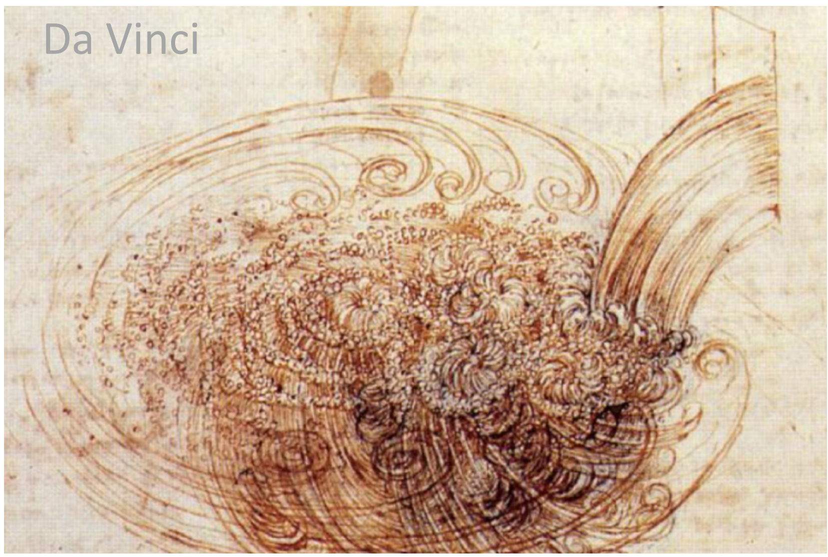
Simulating Plasma Turbulence

D. R. Hatch

ICTP Oct 2018

Turbulence

Turbulence: Enduring Fascination and Challenge

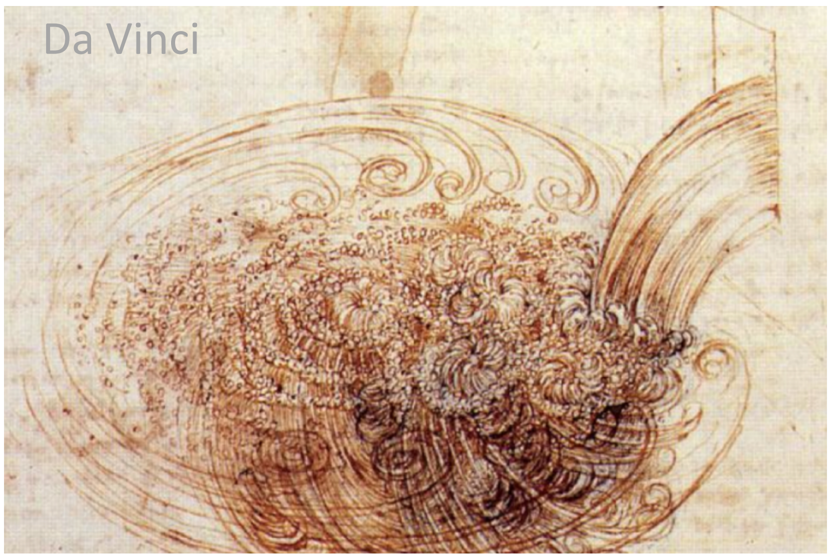


`Turbulence is the most important unsolved problem of classical physics.'

- Richard Feynman -

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \frac{1}{\mathbf{Re}}\nabla^2 \mathbf{u} = \mathbf{g}$$

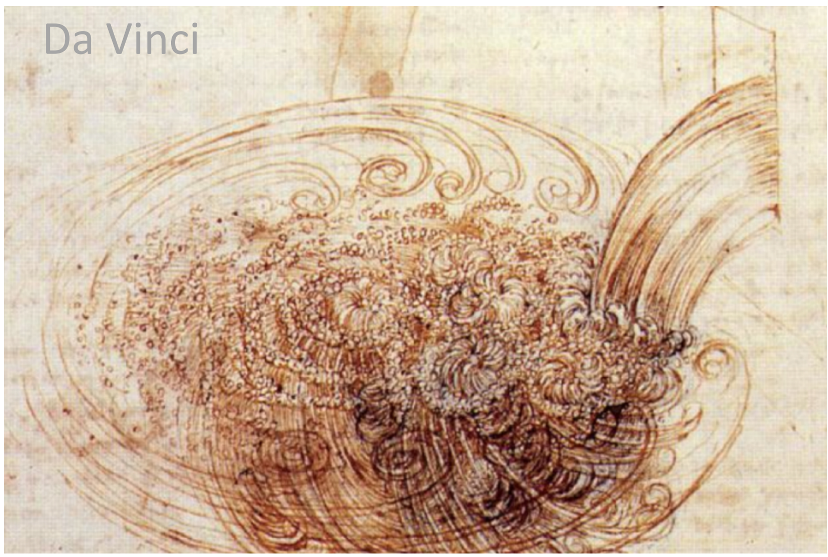
Turbulence: Enduring Fascination and Challenge



'When I die and go to Heaven there are two matters on which I hope for enlightenment. One is quantum electrodynamics and the other is the turbulent motion of fluids. And about the former I am really rather optimistic.' **-Horace Lamb-**

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \frac{1}{\mathbf{Re}}\nabla^2 \mathbf{u} = \mathbf{g}$$

Turbulence: Enduring Fascination and Challenge



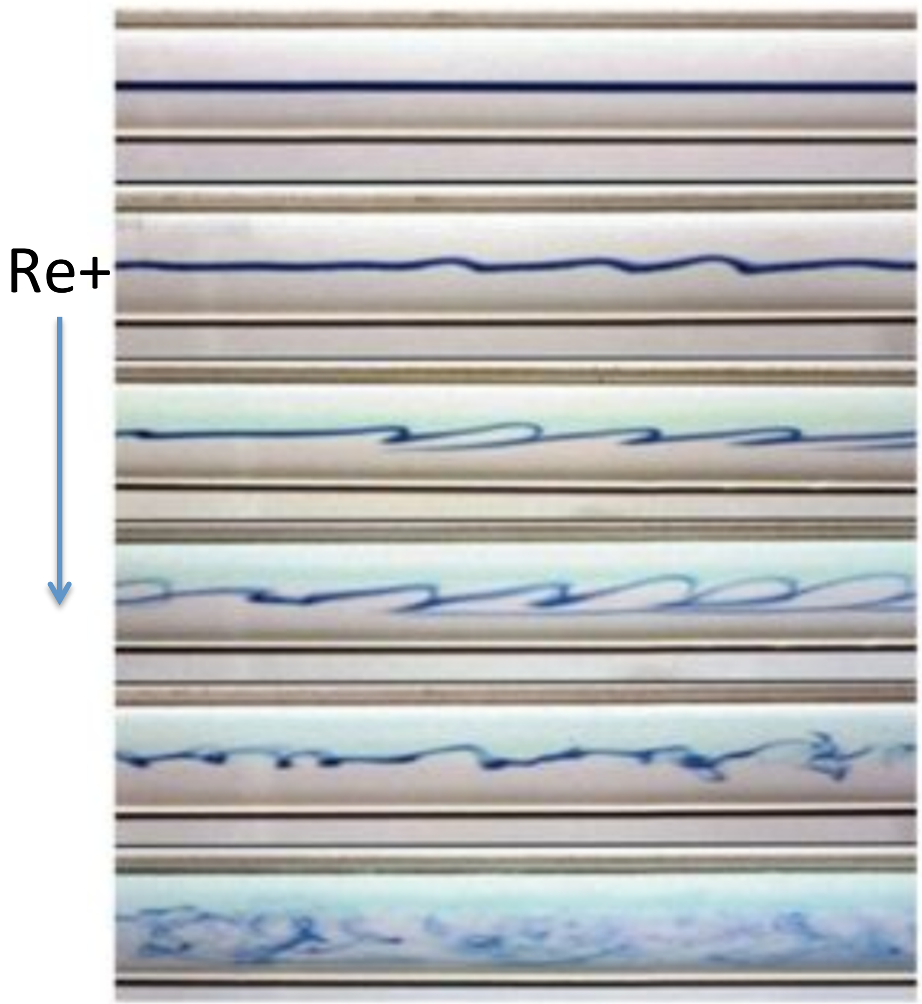
Millenium Prize: Existence and Smoothness of Navier Stokes

'This is the equation which governs the flow of fluids such as water and air. However, there is no proof for the most basic questions one can ask: do solutions exist, and are they unique? Why ask for a proof? Because a proof gives not only certitude, but also understanding.'

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \frac{1}{\mathbf{Re}}\nabla^2 \mathbf{u} = \mathbf{g}$$

Turbulence: Why is it Important?

Dye in Pipe Flow



How long can rolling waters remain impure?

- Turbulence makes things happen:
  - Incredibly effective at mixing / transporting particles, momentum, heat, etc.
  - Much more effective than laminar flow or molecular diffusion (typically factor of ~Re faster—i.e. 10^4-10^7!)

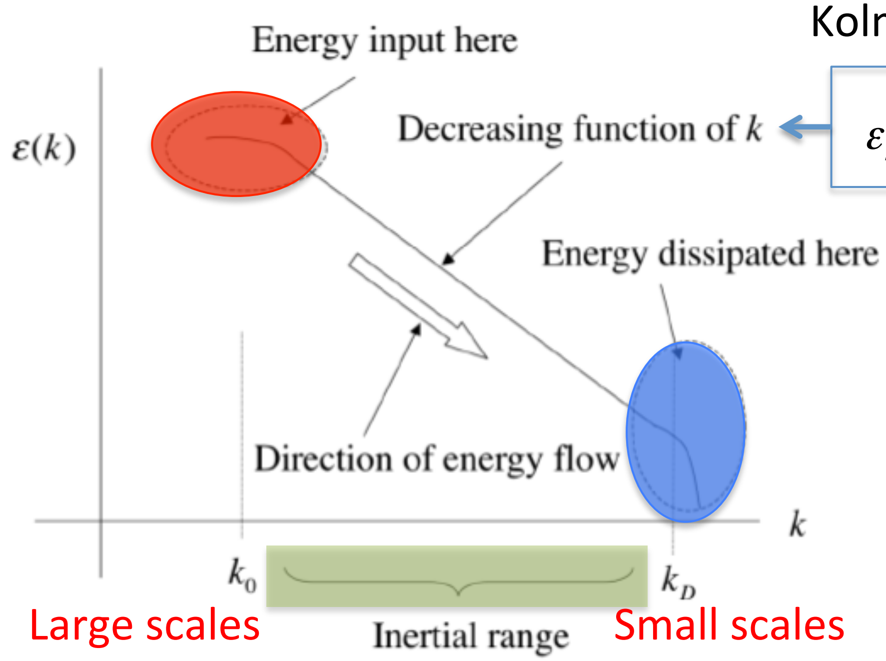
$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \frac{\mathbf{1}}{\mathbf{Re}}\nabla^{\mathbf{2}}\mathbf{u} = \mathbf{g}$$

$$Re = \frac{UL}{\nu}$$

Fundamental Turbulence Paradigm

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \frac{1}{\mathbf{Re}} \nabla^2 \mathbf{u} = \mathbf{g}$$

Turbulence:
Energy
1. Injection
2. Redistribution
3. Dissipation



Kolmogorov

$$\varepsilon_k \propto k^{-5/3}$$

Moments of Distribution Function

$$f_\sigma(\mathbf{x}, \mathbf{v}, t)$$

$$n_\sigma(\mathbf{x}, t) = \int f_\sigma(\mathbf{x}, \mathbf{v}, t) d\mathbf{v} \quad \longrightarrow \quad \rho = \sum_\sigma q_\sigma n_\sigma$$

$$\mathbf{u}_\sigma(\mathbf{x}, t) = \frac{1}{n_\sigma} \int \mathbf{v} f_\sigma(\mathbf{x}, \mathbf{v}, t) d\mathbf{v} \quad \longrightarrow \quad J = \sum_\sigma q_\sigma n_\sigma u_\sigma$$

$$P_\sigma(\mathbf{x}, t) = \frac{m_\sigma}{3} \int v^2 f_\sigma(\mathbf{x}, \mathbf{v}, t) d\mathbf{v}$$

.

.

How To Solve for Distribution Function

$$\frac{\partial f_\sigma}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{q_\sigma}{m_\sigma}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f_\sigma = C_{\sigma\alpha}(f_\sigma)$$

+

Maxwell's Equations

Kinetic Theory = Infinite Hierarchy of Moment Equations

$$\boxed{\frac{\partial f_\sigma}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{q_\sigma}{m_\sigma}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f_\sigma = C_{\sigma\alpha}(f_\sigma)}$$

$$\frac{\partial n_\sigma}{\partial t} + \nabla \cdot (n_\sigma \boxed{\mathbf{u}_\sigma}) = 0$$

Kinetic Theory = Infinite Hierarchy of Moment Equations

$$\frac{\partial f_\sigma}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{q_\sigma}{m_\sigma}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f_\sigma = C_{\sigma\alpha}(f_\sigma)$$

$$\frac{\partial n_\sigma}{\partial t} + \nabla \cdot (n_\sigma \mathbf{u}_\sigma) = 0$$

$$\frac{\partial \mathbf{u}_\sigma}{\partial t} + (\mathbf{u}_\sigma \cdot \nabla)\mathbf{u}_\sigma = \frac{q_\sigma}{m_\sigma}(\mathbf{E} + \mathbf{u}_\sigma \times \mathbf{B}) - \frac{1}{n_\sigma m_\sigma}\nabla P_\sigma - \frac{1}{n_\sigma m_\sigma}\mathbf{R}_{\sigma\alpha}$$

Kinetic Theory = Infinite Hierarchy of Moment Equations

$$\frac{\partial f_\sigma}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{q_\sigma}{m_\sigma}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f_\sigma = C_{\sigma\alpha}(f_\sigma)$$

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$$\frac{N}{2}\frac{dP_\sigma}{dt} + \frac{2+N}{2}P\nabla \cdot \mathbf{u}_\sigma = -\nabla \cdot \mathbf{Q}_\sigma + \mathbf{R}_{\sigma\alpha} \cdot \mathbf{u}_\sigma - \left(\frac{\partial W}{\partial t}\right)_{E\sigma\alpha}$$

$$\frac{\partial \mathbf{Q}_\sigma}{\partial t} = \cdots$$

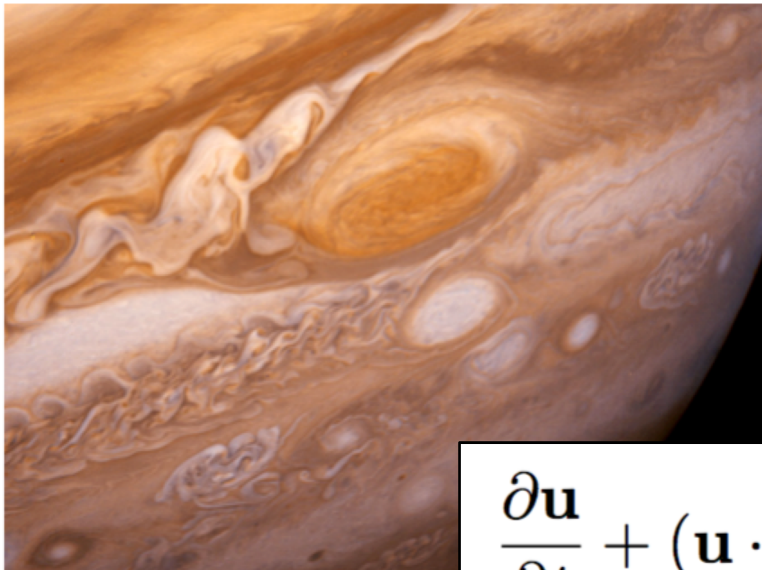
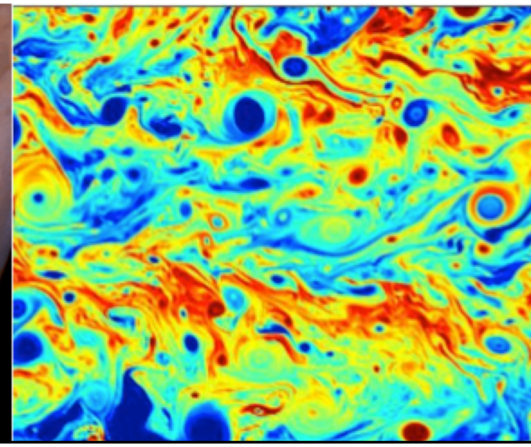
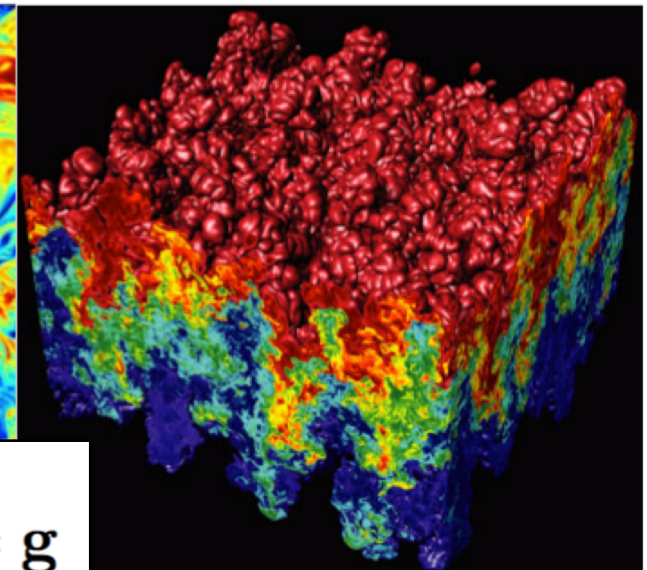
Navier Stokes = Limiting Case of Plasma Fluid Equations

$$\frac{\partial \mathbf{u}_\sigma}{\partial t} + (\mathbf{u}_\sigma \cdot \nabla)\mathbf{u}_\sigma = \frac{q_\sigma}{m_\sigma}(\mathbf{E} + \mathbf{u}_\sigma \times \mathbf{B}) - \frac{1}{n_\sigma m_\sigma}\nabla P_\sigma - \frac{1}{n_\sigma m_\sigma}\mathbf{R}_{\sigma\alpha}$$

q ➔ 0, incompressibility, etc.

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu \nabla^2 \mathbf{u} = \mathbf{g}$$

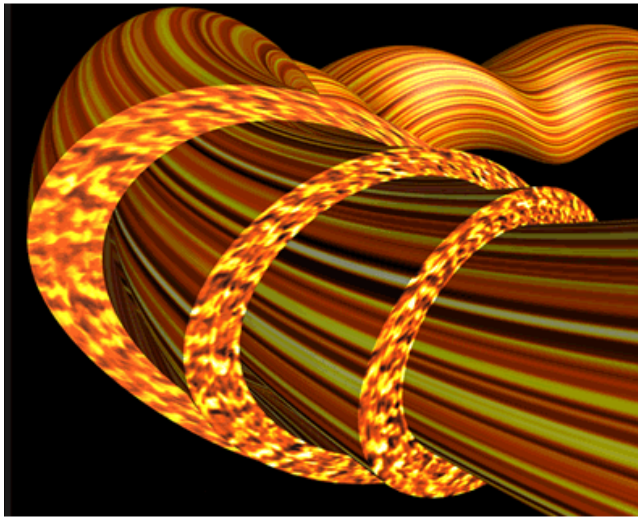
Fluid Dynamics Described by Single Fluid Equation

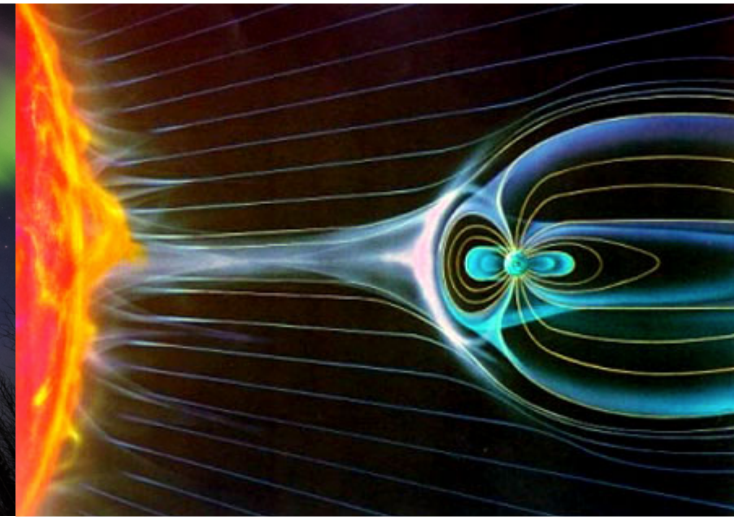




$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \frac{1}{\mathbf{Re}}\nabla^2 \mathbf{u} = \mathbf{g}$$

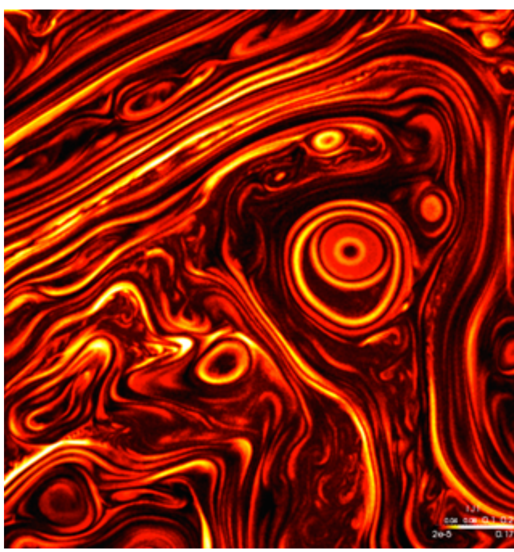
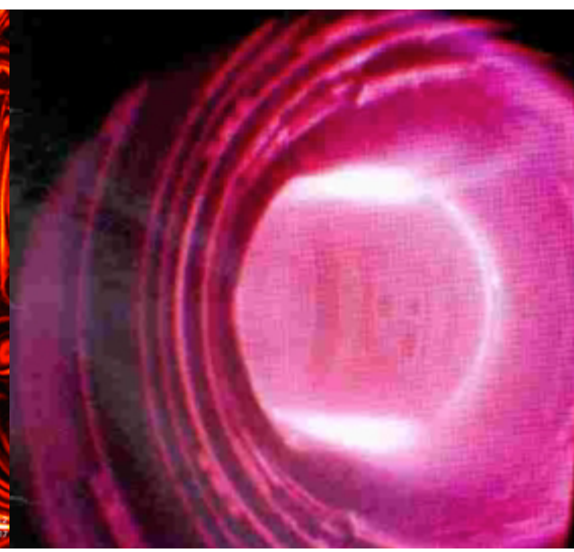


Plasma: Challenge and Opportunity





$$\frac{\partial f_\sigma}{\partial t} + \mathbf{v}\cdot\nabla f + \frac{q_\sigma}{m_\sigma}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\nabla_{\mathbf{v}} f_\sigma = C_{\sigma\alpha}(f_\sigma)$$


Many Non-Fusion Applications

- Foundation for space and astrophysical turbulence
  - G. G. Howes et al. *ApJ*, (2006).
  - A. A. Schekochihin et al. *ApJS*, (2009).
- Solar wind turbulence
  - G. G. Howes, et al. *Phys. Rev. Lett.*, (2011).
  - J. M. TenBarge et al. *Physics of Plasmas*, (2012).
  - D. Told et al. *Phys. Rev. Lett.* (2015).
- Magnetic reconnection
  - J. M. TenBarge, et al. *Physics of Plasmas (*2014).
  - M. J. Pueschel,, et al. *ApJS*, (2014).
- Fundamental turbulence
  - Tatsuno *et al. Phys. Rev. Lett. (2009)*
  - Banon-Navarro et al. *Phys. Rev. Lett. (2011)*
  - Teaca *et al. Phys. Rev. Lett. (2012)*
  - Hatch *et al. Phys. Rev. Lett. (2011,2013)*
- Codes
  - AstroGK (based on fusion code GS2)
  - GENE
  - Others

Compare / Contrast with Fluid Turbulence

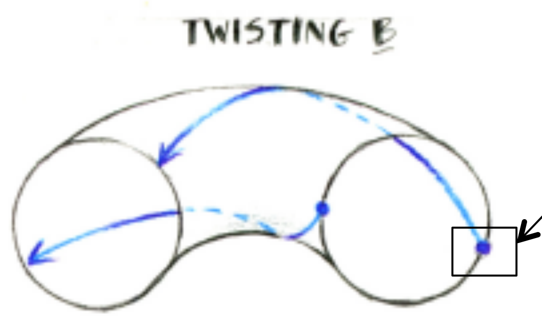
What Drives Turbulence in a Tokamak?

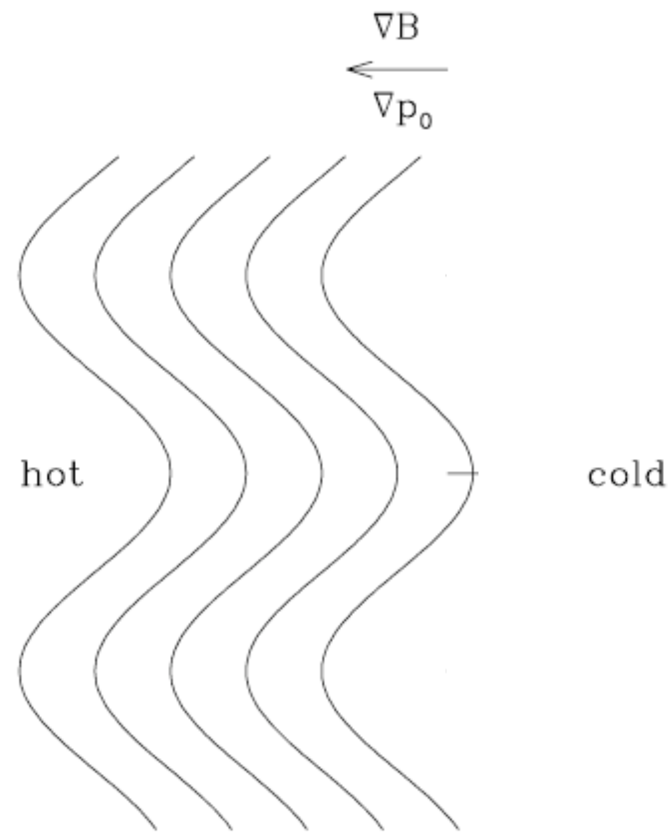
- Several drive mechanisms for fluid turbulence
  - Kelvin-Helmholtz (shear flow)
  - Rayleigh Taylor (density gradient)
- There exist a whole zoo of instabilities (ITG, ETG, TEM, MTM, KBM, etc.), each with both instability and wave-like properties
  - (Jonathan Citrin will discuss further in later lecture)
- Driven by extreme gradients in fusion plasmas (usually gradients in temperature, density)
- I will briefly introduce the ion temperature gradient (ITG) instability
  - Perhaps the most important instability for tokamak transport

Ion Temperature Gradient Instability

Ion Temperature Gradient Instability (ITG)

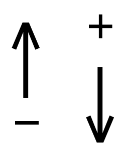
Outboard Side of Torus





Contours of total $p=p_0+\tilde{p}$

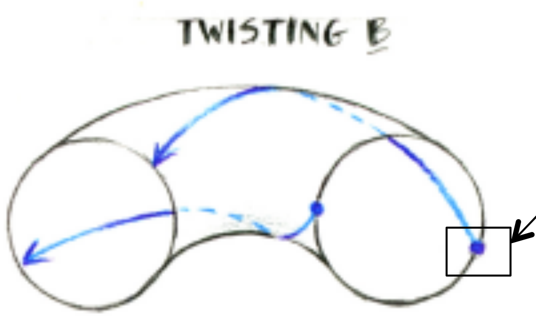
$$v_d = \frac{\frac{1}{2} m_j v_\perp^{\,2}}{q_j B} \frac{B \times \nabla B}{B^2}$$



$$v_E = \frac{E \times B}{B^2}$$

Ion Temperature Gradient Instability

Ion Temperature Gradient
Instability (ITG)



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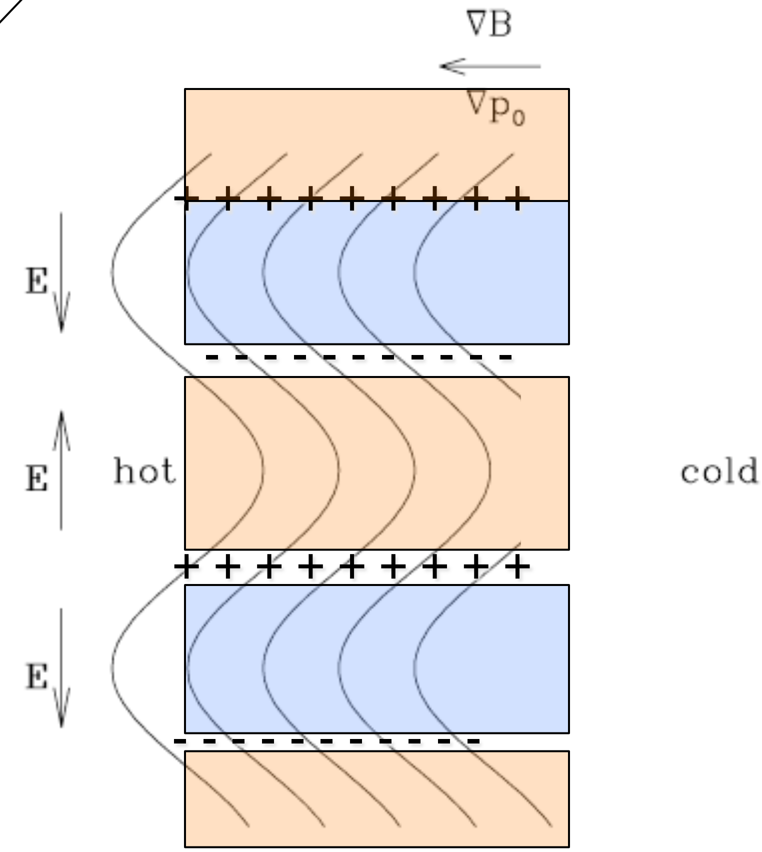

Contours of total $p = p_0 + \tilde{p}$

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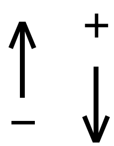

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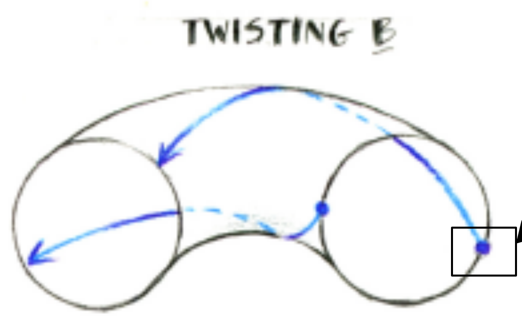
Ion Temperature Gradient Instability

Ion Temperature Gradient Instability (ITG)

Outboard Side of Torus



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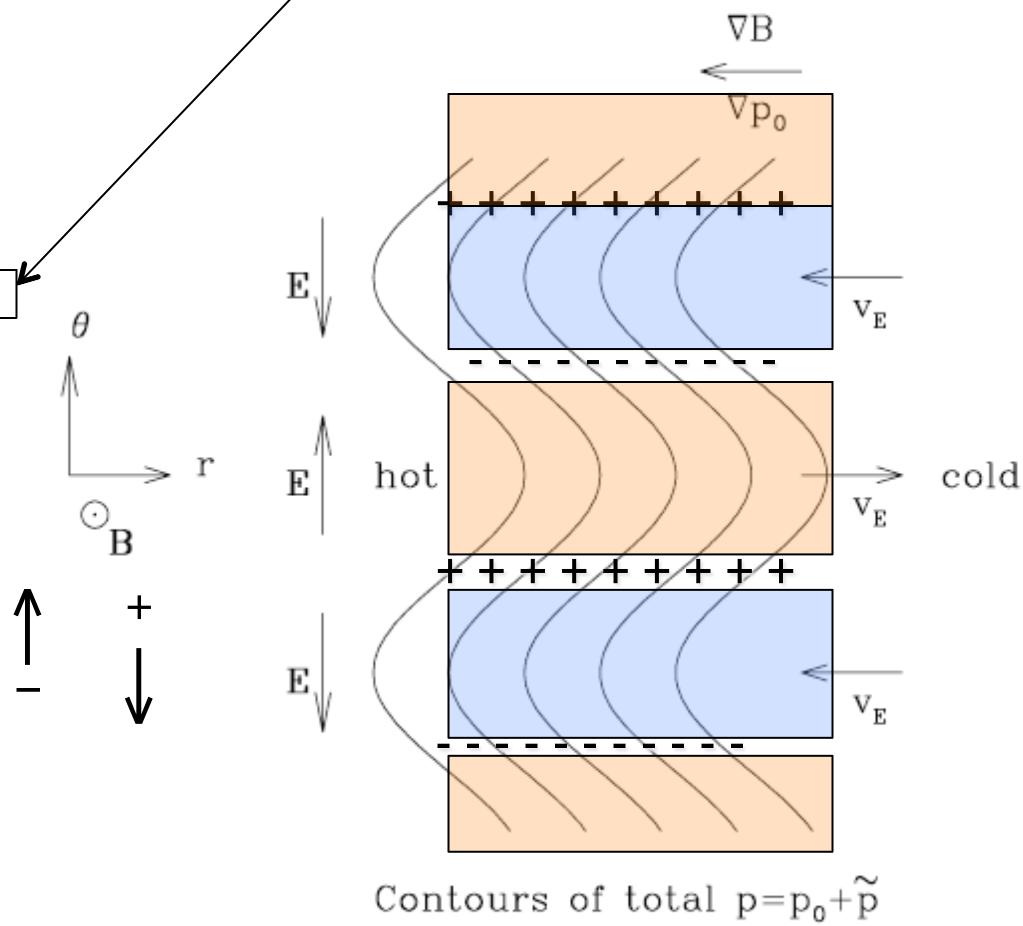

Fundamental Turbulence Paradigm

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Turbulence:
Energy
1. Injection
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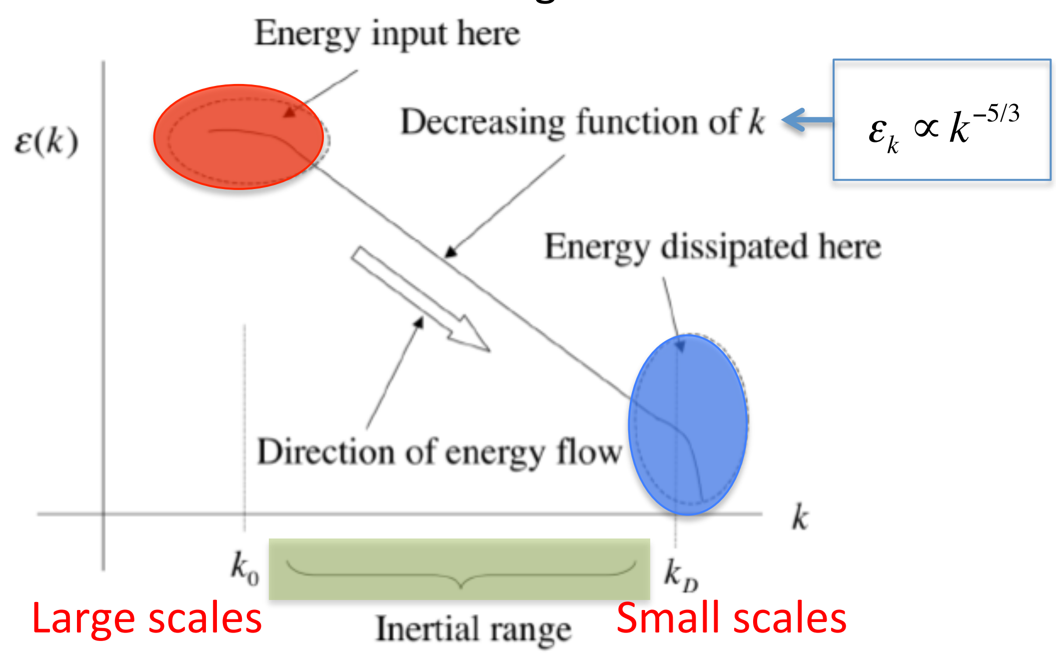
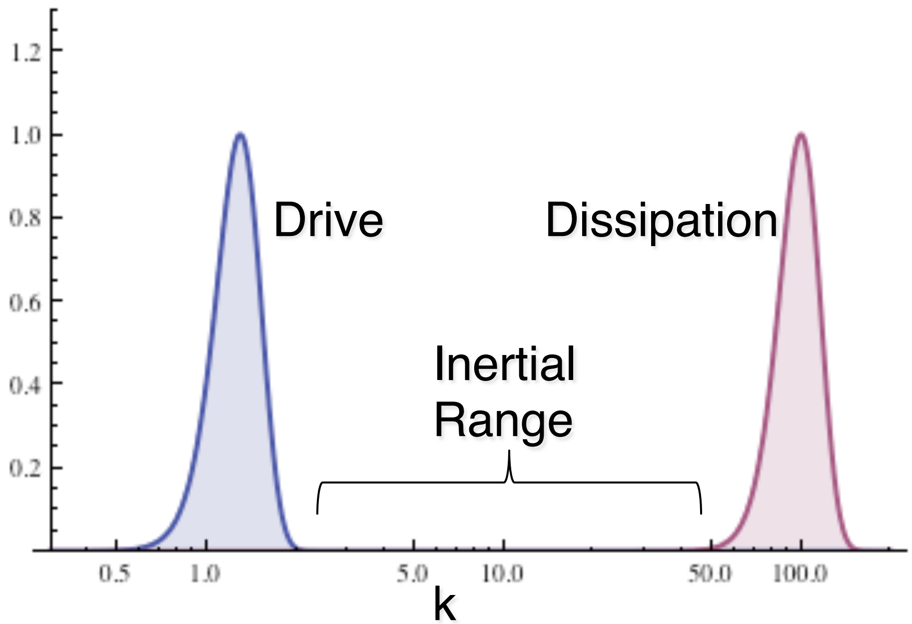
3. Dissipation

Kolmogorov



Fluid Turbulence - Saturation

Hydrodynamic turbulence:
Navier Stokes equation ➔ Kolmogorov picture:



1. Energy drive (stresses) at large scales.
2. Conservative nonlinear energy transfer through inertial range of scales.
3. Dissipation at small scales.

Saturation ➔ Energy drive at large scales balances with dissipation at small scales.

Dissipation mechanisms

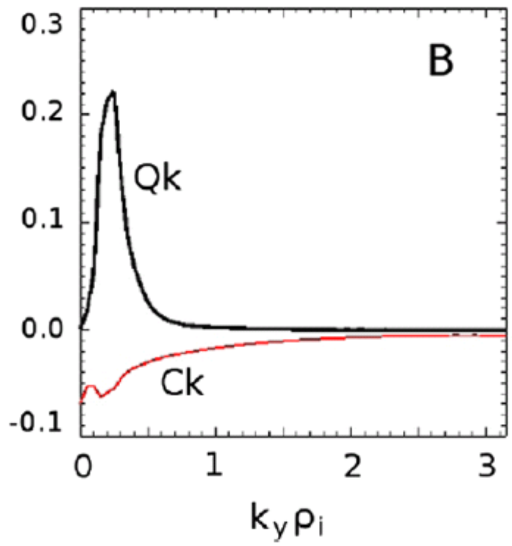
Dissipation in fluid turbulence: $\frac{1}{\mathrm{Re}}\nabla^2 u$

Re=large➔small scale dissipation

Dissipation in kinetic plasma turbulence: $\propto \upsilon \nabla_{\mathrm{v}}{}^2 f$

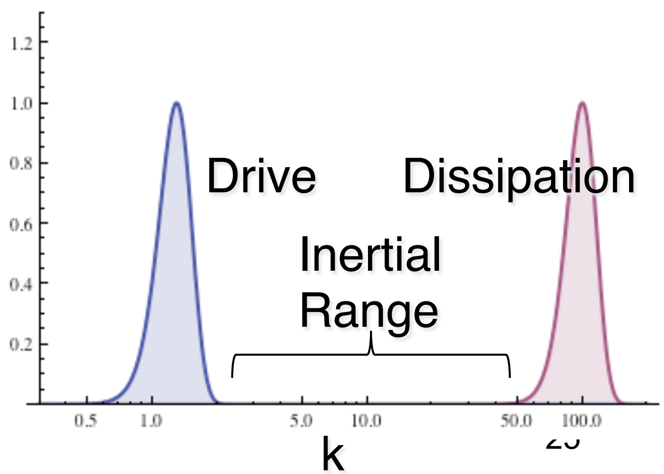
υ =small➔small scale dissipation in **velocity space**

Dissipation in gyrokinetic ITG turbulence

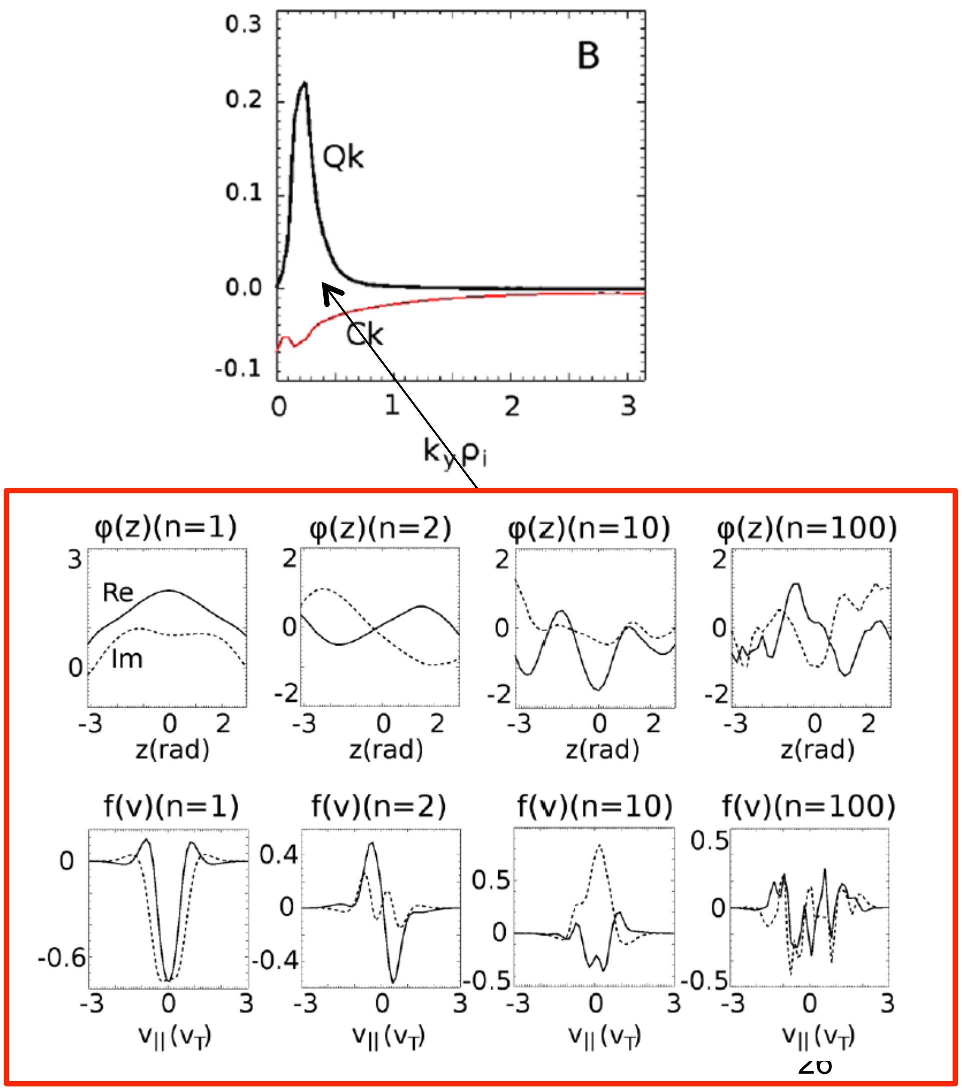


Q=gradient drive
C=collisional dissipation

Contrast:



Dissipation in gyrokinetic ITG turbulence



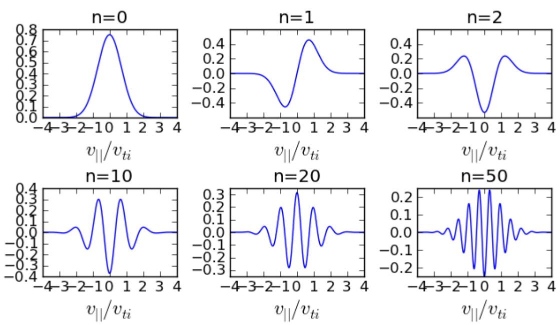
Q=gradient drive
C=collisional dissipation

Small scales develop in velocity space even at drive scales in real space.
➔Scale range of drive and dissipation overlap!
D. R. Hatch et al. PRL, 2011.
(Also a cascade at higher k
Banon Navarro, PRL, 2011.)

Model—`Reduced Gyrokinetics’

Hermite representation:

$$f(v) = \sum_{n=0}^{\infty} \hat{f}_n H_n(v) e^{-v^2}$$



Simple relations between moments and Hermites

$$\phi = D(k_\perp)\hat{f}_0$$

$$u = \frac{\pi^{1/4}}{\sqrt{2}}\hat{f}_1$$

$$p = \frac{\pi^{1/4}}{\sqrt{2}}\hat{f}_2 + \frac{\pi^{1/4}}{2}\hat{f}_0$$

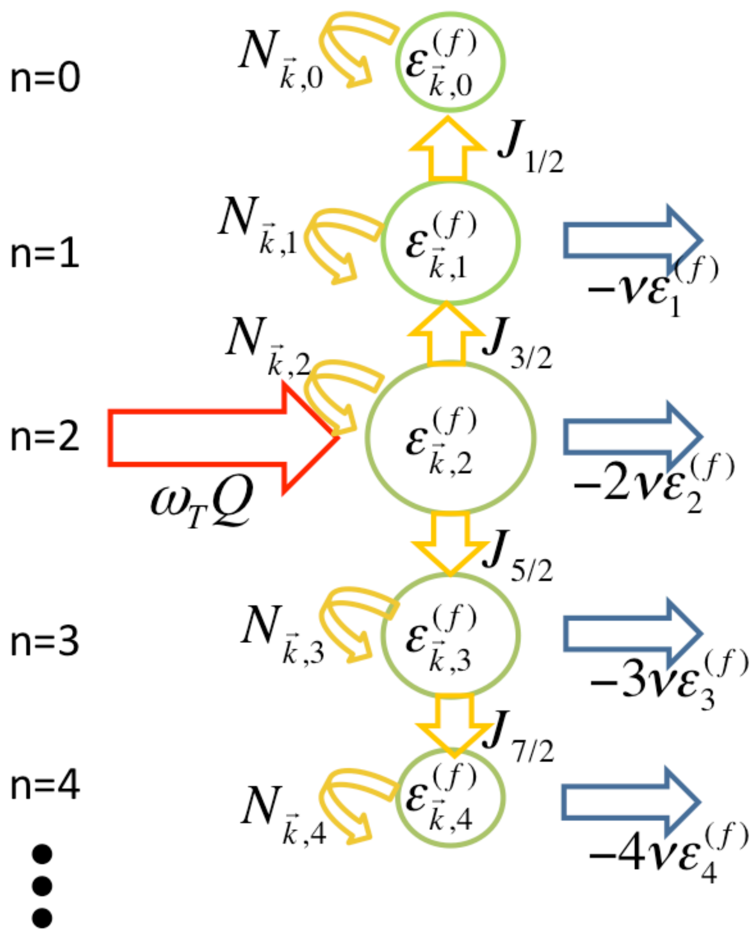
⋮

Equations:

$$\frac{\partial \hat{f}_n}{\partial t} = \frac{\eta_i i k_y}{\pi^{\frac{1}{4}}}\frac{k_\perp^2}{2}\bar{\phi}\delta_{n,0} - \frac{i k_y}{\pi^{\frac{1}{4}}}\bar{\phi}\delta_{n,0} - \frac{\eta_i i k_y}{\sqrt{2}\pi^{\frac{1}{4}}}\bar{\phi}\delta_{n,2}$$
$$-\frac{i k_z}{\pi^{\frac{1}{4}}}\bar{\phi}\delta_{n,1} - i k_z\left(\sqrt{n}\hat{f}_{n-1} + \sqrt{n+1}\hat{f}_{n+1}\right)$$
$$-\nu n \hat{f}_n + \sum_{\boldsymbol{k}'}\left(k_x' k_y - k_x k_y'\right)\bar{\phi}_{\boldsymbol{k}'}\hat{f}_{n,\boldsymbol{k}-\boldsymbol{k}'}.$$

DNA code:
Reduced GK model:
As simple as possible while still capturing dynamics of interest.
Hatch *et al* PRL ’13
Hatch *et al* JPP ’14
Hatch *et al* NJP ‘16

Injection, Redistribution, and Dissipation of Energy in Phase Space



Gyrokinetic energy in Hermite space:

$$\varepsilon^{(f)}_{\mathbf{k},n} = \frac{1}{2}\pi^{1/2}|\hat{f}_{\mathbf{k},n}|^2$$

Energy evolution equation:

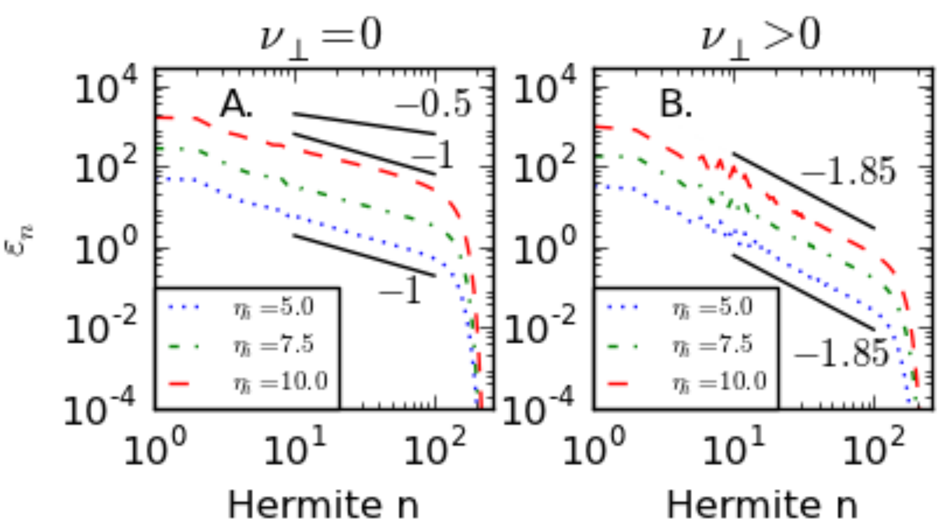
$$\frac{\partial \varepsilon^{(f)}_{\mathbf{k},n}}{\partial t} = \eta_i Q_{\mathbf{k}} \delta_{n,2} - \nu n \varepsilon^{(f)}_{\mathbf{k},n} - \nu_{\perp}(k_{x,y}/k_{\max})^8 + J_{\mathbf{k},n-1/2} - J_{\mathbf{k},n+1/2} + N^{(f)}_{\mathbf{k},n},$$

Description of Hermite Energy Spectra

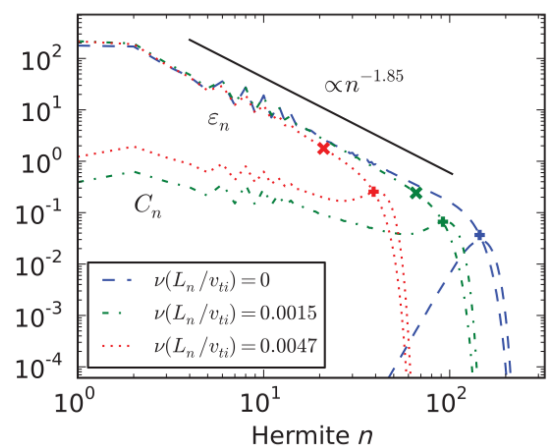
$$\frac{\partial}{\partial n}\langle k_z\rangle_n\sqrt{n}\varepsilon_n = -\nu n\varepsilon_n - \alpha\varepsilon_n$$

$$\varepsilon = c_0 n^{-1-\alpha} e^{-n/n_c}$$

Varying driving gradients and collisionality



Vary collisionality: constant power law, changing n_c



$$n_c = \frac{\langle k_z\rangle_n}{\nu n^{1/2}}$$

Compare Schekochihin *et al.* JPP 2016