Vorticities in relativistic plasmas: from waves to reconnection

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- ***Part I: Waves in relativistic plasmas***
- ***Part II: Electro–Vortical formulation***
- ***Part III: Generalized Connetion and Reconnection***

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Part II:
ELECTRO-VORTICAL FORMULATION

- ***Unified model for relativistic plasmas***

Relativistic Plasma equations

- the rest-frame density of the fluid n.
- the energy density ϵ, pressure p, enthalpy density $h = \epsilon + p$, and temperature T.
- relativistic velocities and the Lorentz factor $\gamma = (1 - \mathbf{v}^2)^{-1/2}$.
- coupled to Maxwell equations via the current density $n\gamma\mathbf{v}$.

Plasma fluid equation

$$m\gamma \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) (f\gamma\mathbf{v}) = q\gamma (\mathbf{E} + \mathbf{v} \times \mathbf{B}) - \frac{1}{n} \nabla p$$

Continuity equation

$$\frac{\partial(\gamma n)}{\partial t} + \nabla \cdot (\gamma n \mathbf{v}) = 0$$

$$f \equiv \frac{h}{mn} = f(T)$$

And an equation of state for pressure and density. For an ideal relativistic gas $f = K_3(m/T)/K_2(m/T)$.

Relativistic plasma fluids - covariant form

It is better to work with the covariant formalism. In this case, equations are manifestly invariant under Lorentz transformations. For flat–spacetimes with $\eta_{\mu\nu} = (-1, 1, 1, 1)$, the plasma fluid 4-velocity $U^\mu = (\gamma, \gamma \mathbf{v})$ satisfies $U_\mu U^\mu = \eta_{\mu\nu} U^\mu U^\nu = -1$.

The equation for the plasma fluid is

$$U^\nu \partial_\nu \left(mf U^\mu \right) = q F^{\mu\nu} U_\nu - \frac{1}{n} \partial^\mu p$$

Also we have the continuity equation

$$\partial_\mu (n U^\mu) = 0$$

and Maxwell equations

$$\partial_\nu F^{\mu\nu} = q n U^\mu$$

Magnetofluid Unification[2]

Instead of solving the previous equations, let us look the big picture.

$$
\begin{aligned}
qF^{\mu\nu}U_\nu - \frac{1}{n}\partial^\mu p &= U^\nu \partial_\nu \left(mfU^\mu\right) \\
&= mU_\nu \left[\partial^\nu (fU^\mu) - \partial^\mu (fU^\nu)\right] + mU_\nu \partial^\mu (fU^\nu) \\
&= mU_\nu S^{\nu\mu} - m\partial^\mu f
\end{aligned}
$$

as $U_\nu \partial^\mu U^\nu = 0$ and

$$
S^{\mu\nu} = \partial^\mu (fU^\nu) - \partial^\nu (fU^\mu)
$$

[2] Mahajan PRL **90**, 035001 (2003); Mahajan & Yoshida, PoP **18**, 055701 (2011).

Magnetofluid Unification

The covariant fluid equation can be cast in the form

$$qU_\nu M^{\mu\nu} = -T\partial^\mu \sigma$$

where the magnetofluid tensor is

$$M^{\mu\nu} = F^{\mu\nu} + \frac{m}{q} S^{\mu\nu}$$

and the entropy density follows

$$-\partial^\mu \sigma = \frac{1}{nT} \left(\partial^\mu p - mn\partial^\mu f\right)$$

Magnetofluid tensor (why is important)

$$M^{\mu\mu} \equiv 0$$

$$M^{0i} \to \xi = \mathbf{E} - \frac{m}{q}\partial_t(f\gamma\mathbf{v}) - \frac{m}{q}\nabla(f\gamma)$$

$$M^{ij} \to \Omega = \mathbf{B} + \frac{m}{q}\nabla\times(f\gamma\mathbf{v})$$

The magnetofluid tensor is the natural extension to the covariant form of the plasma vorticity.

Equation $qU_\nu M^{\mu\nu} = T\partial^\mu\sigma$ is the covariant vorticity equation for the plasma.

$$(\text{For } \mu = 0) \Longrightarrow \mathbf{v}\cdot\xi = -\frac{T}{q\gamma}\frac{\partial\sigma}{\partial t}$$

$$(\text{For } \mu = i) \Longrightarrow \xi + \mathbf{v}\times\Omega = \frac{T}{q\gamma}\nabla\sigma$$

Defining the potential (generalized canonical momentum)

$$\mathcal{P}^\mu = A^\mu + \frac{m}{q} f U^\mu = (\mathcal{P}^0, \mathcal{P})$$

then

$$M^{\mu\nu} = \partial^\mu \mathcal{P}^\nu - \partial^\nu \mathcal{P}^\mu$$

In this way

$$\xi = -\frac{\partial \mathcal{P}}{\partial t} - \nabla \mathcal{P}^0 , \qquad \Omega = \nabla \times \mathcal{P}$$

$$\Longrightarrow \nabla \times \xi = -\frac{\partial \Omega}{\partial t} \Longleftrightarrow \frac{1}{2} \varepsilon_{\alpha\beta\mu\nu} \partial^\beta M^{\mu\nu} = 0$$

$$(\text{For } \mu = 0) \Longrightarrow \mathbf{v} \cdot \xi = -\frac{T}{q\gamma} \frac{\partial \sigma}{\partial t}$$

$$(\text{For } \mu = i) \Longrightarrow \frac{\partial \mathcal{P}}{\partial t} - \mathbf{v} \times \Omega = -\frac{T}{q\gamma} \nabla \sigma - \nabla \mathcal{P}^0$$

This last equation is the potential equation for the vortical dynamics!

Relativistic Electro-Vortic (EV) field[3]

From $qU_\nu M^{\mu\nu} = T\partial^\mu\sigma$ we have the conservation law

$$U_\mu \partial^\mu \sigma = 0$$

Now consider the new EV field

$$\mathcal{M}^{\mu\nu} = \partial^\mu \Pi^\nu - \partial^\nu \Pi^\mu$$

$$\Pi^\mu = A^\mu + \frac{mf}{q} U^\mu + \chi^\mu$$

requiring that

$$U_\nu \mathcal{M}^{\mu\nu} = 0$$

[3] S. M. Mahajan, Phys. Plasmas **23**, 112104 (2016).

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$$U_\nu \mathcal{M}^{\mu\nu} = 0$$

$$U_\nu \mathcal{M}^{\mu\nu} = U_\nu M^{\mu\nu} + U_\nu \partial^\mu \chi^\nu - U_\nu \partial^\nu \chi^\mu = -\frac{T}{q}\partial^\mu \sigma + U_\nu \partial^\mu \chi^\nu - U_\nu \partial^\nu \chi^\mu$$

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Relativistic Electro-Vortic (EV) field

$$-\frac{T}{q}\partial^\mu\sigma + U_\nu\partial^\mu\chi^\nu - U_\nu\partial^\nu\chi^\mu = 0$$

The solution for the EV field comes from a field in the Clebsh form

$$\chi^\mu = \sigma\partial^\mu\phi$$

In this case we get

$$-\frac{T}{q}\partial^\mu\sigma + U_\nu\partial^\mu\sigma\partial^\nu\phi - U_\nu\partial^\nu\sigma\partial^\mu\phi = 0$$

$$\left(-\frac{T}{q} + U_\nu\partial^\nu\phi\right)\partial^\mu\sigma = 0$$

The simplest solution is

$$U_\nu\partial^\nu\phi = \frac{T}{q}$$

Relativistic Electro-Vortic (EV) field. A TOTAL UNIFIED FIELD

Then, we find a general unified form for relativistic plasmas

$$U_\nu \partial^\nu \phi = \frac{T}{q}$$

$$\Pi^\mu = qA^\mu + \frac{mf}{q} U^\mu + \sigma \partial^\mu \phi \,; \quad \mathcal{M}^{\mu\nu} = \partial^\mu \Pi^\nu - \partial^\nu \Pi^\mu \,; \quad U_\nu \mathcal{M}^{\mu\nu} = 0$$

$$\mathcal{M}^{0i} \equiv \Xi = \mathbf{E} - \frac{m}{q} \partial_t (\hat{f} \gamma \mathbf{v}) - \frac{m}{q} \nabla (\hat{f} \gamma) - \partial_t \sigma \nabla \phi - \nabla \sigma \partial_t \phi$$

$$\mathcal{M}^{ij} \equiv \Psi = \mathbf{B} + \frac{m}{q} \nabla \times (\hat{f} \gamma \mathbf{v}) + \nabla \sigma \times \nabla \phi$$

$$\mathbf{v} \cdot \Xi = 0 \,; \qquad \Xi + \mathbf{v} \times \Psi = 0$$

Ξ is the effective electric field and Ψ is the effective magnetic/vortical field. They fulfill an “Ohm’s law”.

Steady-state system (except for ϕ)

$$-\gamma\partial_t\phi + \gamma\mathbf{v}\cdot\nabla\phi = \frac{T}{q} \rightarrow \partial_t\phi = -\frac{\gamma T}{q}\,; \quad \nabla\phi = -\frac{\gamma T\mathbf{v}}{q}$$

$$\Xi + \mathbf{v}\times\Psi = 0$$

$$\Xi = \mathbf{E} - \frac{m}{q}\nabla(\hat{f}\gamma)$$

$$\Psi = \mathbf{B} + \frac{m\gamma}{q}\nabla\hat{f}\times\mathbf{v}$$

$$\hat{f} = f - \frac{T\sigma}{m}$$

We also need Maxwell equations

$$\nabla\times\mathbf{B} = 4\pi qn\gamma\mathbf{v}$$

“Superconducting” state

Generalization of the London equation. Expulsion of the vorticity

$$\Xi = 0\,; \quad \mathbf{E} = 0\,; \quad \nabla(\hat{f}\gamma) = 0\,; \quad \Psi = 0 = \mathbf{B} + \frac{m\gamma}{q}\nabla\hat{f} \times \mathbf{v}$$

$$\frac{1}{4\pi q n}\nabla \times \mathbf{B} = \gamma \mathbf{v}$$

$$0 = \frac{4\pi q^2 n}{m}\mathbf{B} + \nabla\hat{f} \times (\nabla \times \mathbf{B})$$

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$$0 = \frac{4\pi q^2 n}{m}\mathbf{B} + \nabla\hat{f} \times (\nabla \times \mathbf{B})$$

In 1D, such that $\nabla = \hat{e}_x d/dx$ and $\hat{e}_x \cdot \mathbf{B} = 0$, we find the solution

$$\mathbf{B} = \mathbf{B}_0 \exp\left[\int_0^x \frac{dx'}{\lambda^2 \frac{d\ln\hat{f}}{dx'}}\right]$$

where $\lambda = \lambda_0\sqrt{n_0\hat{f}/n}$, with the ambient density n_0 and the skin depth $\lambda_0 = \sqrt{4\pi n_0 q^2/m}$.

Super–Beltrami equilibrium

$$\Xi = 0\,; \quad \mathbf{E} = 0\,; \quad \nabla(\hat{f}\gamma) = 0\,; \quad \Psi = \alpha n \gamma \mathbf{v}$$

with α constant such that $\nabla \cdot \Psi = 0$

Super–Beltrami equilibrium

$$\Xi = 0\,; \quad \mathbf{E} = 0\,; \quad \nabla(\hat{f}\gamma) = 0\,; \quad \Psi = \alpha n\gamma\mathbf{v}$$

with α constant such that $\nabla \cdot \Psi = 0$

$$\alpha n\gamma\mathbf{v} = \mathbf{B} + \frac{m\gamma}{q}\nabla\hat{f} \times \mathbf{v}$$

$$\frac{qn\alpha}{m}\nabla \times \mathbf{B} = \frac{4\pi q^2 n}{m}\mathbf{B} + \nabla\hat{f} \times (\nabla \times \mathbf{B})$$

The typical Beltrami state is $\nabla \times \mathbf{B} = \beta\mathbf{B}$

Super–Beltrami equilibrium. 1D solution

$$\frac{qn\alpha}{m}\nabla \times \mathbf{B} = \frac{4\pi q^2 n}{m}\mathbf{B} + \nabla\hat{f} \times (\nabla \times \mathbf{B})$$

For $\nabla = \hat{e}_x d/dx$ and $\hat{e}_x \cdot \mathbf{B} = 0$, we can find

$$\mathbf{B} = B_0 \left[\cos\left(\int_0^x k_R dx'\right)\hat{e}_y + \sin\left(\int_0^x k_R dx'\right)\hat{e}_z\right]\exp\left(\int_0^x k_I dx'\right)$$

$$k_R = \frac{\alpha\lambda_0^2}{\alpha\lambda_0^2 + \left(\lambda^2 \frac{d\ln\hat{f}}{dx'}\right)^2}$$

$$k_I = \frac{\lambda^2 \frac{d\ln\hat{f}}{dx'}}{\alpha\lambda_0^2 + \left(\lambda^2 \frac{d\ln\hat{f}}{dx'}\right)^2}$$

A more general solution

$$\mathbf{E} = 0\,; \quad \Xi = -\frac{m}{q}\nabla(\hat{f}\gamma)$$

$$\Psi = \alpha n\gamma\mathbf{v} - \frac{m}{q}\frac{\mathbf{v}\times\nabla(\hat{f}\gamma)}{\mathbf{v}\cdot\mathbf{v}}$$

the new term must be divergenceless (this at least is achieved in 1D case for $\hat{e}_x \cdot \mathbf{v} = 0$ and $\nabla = \hat{e}_x d_x$)

A more general solution

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the new term must be divergenceless (this at least is achieved in 1D case for $\hat{e}_x \cdot \mathbf{v} = 0$ and $\nabla = \hat{e}_x d_x$)
This implies that

$$\alpha n\gamma\mathbf{v} - \frac{m}{q}\frac{\mathbf{v}\times\nabla(\hat{f}\gamma)}{\mathbf{v}\cdot\mathbf{v}} = \mathbf{B} + \frac{m}{q}\nabla\hat{f}\times\gamma\mathbf{v}$$

$$\nabla\times\mathbf{B} = 4\pi q n\gamma\mathbf{v}$$

A more general solution

From Maxwell equation

$$\gamma \mathbf{v} = \frac{\nabla \times \mathbf{B}}{4\pi q n} \rightarrow \gamma = \sqrt{1 + \frac{|\nabla \times \mathbf{B}|^2}{16\pi^2 q^2 n^2}}$$

A more general solution

From Maxwell equation

$$\gamma \mathbf{v} = \frac{\nabla \times \mathbf{B}}{4\pi q n} \to \gamma = \sqrt{1 + \frac{|\nabla \times \mathbf{B}|^2}{16\pi^2 q^2 n^2}}$$

Then we find

$$\frac{\alpha q n}{m} \nabla \times \mathbf{B} = \frac{4\pi q^2 n}{m} \mathbf{B} + \nabla \hat{f} \times (\nabla \times \mathbf{B})$$
$$+ \frac{16\pi^2 q^2 n^2}{|\nabla \times \mathbf{B}|^2} \sqrt{1 + \frac{|\nabla \times \mathbf{B}|^2}{16\pi^2 q^2 n^2}} (\nabla \times \mathbf{B}) \times \nabla \left(\hat{f} \sqrt{1 + \frac{|\nabla \times \mathbf{B}|^2}{16\pi^2 q^2 n^2}} \right)$$

In 1D, with $\hat{e}_x \cdot \mathbf{B} = 0$, and in the non–relativistic regime $4\pi q n \gg |d\mathbf{B}/dx|$, we have

$$\frac{q n \alpha}{m} \hat{e}_x \times \frac{d\mathbf{B}}{dx} = \frac{4\pi q^2 n}{m} \mathbf{B} - \left(1 - \frac{16\pi^2 q^2 n^2}{|d\mathbf{B}/dx|^2} \right) \frac{d\hat{f}}{dx} \frac{d\mathbf{B}}{dx}$$

That’s all (for now).
Thanks!