

Alfven Waves

Swadesh Mahajan

The fundamental MHD wave

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Alfven Waves

Alfven Waves are the most ubiquitous and important mode of magneto hydrodynamic (mhd) oscillations.

Typically the applied external mag. field is in the \hat{z} -direction $\underline{B}_{\text{ext}} = \underline{B}_0 \hat{e}_z$. Defining $k_{\parallel} = k \cdot \underline{\hat{e}} / |k|$, the dispersion relation in a homogenous medium is

$$(\omega^2 - k_{\parallel}^2 v_A^2)(\omega^2 - k_{\parallel}^2 v_A^2 - k_{\perp}^2 v_A^2) = 0$$

leading to the two uncoupled independent modes

(1) Shear : $\omega^2 = k_{\parallel}^2 v_A^2$ with $\hat{b} \perp \underline{B}_0$

(2) Compressional : $\omega^2 = k^2 v_A^2$ with $\hat{b} \parallel \underline{B}_0$

In most plasmas of interest (laboratory as well as astrophysical) $k_{\perp}^2 \sim k^2$ is generally large, i.e., $k_{\parallel}^2 \ll k_{\perp}^2$.

We can view k_{\parallel} (k_{\perp}) as indicating the spring const. of the plasma. Since larger spring constants imply greater ability to deal with external perturbations, the shear waves are more likely to go unstable as compared to the compressional waves.

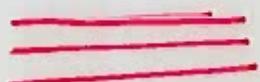
Stability

Small wonder that most of the stability (mHD) literature is full of Alfvén waves; kinks, sausages, pressure driven are Alfvén waves in ^{one} form or another.

Where do Instabilities arise from?

One introduces inhomogeneities of all kinds, density and temp. gradients, magnetic fields which vary, plasmas carrying currents with gradients etc.

Let us look at the level of complications that can exist in the externally applied fields:



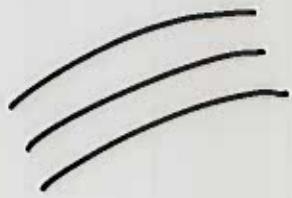
straight field
lines

$$\tilde{B}_{ext} = B_0 \hat{e}_z$$



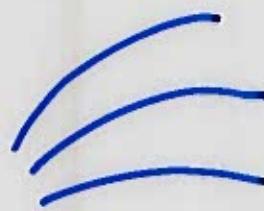
sheared field

$$\tilde{B}_{ext} = B_0 [\hat{e}_z + f(x) \hat{e}_y]$$



Curvature

$$\tilde{\tilde{B}}_{ext} \neq 0$$



shear + curvature

We need to have the field as a function of two independent directions: $(r, \theta) \rightarrow (x, y)$

And as one increases the complications, one enriches the kind of motions the plasmas can sustain.

More on stability

When various gradients are present, one could (very very crudely and schematically) write the dispersion relation as

$$\omega^2 = \langle k_{\parallel}^2 v_A^2 \rangle - \langle D^2 \rangle < 0 \text{ for instability.}$$

where

$$D^2 = D^2 [J_2, J_2', \overset{\curvearrowleft}{x}, p, \frac{\partial p}{\partial x}, \dots]$$

plasma current \uparrow
 current gradients \uparrow
 (shear) \uparrow
 pressure \downarrow
 pressure gradients

$\langle \rangle \rightarrow$ Some appropriately weighted spatial average.

Remembering that $v_A^2 = B^2/4\pi\rho$ is generally large, $\langle \omega^2 \rangle$ could overcome $\langle k_{\parallel}^2 v_A^2 \rangle$ only if k_{\parallel}^2 remained uniformly small in the region where the wave function (the weighting function) is finite; $k_{\parallel}=0$ is the region around where the modes must be localized. Of course, k_{\parallel} is a function of x .

Thus Instabilities often pertain for highly localized modes around $k_{\parallel}=0$ (the rational surfaces in a torus).

It turns out that normal plasmas (without some strong external drive) cannot come up with sufficiently large $\langle D^2 \rangle$ to beat $\langle k_{\parallel}^2 v_A^2 \rangle$ if k_{\parallel} is finite everywhere.

Normally Stable Modes

Unstable modes transfer energy from the plasma to the e.m. fields. The stable modes, on the other hand, can be used to transfer energy from external fields to the plasma. Thus for applications, like plasma heating, intrinsically stable modes are the subjects of interest. Whether it is the heating of Solar Corona or the thermonuclear plasmas, one has to find appropriate agents to do the trick. We shall see that the Alfvén waves provide a such a rich variety of choices that their detailed investigation is strongly recommended. Rest of this lecture will be devoted to precisely this task. I shall assume that this audience is familiar with elementary aspects of Alfvén waves.

Let us build our model keeping a toroidal system in mind (Coronal rings, fusion devices). We first deal with a *theorist's* torus, an opened up torus which



is a cylinder of length $2\pi R$. The direction $S(z/R)$ is ignorable and so is θ for the equilibrium. Thus all equilibrium quantities vary only in the radial direction

Eigen Mode Equation

The magnetic field is sheared $B_{\text{ext}} = \tilde{B}_0 [\hat{e}_z + f(r) \hat{j} G]$, i.e., the \hat{j} component is smaller than \hat{z} (by f) but is a function of r (r). Including gradients, one then manipulates the mhd equations to arrive at the mode equation [$\hat{E}_{||} = \hat{b}$, $\hat{r}, \hat{e}_\perp = \hat{b} \times \hat{r}$ are the unit vectors]

$$\frac{d}{dr} \frac{F}{D} \frac{1}{r} \frac{d}{dr} r E_\perp + F E_\perp + \bar{g} E_\perp = 0 \quad (1)$$

where

$$F = \frac{\omega^2}{V_A^2} - k_{||}^2$$

$$D = \frac{\omega^2}{V_A^2} - k_{||}^2 - k_z^2$$

and g is some complication function of equilibrium current gradients etc. $k_z^2 \sim m^2/r^2$, $k_{||}^2$ and V_A^2 are both functions of r .

- 1) If $g=0$, and $k_{||}^2, V_A^2$ are constants (no shear no gradients), then $F=0$, the shear mode factors out and is not coupled with the compressional mode at all. But any one of these complications couples them, and, as we shall see, introduces interesting variations.
- 2) Notice that in the simplest case, the shear wave has no \perp propagation, and thus cannot be in touch with the world at large

Properties of the Eigenmode Equation

Shear waves, however, can be reached from outside through either the compressional wave, or through kinetic effects which give a radial propagation. In the latter case, the shear wave is generally called: The Kinetic Alfvén Wave (KAW). We shall introduce kinetic effects soon.

- 3). For most interesting cases, $\omega^2/v_A^2 - k_\parallel^2 \ll k_\perp^2$, and \mathcal{D} can be replaced by $-k_\perp^2$. Notice that this approximation does not throw the compressional wave away, but makes it into a cipple. The simplified system is

$$\mathcal{L} E_\perp \equiv \frac{d}{dr} \left(-\frac{1}{k_\perp^2} \right) F \frac{1}{r} \frac{d}{dr} r E_\perp + F E_\perp + \bar{g} E_\perp = 0$$

- (4) Equation one has a singularity:

$$F(r) = \frac{\omega^2}{v_A^2(r)} - k_\parallel^2(r) = 0 \quad \text{at } r=r_0$$

$$F(r) = F'(r)_{r=r_0} (r-r_0) = Kx$$

$$r \approx r_0$$

$$\Rightarrow \frac{d}{dx} x \frac{dE_\perp}{dx} - k_\perp^2 x E_\perp - k_\perp^2 g E_\perp = 0$$

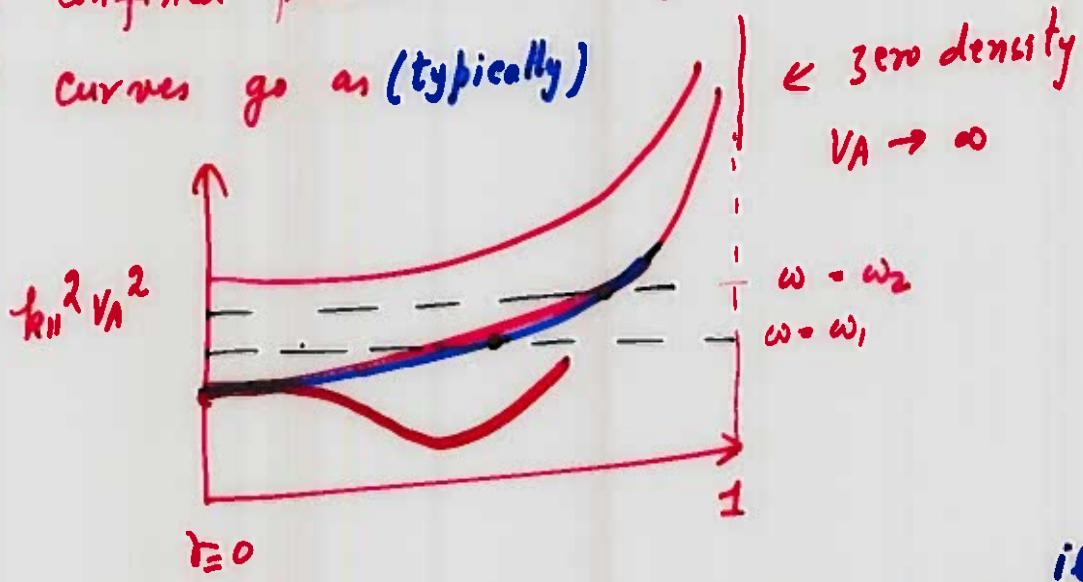
$$\Rightarrow x \rightarrow 0 \quad E_\perp \sim \ln x$$

\Rightarrow No square integrable solutions.

Magneto hydrodynamic Continuum

Thus we arrive at the 'well known' mhd continuum. You pick up any ω ; the response is very large where $\omega^2 = k_{\parallel}^2 v_A^2$, and negligibly small elsewhere: the system responds to all ω .

For a confined plasma with magnetic shear, typical $k_{\parallel}^2 v_A^2$ curves go as (typically)



$$k_{\parallel} \approx \frac{L}{R} + \frac{m}{qR} \quad \text{for a tokamak } E \approx c \quad \text{estimate}$$

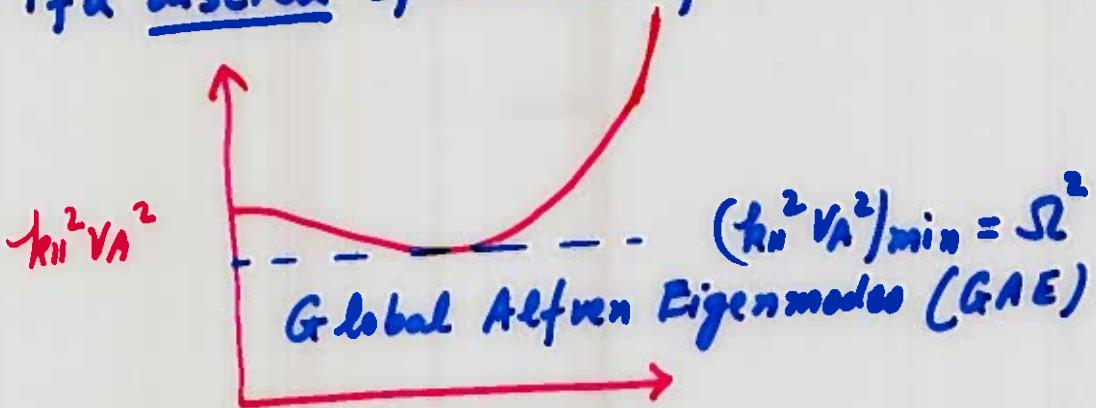
$q \propto B_r / B_z$, generally increases outwards.

The existence of the continuum implies that some additional additional piece of physics is needed; we do not entertain measurable quantities to be infinite.

It turns out ^{that the} inclusion of kinetic effects, for example, electron dynamics in the parallel direction would remove this singularity. Notice, that in MHD (ideal) $E_{\parallel} \equiv 0$. Collisionless dissipation through electron Landau damping allows $E_{\parallel} \neq 0$.

Discrete Modes in MHD [One-Dimensional]

Before introducing K-E (which we shall soon do), let us examine if a discrete spectrum is possible in mhd.



Let us suppose that $(k_{\parallel}^2 V_A^2)$ has a minimum at $r = r_0$.

Thus if $\omega^2 < \Omega^2$, then the differential equation (1) has no singularity. Is it then possible to have well-defined square integrable eigenmodes?

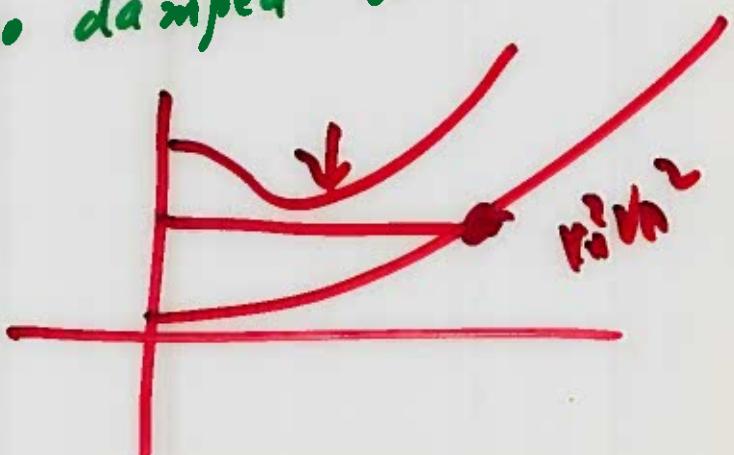
Under suitable conditions, yes. When these modes occur, they normally occupy large parts of the plasma, and thus are bound to be important.

Investigations of these modes is one of the main themes of this lecture. We shall solve the system for the spectrum and for the eigenmodes.

Since damping is very important to understand characteristics of these modes; their utility towards heating, we must solve the system with kinetic effects. A unified theory of KAW and GAE.

1) Although, in the last slide, the damping aspects coming through S are emphasized, it is the creation of the KAW which is even more spectacular. We do not need the imaginary part (as long as we know how to use causality properly) for KAW. -
If the "response were purely adiabatic for example, even then the KAW is generated.

2) The GAE were prematurely shot down as too damped to be interesting.



Unified Theory

Electron parallel dynamics modifies the equations substantially.

$$\lambda E_1 \equiv -\frac{d}{dr} \frac{F}{k_B T} + \frac{d}{dr} r E_1 + F E_1 + \bar{g} E_1 = \left(\frac{d}{dr} r \frac{d}{dr} - r k_B^2 \right) \frac{\frac{k_B^2 \rho_s^2}{S} \frac{1}{r k_B^2} \left(\frac{1}{r^2} \frac{d}{dr} r \frac{d}{dr} - k_B^2 \right) E_1}{(2)}$$

$$S = 1 + \frac{\omega}{k_B T e} \approx \left(\frac{\omega}{k_B T e} \right), \quad v_e = \text{electron thermal speed.}$$

where the right hand side represents the effects of a nonzero E_{\parallel} .

1) Because of the 4th radial derivative in (2), the equation is no more singular even for the regular $k_B^2 V_A^2$ profiles.

2) The existence of S [which contains wave-particle interaction] introduces collisionless damping or collisional damping on appropriate changes in S .

3) We must define the hot and cold regions of the plasma. That will determine how the \pm -functions will behave. Since $\omega \sim k_B V_A$

$$\frac{\omega}{k_B T e} = \frac{V_A}{v_e} = \frac{B_0}{4\pi n_0 m_i} \sqrt{\frac{m_e}{2T_e}} = \sqrt{\frac{m_e}{m_i}} \cdot \frac{1}{\beta} v_e,$$

β = is the plasma beta = $\frac{\text{plasma pressure}}{\text{mag. field pressure}}$.

Hot region :

$$\boxed{\beta > \frac{m_e}{m_i}}$$

Propagating KAW towards high density.

Cold region :

$$\beta < \frac{m_e}{m_i} \rightarrow \text{kinetic surface modes}$$

Unified Theory

Plasma surface modes are rather complicated depending upon the edge conditions. We shall deal primarily with the hot plasma modes. To develop a solvable model, let us expand our sensitive function F about the minimum of $(k_{\parallel}^2 V_A^2)_{\min} = \Omega^2$, i.e., around $r = r_0$.

- (1) We shall look for modes peaking about $r = r_0$.
- (2) All quantities except fast varying $[F]$ will be just evaluated at $r = r_0$.
- (3) New differential equation will emerge in $x = \frac{r - r_0}{r_0}$.
- (4) $F = F_0(r=r_0) + \frac{1}{2} F''|_{r=r_0} (r-r_0)^2 = F_0 + (\cdot) x^2$.

After a fair amount of tedious but straightforward algebra, we can obtain the differential equation

$$\frac{d}{dx} (x^2 - \mu) \frac{dE_L}{dx} = k_{\perp}^2 a^2 (x^2 - \mu) E_L + g_0 E_L \quad (3)$$

$$= \bar{F} \left(\frac{d}{dx^2} - k_{\perp}^2 a^2 \right)^2 E_L$$

The right hand side again represents \parallel electron motion.

$\mu = \frac{F_0}{a^2}$ is the effective eigenvalue, $\bar{F} = \left(\frac{k_{\perp}^2 f_0}{a^2} \frac{1}{S} \right)_{r=r_0}$, and $\lambda^2 = -a^2/2 F''$, while a is the plasma radius. \bar{F} measure the kinetic effects and g_0 the effects of \parallel current. Let us also remind ourselves that $F_0 = \frac{1}{V_A^2(r_0)} [\omega^2 - k_{\parallel}^2 V_A^2]_{r=r_0} \approx \omega^2 - \Omega^2$

Contd.

Equation (3) needs to be solved to delineate the nature of the modes. Notice that if $\bar{r}=0$, and $F_0 > 0$ ($\mu > 0$), the equation is again singular, thus mhd continuum could be recovered.

Eqn. (3) is best analyzed in Fourier space, where it becomes a second order equation. Define

$$E = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} dx E_s(x) e^{ipx}$$

$$E_s = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} dp \tilde{e}^{-ipm} E$$

then with appropriate rescaling $y = k_x a x$, $\tau = k_x^2 a^2 \tau$, and the eigenvalue $E = k_x^2 a^2 \mu$, one obtains

$$\frac{dy}{dy} (1+y^2) \frac{dE}{dy} + [E(1+y^2) + g_0 - \tau(1+y^2)^2] E = 0 \quad (4)$$

which can be cast into a Schrodinger like equation [$y = (1+y^2)^{1/2} E$]

$$\frac{d^2\psi}{dy^2} + [E - V(y)] \psi = 0 \quad (5)$$

Kinetic term.

with

$$V(y) = -\frac{g_0}{1+y^2} + \frac{1}{(1+y^2)^2} + \boxed{\sigma(1+y^2)} \quad (6)$$

and has to be solved as an eigenvalue problem with E as the eigenvalue.

Effective Potential $V(y)$

Effective potential $V(y)$ has a wealth of information:

(a) Let us begin by considering only the mhd part

$$(\sigma=0). \quad v(y) = -\frac{g_0}{1+y^2} + \frac{1}{(1+y^2)^2}$$

a) As $y \rightarrow \infty$, $v(y) \rightarrow 0$ as $-g_0/y^2$

b) At $y=0$, $v(0) = -g_0 + 1$

c) If $g_0 < 0$, then $v(y)$ has an effective hill at $y=0$; No confined modes are possible, and only continuum modes may exist.

d) Let us calculate the extrema of $v(y)$ [$g_0 > 0$]

$$v(y)=0 = 2y \left[\frac{g_0}{(1+y^2)^2} - \frac{2}{(1+y^2)^3} \right] \quad (\dagger)$$

$$y=0, \quad 1+y^2 = 2/g_0.$$

If $g_0 \geq 2$, only one extremum at $y=0$ (minimum)

If $g_0 \leq 2$, there are three extrema, $y=0$ is a maxima and two symmetric minima $\bar{y} = \pm(2/g_0 - 1)^{1/2}$.

(e) For $g_0 \geq 2$, we have a potential well at $y=0$, $v(y)$ is -ve everywhere; the equation $E-v(y)=0$, has real turning points for $E < 0$ ($\mu < 0$) implying bound states for $E < 0$. This is the discrete spectrum of Alfvén waves (AE).

Effective Potential $V(y)$

For $g_{0,1/2}$, there are not really turning points for $E > 0$, and hence the spectrum is continuous. Complete analogy with the hydrogen atom spectrum

- 1) Potential is $-ve$ everywhere
- 2) potential goes to zero slowly as $y \rightarrow \infty$
- 3) Discrete spectrum for $E < 0$ and continuous for $E > 0$.
- 4) $E = 0$ is the accumulation point for the countable infinity of eigenmodes

We have thus shown that under suitable conditions, (plasma parameters along with the mode numbers obeying certain criteria, $g_{0,1/2}$), magneto hydrodynamic equations yield a discrete spectrum. These modes are the Global Alfvén Eigenmodes. We shall presently show that they are global, i.e., occupy large parts of the plasma.

- 5) We still have to show that the set of modes is denumerably infinite. It is sufficient to write the WKB quantization condition for Eq.(5)

$$2|E|^{\frac{1}{2}} \int_{y_+}^{y_t} [-|E| - V(y)]^{\frac{1}{2}} dy = (n + \frac{1}{2})\pi \quad (8)$$

which is good for large n . y_t is the turning point (symmetric). Clearly for $n \rightarrow \infty$ to be possible, it is essential that the W.K.B integral diverge as $y_+ \rightarrow \infty$.

IV. GENERAL FEATURES

An examination of the effective potential $V(x)$ [Eq. (24)] appearing in the mode Eq. (23) yields a wealth of information about the nature of the eigenmodes. To maintain continuity with the published literature, we again begin with the MHD limit $\sigma = 0$. In Fig. 1, we have plotted $V(x)$ as a function of x for several values of g_0 . It is straightforward to see: (1) For $g_0 < 0$, $V(x)$ is a monotonically decreasing function of x , resulting in an effective potential hill at $x = 0$. Thus no localized eigenmodes around $x = 0$ are possible for any ϵ . Only the continuum modes exist. (2) For $g_0 > 2$, $V(x)$ has a minimum at $x = 0$ [$V(0) = -g_0 + 1$], and approaches zero as x^{-2} as x goes to infinity implying a potential well at $x = 0$. Since the potential $V(x)$ is negative everywhere, the equation $\epsilon - V(x) = 0$ has real solutions if and only if $\epsilon < 0$. Thus, the real turning points, and hence the bound states, exist only for $\epsilon < 0$. This is the discrete spectrum of GAE. Details for this case can be seen in Ref. 8. Since there are no real turning points for $\epsilon > 0$, there are no bound states, and only a continuum prevails; $\epsilon = 0$ defines the lower edge of this well-known shear Alfvén continuum. (3) For $0 < g_0 < 2$, $V(x)$ has considerably more structure with a maximum at $x = 0$ and two minima at $x = \pm [(2/g_0) - 1]^{1/2}$. Clearly for $\epsilon > 0$, no bound states are possible. This proves that regardless of the value of g_0 , $\epsilon > 0$ always corresponds to the continuum modes. In this range of g_0 , it is a little hard to determine the criterion which allows the existence of discrete modes for $\epsilon < 0$. The question has been dealt with in detail in Refs. 7 and 8, where it is shown that $g_0 > (1/4)$ is required for the discrete spectrum to be possible. The reason is, that although minimas of the potential exist, they are not deep enough to contain a mode.

The preceding discussion essentially sums up the known results, which, of course, trivially follow from a study of Eqs. (23) and (24) with $\sigma = 0$.

Figure 2 contains plots of $V(x)$ vs x for several values of g_0 , and for a finite but small value of $\sigma \approx 10^{-2}$. We have chosen σ to be real for simplicity. The general nature of the following discussion will hold for a realistic complex σ .

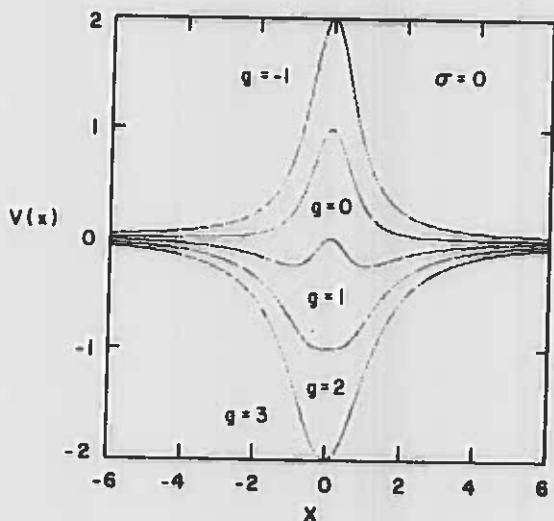


FIG. 1. $V(x)$ vs x for the MHD limit $\sigma = 0$. The curves have different values of $g_0 = g$. Notice that $V(x) \rightarrow 0$ as $x \rightarrow \infty$ for all the curves.

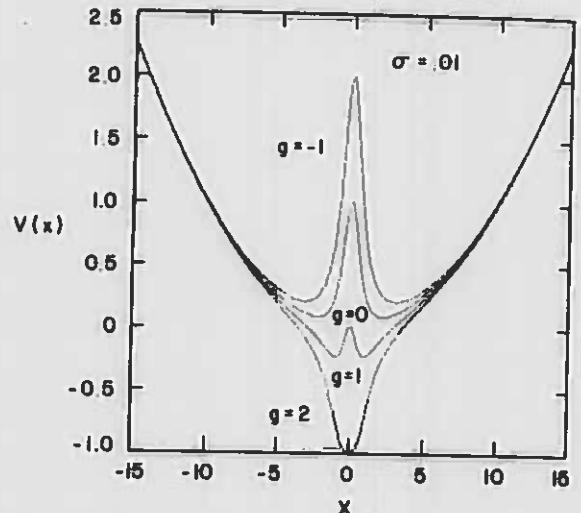


FIG. 2. $V(x)$ vs x for the dissipative case, $\sigma = 10^{-2}$. The potential well at large x is responsible for the discrete spectrum.

There are several features which distinguish Fig. 2 from Fig. 1 ($\sigma = 0$). The most important is the behavior of $V(x)$ as $x \rightarrow \infty$. While for $\sigma = 0$, $V(x) \rightarrow 0$ as $x \rightarrow \infty$, for finite σ , $V(x) \rightarrow x^2$. For large x , the potential is like that of a simple harmonic oscillator and implies bound states for all values g_0 , because the equation $\epsilon - V(x) = 0$ always has real solutions. Thus the erstwhile MHD continuum for $\epsilon > 0$ has changed over to a discrete spectrum. Depending upon the value of g_0 , the lowest eigenmode may be characterized by an eigenvalue $\epsilon < 0$. Thus, the distinction between GAE and the continuum has disappeared. They are the parts of the same discrete spectrum; the lowest eigenvalue will be positive or negative depending upon the value of g_0 , which is due to the presence of equilibrium currents and ω/ω_{ci} effects.

From Fig. 2, it is also obvious that for positive g_0 , the lowest mode will be localized in the potential well around $x = 0$, and will not sample the form of the potential for large x . Thus its character is only mildly affected by the presence of σ . Since σ is due to the dissipative processes (for example Landau damping for a collisionless case), the mode will have very slight damping.

The case for $g_0 < 0$ is altogether different. Since the modes are formed because of the turning points at large x (due to the term proportional to σ), their mode structure as well as the eigenvalue strongly depends upon σ . As a result, these modes are considerably more damped.

We remind the reader that Eqs. (23) and (24) are in the Fourier space x . The real spatial variable is $y = (r - r_0)/\epsilon$. Since $p = k_x x$ and y are conjugate Fourier variables, $k_x \Delta x \Delta y \sim 1$, where Δx and Δy are, respectively, the width in x and y . Therefore, the GAE modes (now corresponding to the lowest modes for certain values of g_0), which are narrow in x , are relatively broad in y , possibly extending over most of the plasma. This justifies their name, global Alfvén eigenmodes. The exact width, however, is a function of the n number; $\Delta r \approx r_0/|m|$, and can become quite small if $|m| > 1$. The discrete modes lying in the MHD continuum, however, are comparatively broad in x (because of the smallness of σ) and therefore narrow in y . Thus, in real space, these modes are strongly localized around $r \approx r_0$ ($y = 0$). This is to be

Infinity of Eigenmodes

The integral does diverge because $V(y)^{1/2} \rightarrow y^{-1}$ ($y \rightarrow 0$) and allowing the system to diverge if $y_+ \rightarrow \infty$. This also happens because as $\epsilon \rightarrow 0$, the turning points go to infinity. A potential going to infinity faster than y^{-2} will not have this feature.

- (6) For large n , the eigenvalues can be written down as a simple expression

$$E_n = \left(\frac{4}{\epsilon}\right)^2 e^{-\frac{(2n+1)\pi}{2g^{1/2}}} \quad (9)$$

$$\frac{E_{n+1}}{E_n} = e^{-\pi/g^{1/2}} \quad \text{independent of } n.$$

- (7) It is seldom that such high n eigenmodes are interesting. In realistic plasmas, they are wave-like indistinguishable from the continuum modes.

- (8) Before we proceed to obtain the eigenvalue for the lowest $n=0$ (and the most important one) mode, we must comment about the region $2 > g > 0$. In this range of g_0 , the potential is sufficiently complicated [Fig. 1] that much more careful analysis is needed to determine whether the discrete spectrum exists or not. It can be shown that $a > v_0$ is essential to GAF.

Effects of σ

Till we now we discussed pure mhd. let us now consider the entire potential

$$V(y) = -\frac{g_0}{1+y^2} + \frac{1}{(1+y^2)^2} + \sigma(1+y^2)$$

σ is generally a very small number, $\sigma \approx 10^{-4}$. Thus σ starts effecting the potential for comparatively large y . For $g_0 < \frac{1}{\sigma}$ (no mhd confinement), the modes will be confined by the term σy^2 (simple harmonic oscillator), but the width of these modes will be large

$$\sigma(\delta y)^2 \sim 1 \quad \delta y \sim \frac{1}{\sigma y_2}$$

These modes thus will be kinetically confined ~~but~~ but are wide in δy . Since x is the conjugate variable variable to y

$$\delta x \sim \frac{1}{\delta y} \sim \sigma^{1/2} \ll 1$$

thus they will be very peaked in real space. Without

σ they were only nonzero at $x=0$, now kinetic effects give them a finite but small width.

MHD Confined modes on the other hand are

Confined by $g_0 \sim 1$

$$\delta y \sim 1$$

$$\delta x \sim \frac{1}{\delta y} \sim 1$$

Hence the name global, 1 is the size of the plasma.

Variational Principle, the $n=0$ mode.

The fundamental mode ($n=0$) dispersion relation is readily obtained by using variational techniques. (The potential does not allow an easy analytical approach).

For this purpose Eq.(4) (rather than the Schrodinger like Eq.(3)) is more suitable. We notice that [$\langle \cdot \rangle = \int_{-\infty}^{+\infty} dy \cdot \cdot \cdot$]

$$S = -\langle (1+y^2) E^2 \rangle + \epsilon \langle (1+y^2) E^2 \rangle + g_0 \langle E^2 \rangle - \sigma \langle (1+y^2)^2 E^2 \rangle \quad (10)$$

is variational in the sense that $\delta S=0$ reproduces our original Eq.(4). Choose a trial function

$$E = e^{-\alpha y^2/2} \quad \text{Re } \alpha > 0$$

where α is a variational parameter. Eigenvalue ϵ and α are now obtained by a simultaneous solution of

$$S(\kappa) = 0, \text{ and } \frac{\partial S}{\partial \kappa} = 0 \quad (11)$$

With the usual amount of algebra, one can obtain the dispersion relation when $\kappa \approx 1$ for $G \propto E$, and when $\kappa \ll 1$ for the old continuum modes which have become eigenmodes due to σ . Remember that with σ , there is no continuum, the entire spectrum is discrete.

Dispersion Relation for GAE

We are dealing with the hot region of plasma:

$$\frac{\omega}{(k_{\parallel 0})v_{ce}} = \frac{v_A}{v_{ce}} \ll 1$$

$$S \approx 1 + i\pi \frac{v_A}{v_{ce}}$$

With above expression for S , one can obtains [r=r₀]

$$\frac{\omega^2}{V_A^2} = k_{\perp}^2 \left[1 + \tau k_{\perp}^2 p_s^2 \left(1 - i\pi \frac{v_A}{v_{ce}} \right) \right] - \bar{E}_0$$

with changes in real part and an imaginary part.
The real frequency is still $\sim (k_{\perp}^2 V_A^2)^{1/2}_{r=r_0}$.

$$\frac{\text{Im } \omega}{\text{Re } \omega} \approx -\frac{\pi}{2} \frac{m^2}{r_0^2} p_s^2 \pi^{1/2} \frac{v_A}{v_{ce}} \ll 1$$

$$K = \frac{1}{(\delta y)^2} \sim 1 \quad \text{as expected.}$$

The GAE are characterized by a frequency small than Ω , and are weakly ($\sigma \ll 1$) damped, and cover most of the plasma.

[These are experimentally found and detected precisely at the predicted frequencies with predicted damping]. The KAW mode is narrow, and is highly damped ($\sigma \gg 1$). $\Delta x \sim (p_s/a)^{1/2} \ll 1$.

Global Modes and the K.A.W

We found that certain conditions, global modes are found in MHD theory. If found, they are

b) Global and weakly damped.

The kinetic waves are local and comparatively strongly damped

$$\frac{(\tau/\omega)_{\text{K.A.W}}}{(\tau/\omega)_{\text{GAE}}} \approx \frac{\sigma^{\frac{1}{2}}}{\Gamma} \sim \frac{1}{\sigma^{\frac{1}{2}}} \sim 10^{-2}$$

Highly damped modes are difficult to excite from external antennas or other energy sources, (say a drive from fast particles like fusion K's) and tend to be poor agents for plasma heating (although the eventual dissipation of energy to particles takes place through damping). Typical impedances associated with modes go as

$$Z \sim \frac{\omega^2}{\Gamma}$$

and this figure of merit is high for GAE and not so high for KAW. KAW is excellent for local dissipation of energy.

Toroidal Alfvén Eigenmode

We shall now discuss another arrival in the Alfvén zoo which has attracted much attention in the recent past. I shall not either extol its virtues nor enumerate its vices, but explain what it is, and how it comes to be. Till now, we had considered the equilibrium to be trivial in $S(z)$ and θ , and had only r variation. Magnetic fields, however, are not generally that simple and they can have θ variation also (for example). Let us look at the tokamak field

$$\tilde{B}_\theta = \frac{B_0 \hat{z}}{1 + \epsilon \cos\theta} + \epsilon \hat{y} f(x)$$

$\epsilon = r/R$ is the inverse aspect ratio.

For this geometry, m ceases to be a good quantum number, and our determining differential equations have a two dimensional character. For each

$$\tilde{B}_\theta = \hat{z} B_0 [1 - \epsilon \cos\theta] + \epsilon \hat{y} f(x)$$

a Fourier analysis of the perturbation will result in coupling E_m with E_{m+1} .

Can new features emerge in such a case?

Yes, indeed they do.

Toroidal Modes

The new geometry, of course, manifests itself in various ways: but let us examine its most profound impact. The Alfvén speed $v_A \propto B$, becomes a function of θ .

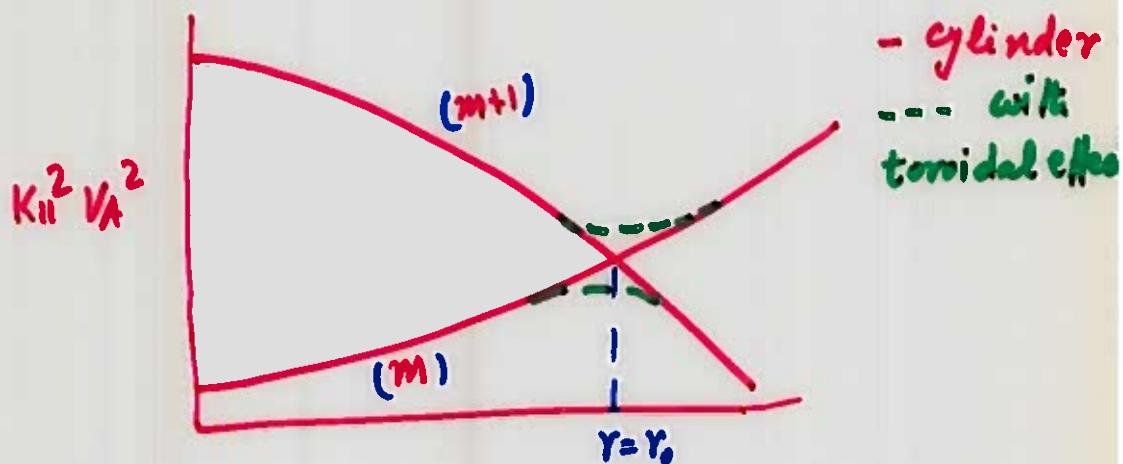
Terms like

$$\nabla \frac{\omega^2}{v_{A0}^2} E \rightarrow \frac{\omega^2}{v_{A0}^2} [1 + \epsilon \cos \theta] E$$

on Fourier analysis

$$\frac{\omega^2}{v_{A0}^2} [1 + \epsilon \cos \theta] E \rightarrow \frac{\omega^2}{v_{A0}^2} E_m + \frac{\epsilon}{2} \frac{\omega^2}{v_{A0}^2} [E_{m+1} + E_{m-1}]$$

Thus the azimuthal (toroidal) harmonics are coupled. Can we analytically handle this situation? We just show we can. Let us first go back to our $k_{\parallel}^2 v_A^2$ versus r picture.



Remember $K_{\parallel} = \frac{l}{R} + \frac{m}{qR}$

At $r = r_0$, $K_{\parallel}^2(m_1) = K_{\parallel}^2(m_2)$

$$\rightarrow \frac{l^2}{R^2} + \frac{ml}{qR} = - \left[\frac{l^2}{R^2} + \frac{m_2 l}{qR} \right] \Rightarrow m_1 + m_2 = 2lq$$

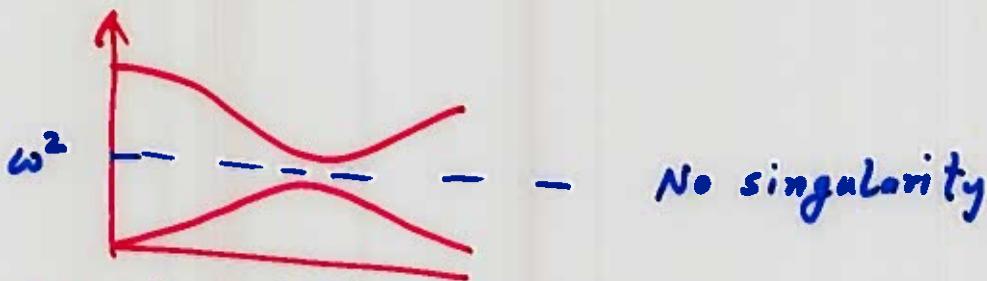
is the condition for intersection.

Gap Formation

When toroidicity is switched on, the two curves are coupled and bifurcation takes place

$$f_1(\mathbf{R}) = 0 \quad f_2(\mathbf{R}) = 0 \quad \text{cylindrical}$$

$$f_1(\mathbf{x}) \quad f_2(\mathbf{x}) = 0 \quad \text{toroidal}$$



and hence no possibility of forming an eigenmode

$$(\hbar n^2 k_h^2)_m^* < \omega^2 < (\hbar n^2 k_h^2)_{m+1}^* \quad * \text{ toroidicity modified}$$

For large m numbers, a large no. of gaps can come and an ω^2 threading these gaps will correspond to an eigenmode.

Thus toroidicity sets the stage for a further investigation.

Let us now look at the toroidal (gap) problem in the simplest possible approximation. Let there be a single gap corresponding to $m = m_1$ and $(m+1) = m_2$, whose coupling with another is only toroidal effect kept. We now derive the conditions under which a gap eigenmode may exist.

Gap Modes

Deal with the simplest coupled system

$$\frac{d}{dr} F_1 \frac{d\phi_1}{dr} - K_1^2 F_1 G_1 \phi_1 = \bar{\epsilon} \left(\frac{d^2}{dr^2} - k_2^2 \right) \phi_2$$

$$\frac{d}{dr} F_2 \frac{d\phi_2}{dr} - K_2^2 F_2 G_2 \phi_2 = \bar{\epsilon} \left[\frac{d^2}{dr^2} - K_1^2 \right] \phi_1$$

with $F_{1(2)} = \frac{\omega^2}{V_{A2}} - \frac{k_1^2}{k_{1(2)}}$, $K_{1(2)}^2 = k_{1(2)}^2$, and

$\bar{\epsilon}$ is the toroidal coupling. At $r=r_0$, $k_1^2 = k_2^2$, and is the point of crossing. Let us ignore density gradients for simplicity. $x(r-r_0)$

$$k_1^2 = k_1^2(r=r_0) + x \frac{d}{dx} k_1^2 |_{r=r_0} = k_0^2 - \alpha_1 x \\ k_2^2 = \dots = k_0^2 + \alpha_2 x$$

$$F_1 = \Delta - \alpha_1 x, \quad F_2 = \Delta + \alpha_2 x \quad \alpha_1, \alpha_2 > 0$$

$$\Delta = \left(\frac{\omega^2}{V_{A2}} - k^2 \right)_{r=r_0}. \quad \text{Evaluate all other}$$

quantities at $r=r_0$, and one obtains

$$\frac{d}{dx} [\Delta - \alpha_1 x] \frac{d\phi_1}{dx} - K_1^2 [\Delta - \alpha_1 x] \phi_1 - G_1 \phi_1 = \bar{\epsilon} \left[\frac{d^2}{dx^2} - k_2^2 \right] \phi_2$$

$$\frac{d}{dx} [\Delta + \alpha_2 x] \frac{d\phi_2}{dx} - K_2^2 [\Delta + \alpha_2 x] \phi_2 - G_2 \phi_2 = \bar{\epsilon} \left[\frac{d^2}{dx^2} - K_1^2 \right] \phi_1$$

Fourier transform, rescale the variables (dependent) as well as independent, reasonable algebra, reasonable approximations etc lead to the set

Gap Modes, Existence of

$$\left[\frac{d}{dy} + i \left(\hat{\Delta} + \frac{\hat{G}}{1+y^2} \right) \right] \psi_1 = -i \hat{E} f(y) \psi_2$$

$$\left[\frac{d}{dy} - i \left(\hat{\Delta} + \frac{\hat{G}}{1+y^2} \right) \right] \psi_2 = i \hat{E} f(y) \psi_1$$

where $f(y) = \left(\frac{y^2+1}{y^2+\mu} \right)^{1/2}$.

Two equations can be combined to lead to a single equation:

$$\frac{d}{dy} f(y) \frac{d\psi_2}{dy} + f \left[\left(\hat{\Delta} + \left(\frac{\hat{G}}{1+y^2} \right) \right)^2 - \hat{E}^2 \right] \psi_2 = i \left(\frac{\hat{G}f}{1+y^2} \right)' \psi_2 \quad (a)$$

$$+ i(Ef)' \psi_2$$

It is generally assumed that the gap formation, i.e., existence of \hat{E} is enough to form eigenmodes.

We show that it is not so. Let $\hat{G} = 0$ [effects of equilibrium currents], then manipulation of

Eq. a leads to

$$\hat{\Delta}^2 - \hat{E}^2 = \frac{\langle f(y) \left| \frac{d\psi_2}{dy} \right|^2 \rangle}{\langle f(y) |\psi_2|^2 \rangle} > 0 \quad (b)$$

We also deduce that $y \rightarrow \infty$, for the ψ_2 to decay we must require

$$\hat{E} - \hat{\Delta}^2 > 0 \quad (c)$$

Thus no ~~bound~~ square integrable modes are possible. Toroidicity may be (is) necessary ~~and~~ but ^{is} not sufficient.

Gap Modes, Existence of

We now switch G on. The condition c [$\hat{E}^2 - \delta^2 > 0$] remains unaltered, but b changes to

$$\delta^2 - \hat{E}^2 = \frac{\langle f | \frac{1}{2g} | \psi_1 \rangle^2}{\langle f | \psi_1^2 \rangle} - \frac{2G \langle f | \frac{1}{1+y^2} | \psi_1 \rangle + G^2 \langle f | \frac{1}{(1+y^2)^2} | \psi_1 \rangle}{\langle f | \psi_1^2 \rangle}$$

Thus \underline{b} and \underline{c} can be made compatible if and only if $G \neq 0$, i.e., the effects of the equilibrium current are essential to form gap modes in MHD.

Existence can also be granted by the kinetic effects, but then the mode is hardly distinguishable from the normal KAW.

It turns out the Gap Mode has an intermediate damping; somewhere inbetween the GAE and KAW.

Damping of the TAE (gap mode) has been a controversial subject. A proper understanding of damping is crucial for understanding the importance of these modes which are essential in any system where the mag. field has a two-dimensional variation.

Summary

There is a large variety of eigenmodes corresponding to the fundamental shear Alfvén Motion of an inhomogeneous plasma.

These modes depending on their structure (for example, local or global) ^{and} on their damping can be used as vehicles for ~~various~~ appropriate plasma applications.

Global modes (GAE, TAE) can be readily excited by external antennas and can be used to heat plasmas. Coronal heating by these modes may be of great relevance to students of the Sun and its Corona.

The destabilization (GAE, TAE) by Fusion α -particles in a fusion reactor may result in an anomalous loss of α particles with severe consequences for ignition (sustained).

Further understanding and application of these would surely clarify their consequences.