergravity. This might lead to the solution of the long standing problem of the classification of localizing backgrounds for this theory.
- Find the fate of the topological emergent structures of supergravity at quantum level.

An effective topological sigma model on the space of supersymmetric vacua?

- We have seen that the classical supersymmetric vacua of $d = 2$ $N = (4, 4)$ can be parametrized by "on shell" topological multiplets $\mathbb{H}_a$ and topological gravity backgrounds $\{g_{\mu\nu}, \gamma^\mu\}$, with

$$\eta^{ab}\,\mathbb{H}_a\,\mathbb{H}_b = \gamma^2 + \star\phi_{Lorentz} \qquad \mathcal{D}_\gamma \mathbb{H}_a = 0$$

- Could one construct a non-linear topological sigma models with coordinates $\mathbb{H}_a$ coupled to topological gravity which describes, in an effective way, the quantum fluctuations of supergravity?