D0-D8 BRANE PARTITION FUNCTIONS

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Trieste July 13^{th} 2018

Based on upcoming work with N. Nekrasov

OR:

MAGNIFICENT
FOUR
WITH COLOR

Main Goal: present a conjecture that the twisted partition function

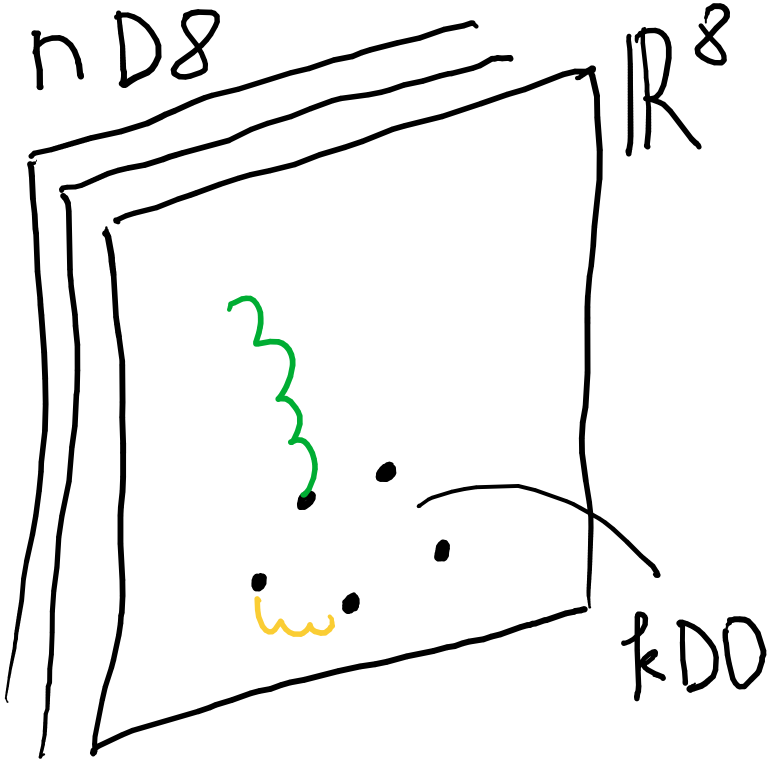
$$Z_k^{(n\,D8)}(g) = \mathrm{tr}_{\mathcal{H}_k} (-1)^F g\, e^{-\beta H}$$

for a system of k slowly moving D0-branes in the background of n D8-branes along $\mathbb{R}_t \times \mathbb{R}^8$ with constant B-field turned on

admits the free-field representation

$$\sum_{k=0}^{\infty} p^k Z_k^{(n\,D8)}(g) = \text{a simple function } F_n.$$

Plethystic Exp of



ADHM quantum mechanics

D0-D8 system is supersymmetric both with and without B-field. We take

$$B = \sum_{a=1}^{4} b_a \, dx_{2a-1} \wedge dx_{2a} \quad \text{along } \mathbb{R}^8.$$

Theory on worldvolume of D0 $\left(\underset{\text{Euclidean}}{\mathbb{R}_t \to S^1_\beta}\right)$ is gauged supersymmetric quantum mechanics with gauge group $U(k)$ and $\mathcal{N}=2$ susy.

Its field content comes from massless D0-D0 and D0-D8 strings.

Denote its global symmetry group by $G_F \ni g$.

SYMMETRIES

To preserve some susy, take:

D8 Chan-Paton bundle N

not manifest in geometry

Speculative interpretation:
$\overline{D8}$ Chan-Paton bundle M

$$G_F = SU(4) \times U(n)_N \times U(n)_M$$

with complexified Cartan torus parametrized by

$$\boxed{\prod_{a=1}^{4} q_a = 1} \leftarrow \{q_a\}_{a=1}^{4}, \{n_\alpha\}_{\alpha=1}^{n}, \{m_\alpha\}_{\alpha=1}^{n}$$

Remark: $SO(1,9) \to SO(8) \to SU(4)$

$$8_s \to 4 + \bar{4}$$

$$8_c \to 6 + 1 + 1$$

FIELD CONTENT

$\mathcal{N}=2$ susy QM multiplet	stringy origin	Fields	$U(k)\times G_F$ reps	Vector Spaces	anomalous global $U(1)$
Vector	D0-D0	A_t, Φ_q fermions η, χ	Adj_k	$\mathrm{End}(K)$	± 2 bosons -1 fermions
Chiral	D0-D0	B_a, $B_a^\dagger$ (& fermions)	$(\mathrm{Adj}_k, \underline{4})$	$\mathrm{End}(K)$	0 bosons
	D0-D8 NS R	$I, I^\dagger$ (fermions)	$k\times\bar{n}$ bifundamental of $U(k)\times U(n)_N$	$\mathrm{Hom}(N,K)$	$+1$ fermions
Fermi	D0-D0	χ_{ab}	$(\mathrm{Adj}_k, \underline{6})$	$\Pi\,\mathrm{End}(K)$	-1
	D0-$\overline{\text{D8}}$ R	Υ	$k\times\bar{n}$ bifund of $U(k)\times U(n)_M$	$\Pi\,\mathrm{Hom}(M,K)$	-1

★ Scalar potential on Higgs branch:

$$V = \sum_{1\le a<b\le 4} \operatorname{tr} s_{ab} s_{ab}^\dagger + \operatorname{tr}(\mu_{\mathbb{R}} - \zeta)^2$$

FY term $\zeta > 0$

$$s_{ab} = [B_a, B_b] + \frac{1}{2}\varepsilon_{abcd}[B_c, B_d]^\dagger = \frac{1}{2}\varepsilon_{abcd}\, s_{cd}^\dagger$$

$$\mu_{\mathbb{R}} = \sum_{a=1}^{4}[B_a, B_a^\dagger] + II^\dagger.$$

SU(4) structure

★ on $\mathcal{M} = V^{-1}(0)/U(k)$ we have

$$[B_a, B_b] = 0, \qquad \operatorname{tr} II^\dagger = k\zeta > 0$$

and stability condition holds:

$\forall v \in K \ \exists F \in \mathbb{C}[z_1,\dots,z_4],\ w \in N$ such that

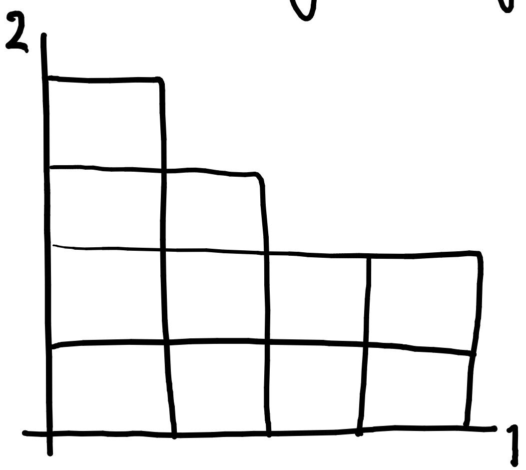
$$F(B_1,\dots,B_4)\, I\, w = v$$

⇝ colored solid partitions

SOLID PARTITIONS

We can think recursively:

2d Young diagrams (4d gauge theory)

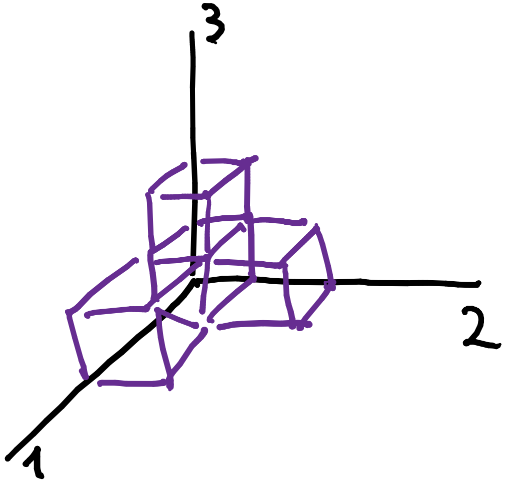


$$Y = (n_1, \dots, n_\ell) \qquad n_i \in \mathbb{Z}_{\geq 0},\ n_i \geq n_{i+1}$$

size $\quad |Y| = \sum_{i=1}^{\ell} n_i$

$$Y_1 \subseteq Y_2 \quad \text{iff} \quad n_i^{(1)} \leq n_i^{(2)} \ \forall i.$$

3d plane partitions (6d gauge theory)



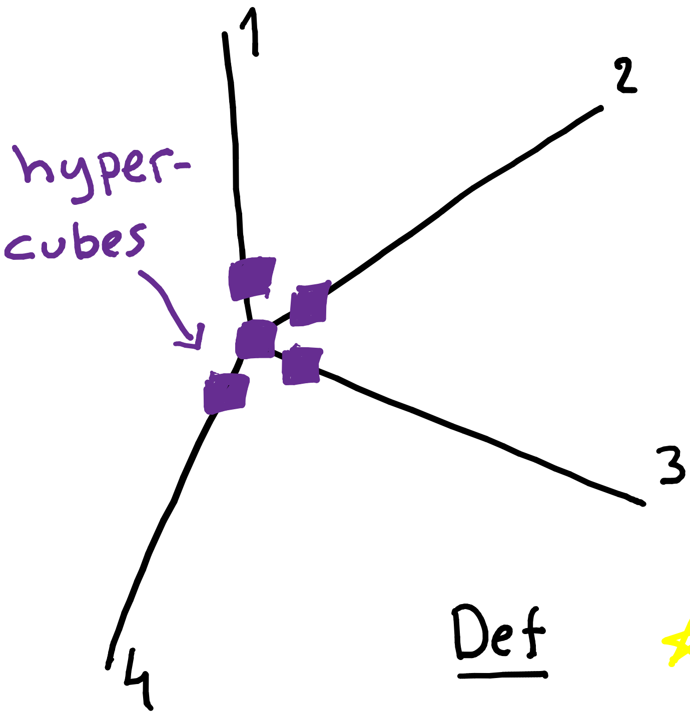
$$\rho = (Y_1, \dots, Y_\ell) \quad \text{s.t.} \quad Y_{i+1} \subseteq Y_i$$

Young diagrams

$$|\rho| = \sum_{i=1}^{\ell} |Y_i|$$

$$\rho_1 \subseteq \rho_2 \quad \text{iff} \quad Y_i^{(1)} \subseteq Y_i^{(2)} \; \forall i$$

4d solid partitions (8d gauge theory)



$$\pi = (\rho_1, \dots, \rho_\ell) \quad \text{s.t. } \rho_{i+1} \le \rho_i$$

plane partitions

$$|\pi| = \sum_{i=1}^{\ell} |\rho_i|$$

<u>Def</u> ☆ colored solid partition of rank n

$$\vec{\pi} = (\pi_1, \dots, \pi_n) \qquad \pi_i : \text{solid partitions}$$

☆ character $K_{\vec{\pi}}(\{q_a\},\{n_\alpha\}) = \sum_{\alpha=1}^{n} n_\alpha \sum_{(A,B,C,D)\in\pi_\alpha} q_1^{A-1} q_2^{B-1} q_3^{C-1} q_4^{D-1}$

<u>Obs</u> generating function known in 2d, 3d

in 4d only known up to size ~ 72

Index computation

$$Z_k^{(nD8)}(g) = \int [d\phi]\, e^{-S_\Omega[\phi]}$$

weakly gauge G_F and apply localization

$\rightsquigarrow$ rational function from one-loop determinants

PB $\quad q_{ab} \equiv q_a q_b$

$$\boxed{\chi_k = \prod_{i \neq j}^{k} \frac{(x_j - x_i) \prod_{a<b}^{3} (x_j - q_{ab} x_i)}{\prod_{a=1}^{4} (x_j - q_a x_i)} \prod_{i=1}^{k} \prod_{\alpha=1}^{n} \frac{m_\alpha - x_i}{n_\alpha - x_i}}$$

$x_A = \exp \sqrt{-1}\, u_A$

$$Z_k^{(nD8)}(g) = \frac{1}{k!} \sum_{u_* \in \mathfrak{M}_{sing}} \text{JK-Res}_{u_*, \zeta}\, \chi_k\, du$$

χ_k defines at u_* a hyperplane arrangement and the residue depends on FY ζ through its chamber

standard iterated Residue

$$= \sum_{\substack{\text{colored solid} \\ \text{partitions } \vec{\pi} \text{ of size } |\vec{\pi}| = k}} \underset{F(\vec{\pi})}{\text{Res}}\, \chi_k\, du$$

Flag F is given by the ordered content of $\vec{\pi}$ and we use Weyl invariance of χ_k to get rid of $|W| = k!$

Conjecture: Z has a free-field representation

<u>Def</u> Plethystic exponent of $f(x_1,\dots,x_r)$ is

$$\mathrm{PE}\, f(x_1,\dots,x_r) = \exp \sum_{m=1}^{\infty} \frac{1}{m} f(x_1^m,\dots,x_r^m).$$

$$\sum_{p=0}^{\infty} p^k Z_k^{(nD8)}(\{q_a\},\{n_\alpha\},\{m_\alpha\}) = \mathrm{PE}\; F_n\Big(q_1,\dots,q_4, \prod_{\alpha=1}^{n} \frac{n_\alpha}{m_\alpha}, -p\Big)$$

where $[t] = t^{1/2} - t^{-1/2}$ and

$$F_n(q_1,\dots,q_4, S_{eff}, p) = \frac{[q_{12}][q_{13}][q_{23}]}{[q_1][q_2][q_3][q_4]} \frac{[S_{eff}]}{[p\sqrt{S_{eff}}][p/\sqrt{S_{eff}}]}$$

Mathematical Properties

maximal torus of G_F

Tangent space to the moduli space at a $\mathbb{T}$-fixed point

= colored solid partition

$$T = (N-M)^* K - \prod_{a=1}^{3}(1-q_a) K K^*$$

seen as a virtual character $\left(N = \sum_{\alpha=1}^{n} n_\alpha , \; M = \sum_{\alpha=1}^{n} m_\alpha \right)$

($*$ means replace q_a by q_a^{-1}, $n_\alpha \to n_\alpha^{-1}$, $m_\alpha \to m_\alpha^{-1}$)

Fact 1 for generic $\{m_\alpha\}, \{n_\alpha\}$, T is movable, namely it does not contain ± 1 factors in the sum.

Fact 2 equivalence of tangent space picture and residue and sign rule

$$\text{Iterated-Res } \chi_k = (-1)^{h(\vec{\pi})} \, \hat{a}(T_{\vec{\pi}})$$

colored solid partition $\vec{\pi}$

a certain well-defined sign that has to do with orientations

★ $\hat{a}$ - map acts on monomials r, s as

$\hat{a}(r) = 1/[r]$ and s.t. $\hat{a}(r+s) = \hat{a}(r)\hat{a}(s)$.

It converts T to a product of weights in equivariant K-theory.

Thank You