Nuclear Structure
(II) Collective models

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NSDD Workshop, Trieste, October 2018

Collective nuclear models

(Rigid) rotor model

(Harmonic quadrupole) vibrator model

Liquid-drop model of vibrations and rotations

Interacting boson model

Rotation of a symmetric top

Energy spectrum:

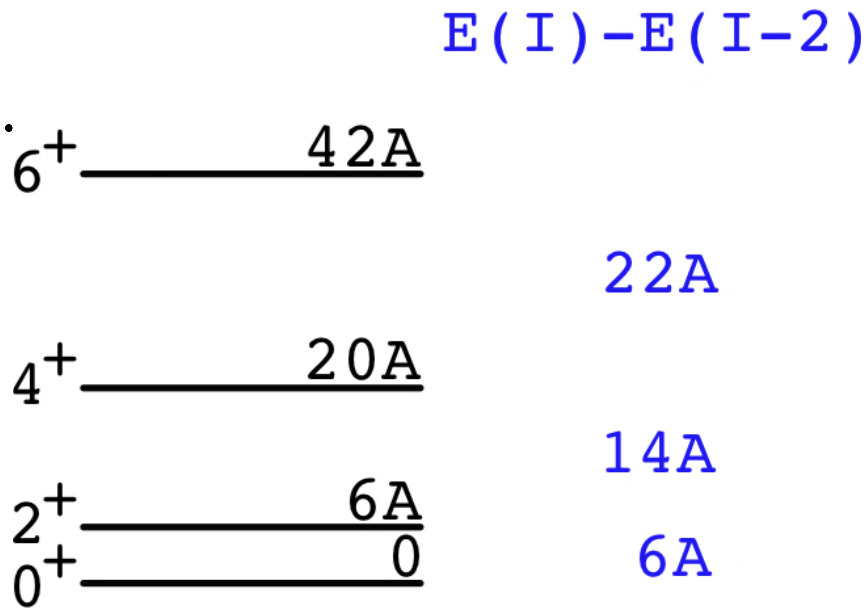
$$E_{\text{rot}}(I) = \frac{\hbar^2}{2\mathfrak{I}} I(I+1)$$

$$\equiv A\,I(I+1), \quad I^\pi = 0^+, 2^+, \ldots$$

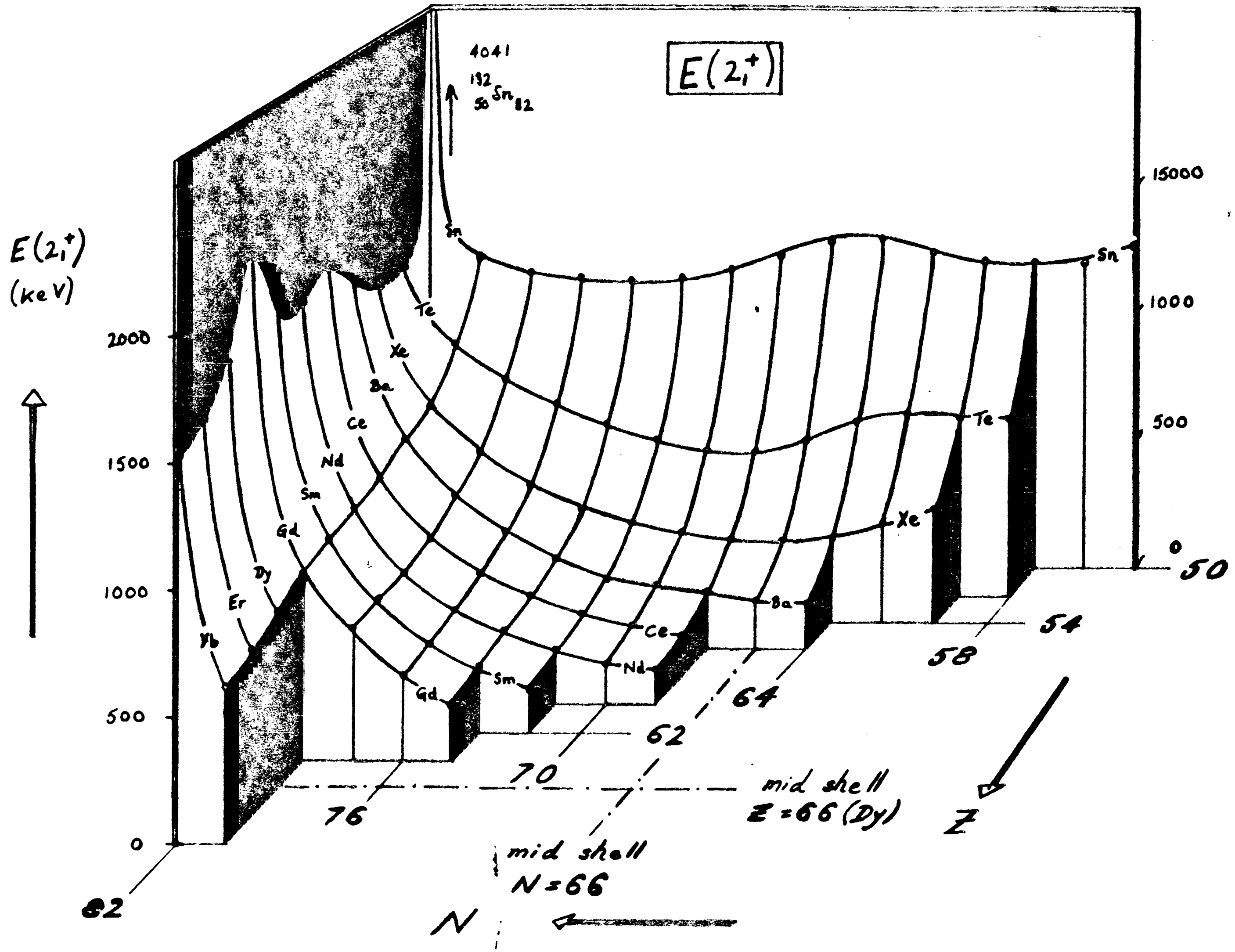
Large deformation $\Rightarrow$ large $\mathfrak{I} \Rightarrow$ low $E_x(2^+)$.

R_{42} energy ratio:

$$E_{\text{rot}}(4^+)/E_{\text{rot}}(2^+) = 3.333\ldots$$

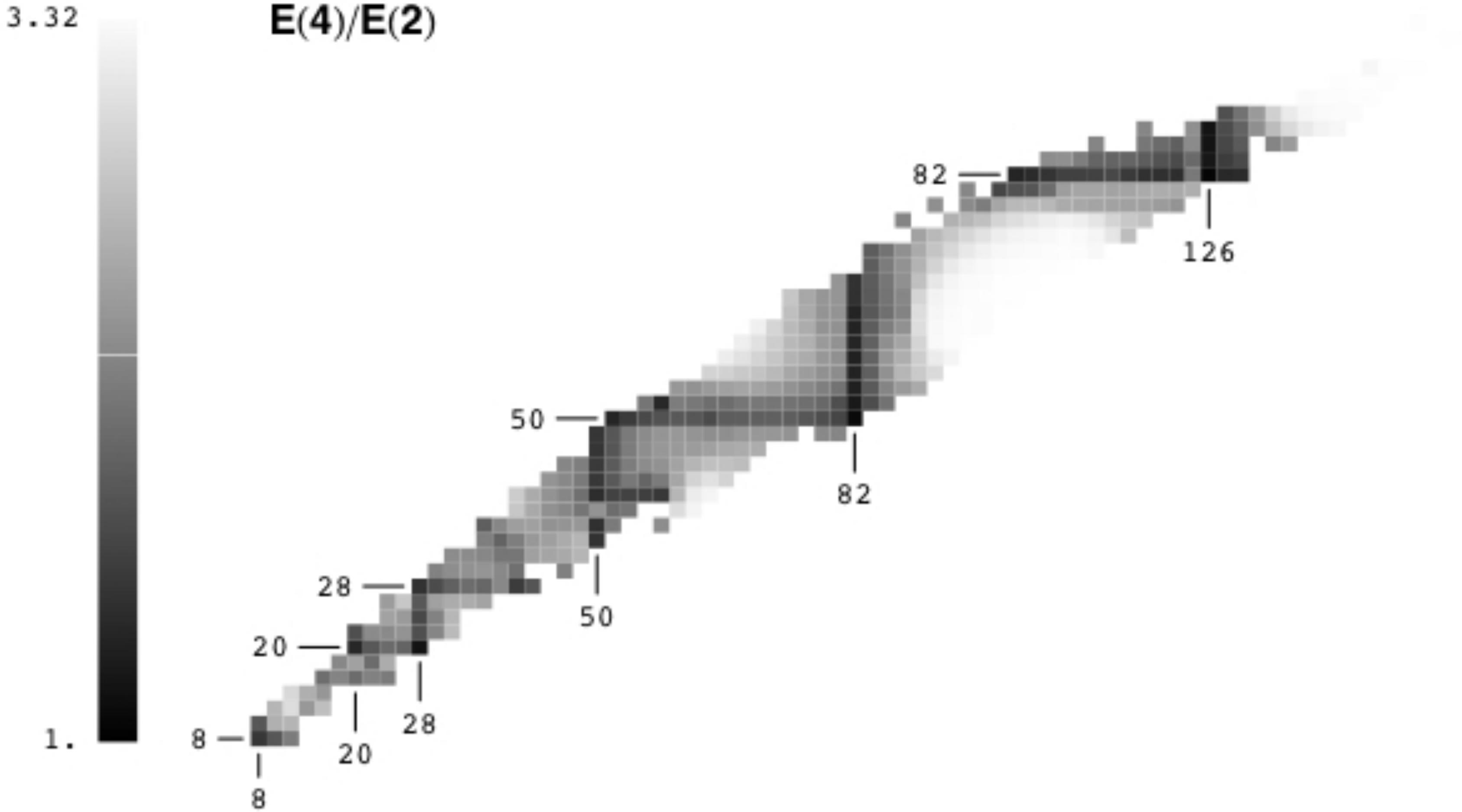


Evolution of $E_x(2^+)$



J.L. Wood, private communication

The ratio R_{42}



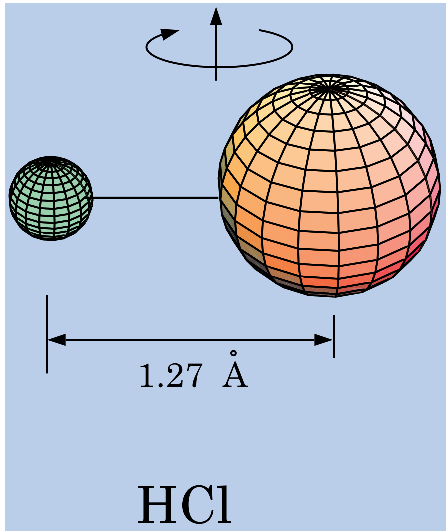
Rotation of an asymmetric top

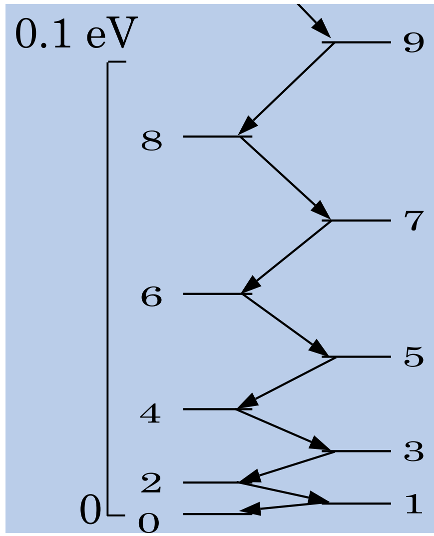
Energy spectrum:

$$E_{\text{rot}}\left(I^{\pi}\right)=\frac{\hbar^{2}}{2\Im}I\left(I+1\right)$$

$$I^{\pi}=0^{+},1^{-},2^{+},3^{-},4^{+},\ldots$$

Reflection symmetry only allows even I with positive parity π.





Nuclear shapes

Shapes can be characterized by variables $\alpha_{\lambda\mu}$ in a surface parameterization:

$$R(\theta,\varphi) = R_0\left(1 + \sum_{\lambda}\sum_{\mu=-\lambda}^{+\lambda}\alpha_{\lambda\mu}Y_{\lambda\mu}^{*}(\theta,\varphi)\right)$$

λ=0: compression (high energy)

λ=1: translation (not an intrinsic deformation)

λ=2: quadrupole deformation

λ=3: octupole deformation

Lord Rayleigh, Proc. Roy. Soc. (London) **29** (1879) 71

Quadrupole shapes

Since the surface $R(\theta,\varphi)$ is real: $\left(\alpha_{\lambda\mu}\right)^{*} = (-1)^{\mu}\alpha_{\lambda-\mu}$

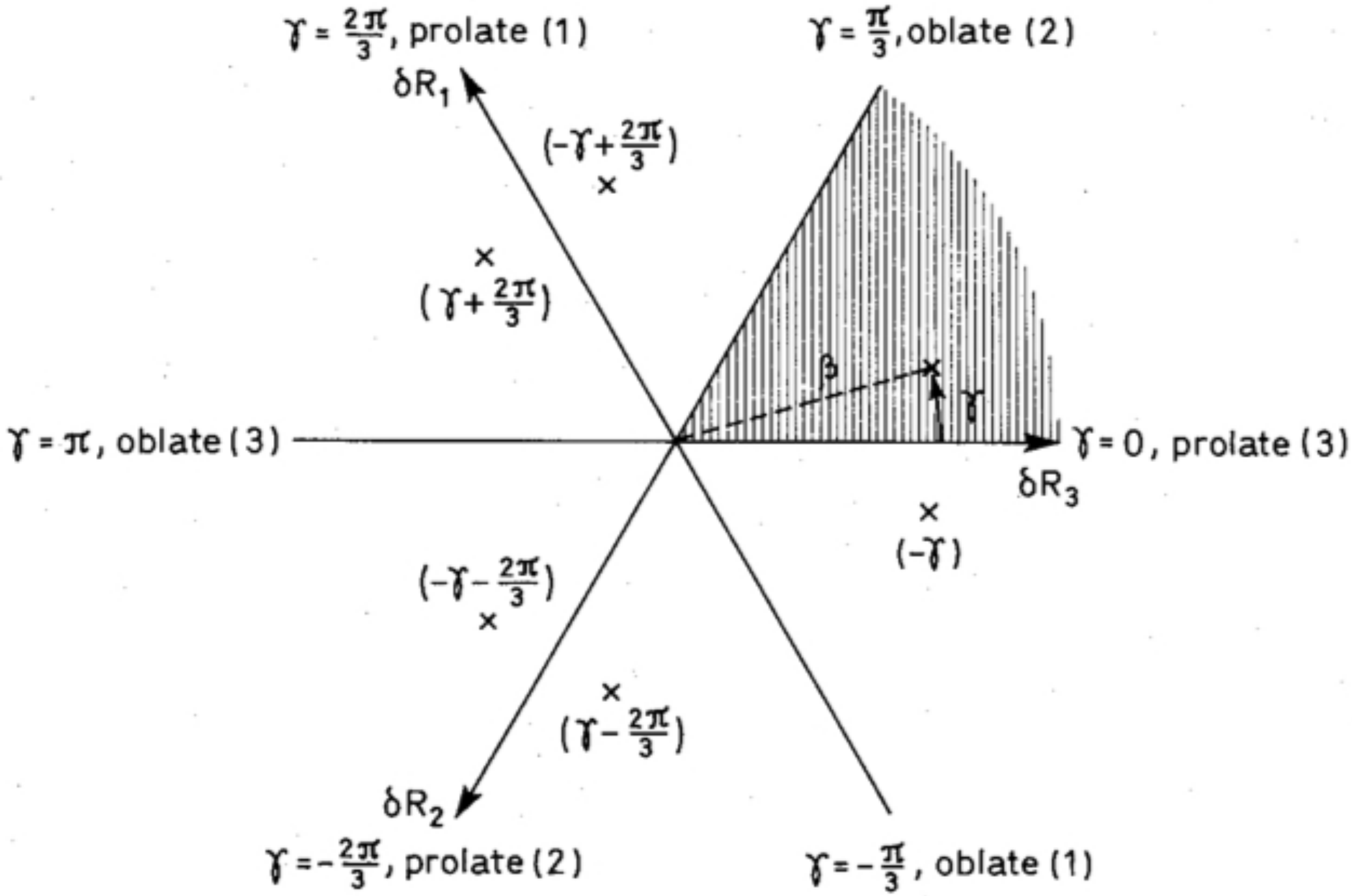
➔ Five independent quadrupole variables (λ=2).

Equivalent to three Euler angles and two intrinsic variables β and γ:

$$\alpha_{2\mu} = \sum_{\nu} a_{2\nu} D^{2}_{\mu\nu}(\Omega), \quad a_{21} = a_{2-1} = 0, \quad a_{22} = a_{2-2}$$

$$a_{20} = \beta\cos\gamma, \quad a_{22} = \frac{1}{\sqrt{2}}\beta\sin\gamma$$

The (β,γ) plane



Modes of nuclear vibration

Nucleus is considered as a droplet of nuclear matter with an equilibrium shape. Vibrations are modes of excitation around that shape.

Character of vibrations depends on symmetry of equilibrium shape. Two important cases in nuclei:

- *Spherical equilibrium shape*
- *Spheroidal equilibrium shape*

A. Bohr, K. Dansk. Vidensk. Mat.-Fys. Meddr. **26** (1952) no 14
A. Bohr & B. R. Mottelson, K. Dansk. Vidensk. Mat.-Fys. Meddr. **27** (1953) no 16

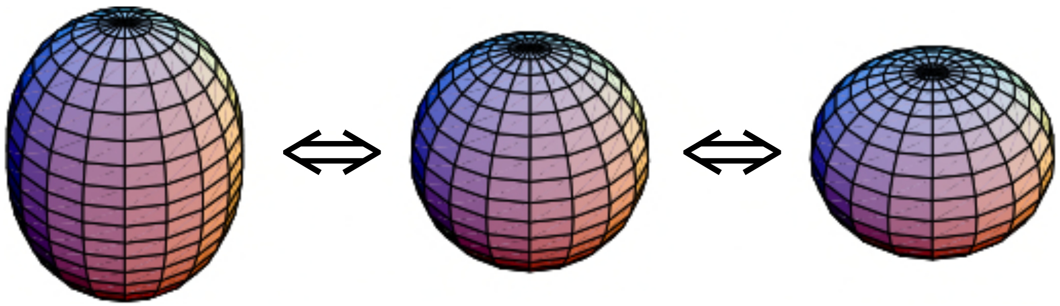
Vibrations about a spherical shape

Vibrations are characterized by λ in the surface parameterization:

λ=0: compression (high energy)

λ=1: translation (not an intrinsic excitation)

λ=2: quadrupole vibration



λ=3: octupole vibration

Spherical quadrupole vibrations

Energy spectrum:

$$E_{\mathrm{vib}}(n) = \left(n + \tfrac{5}{2}\right)\hbar\omega,\ n = 0,1\ldots$$

R_{42} energy ratio:

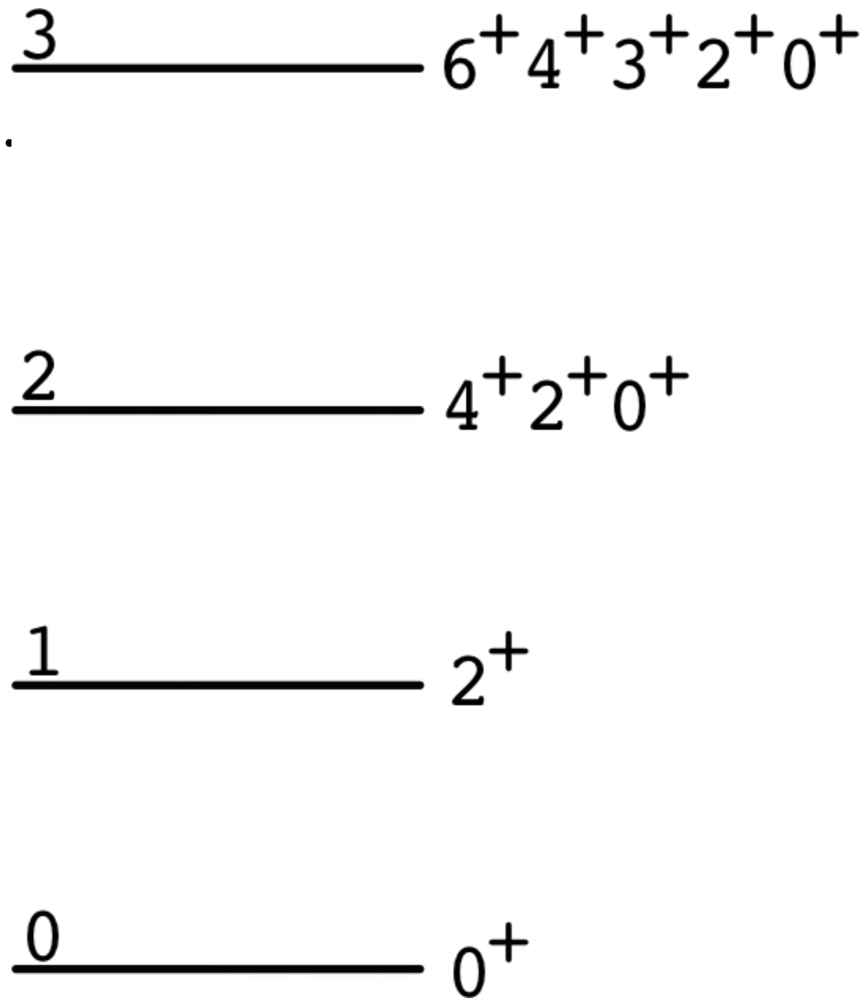
$$E_{\mathrm{vib}}\left(4^+\right)/E_{\mathrm{vib}}\left(2^+\right) = 2$$

E2 transitions:

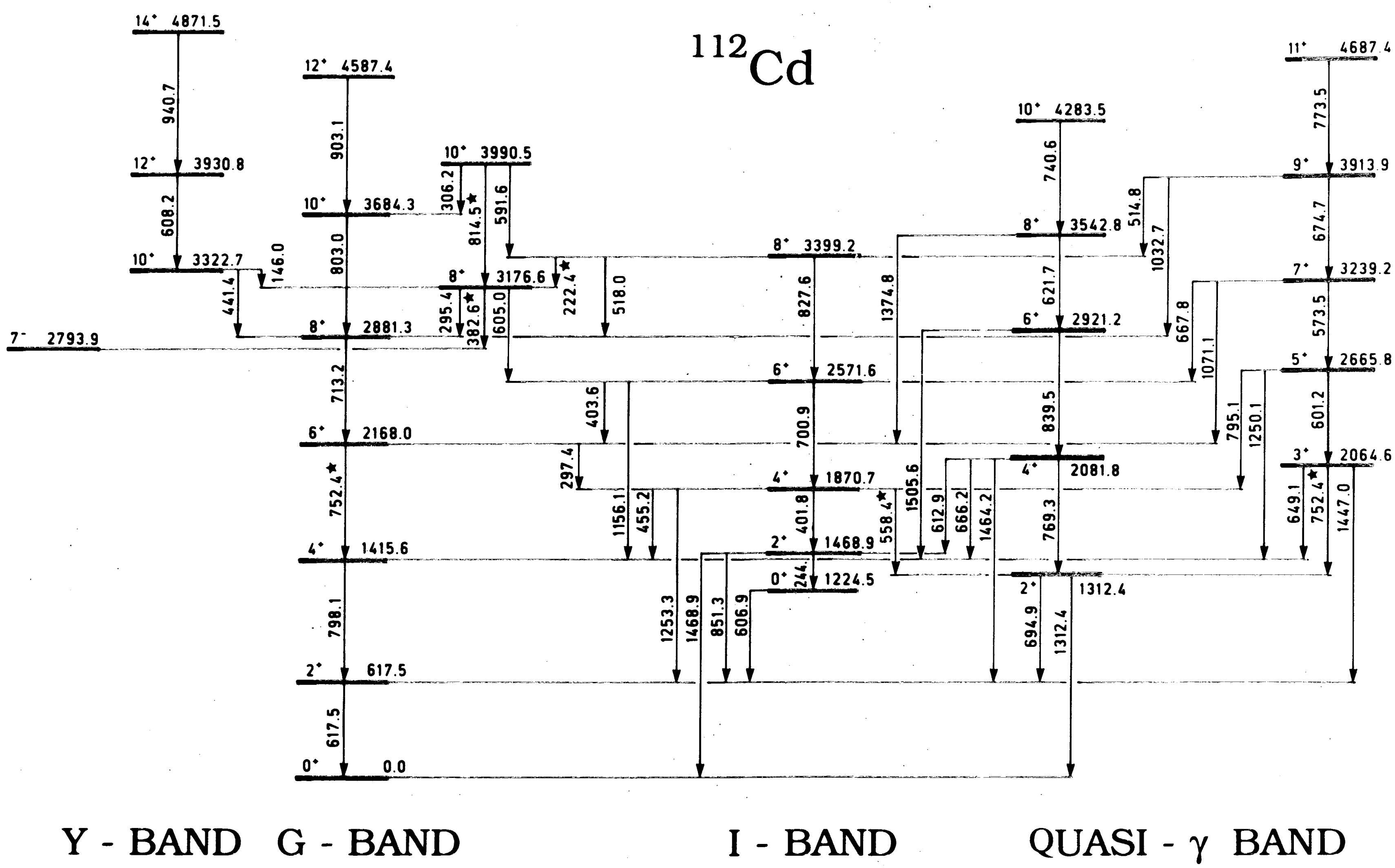
$$B\left(\mathrm{E2}; 2_1^+ \rightarrow 0_1^+\right) = \alpha^2$$

$$B\left(\mathrm{E2}; 2_2^+ \rightarrow 0_1^+\right) = 0$$

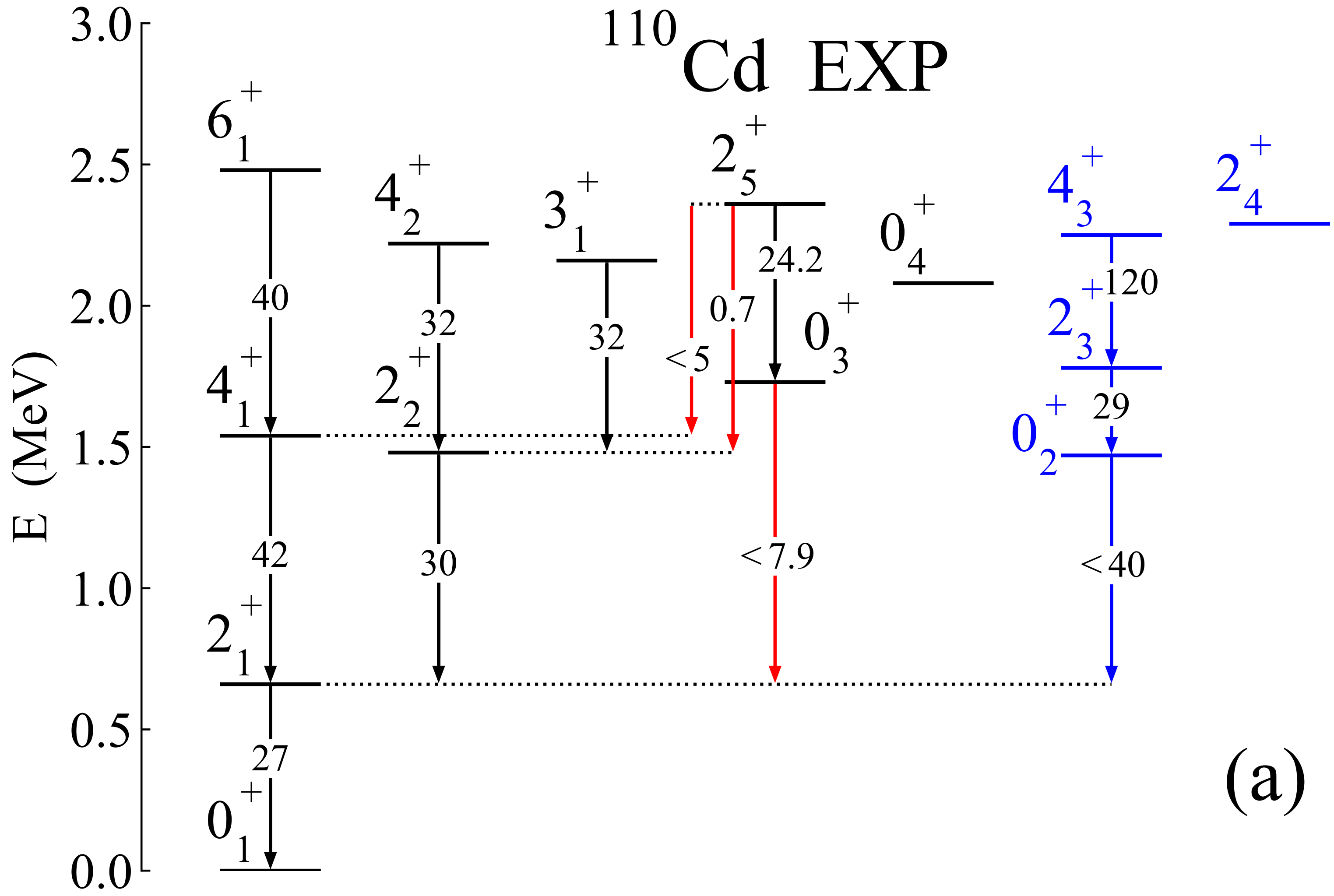
$$B\left(\mathrm{E2}; n = 2 \rightarrow n = 1\right) = 2\alpha^2$$



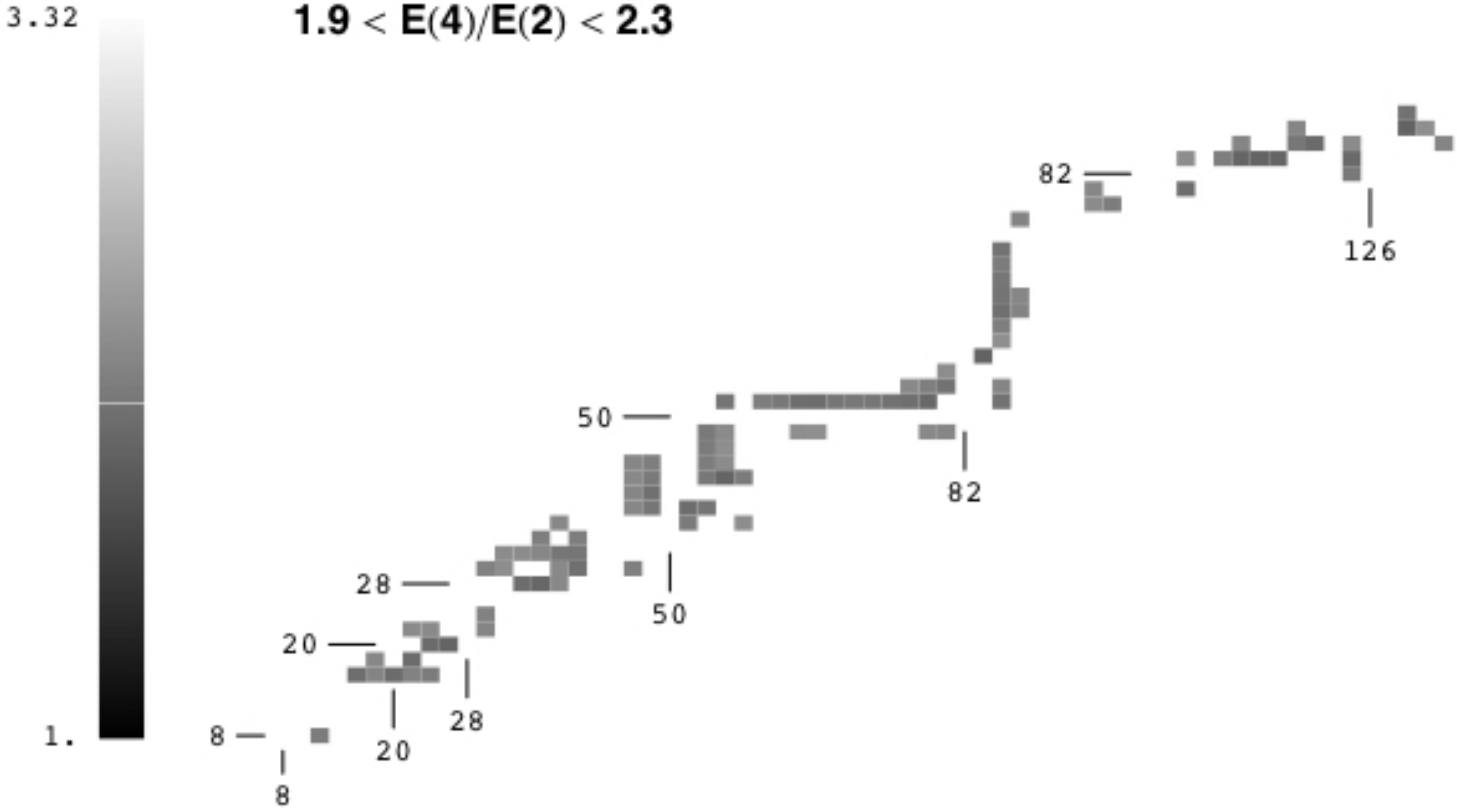
Example of ^{112}Cd



Example of ^{110}Cd



Possible vibrational nuclei from R_{42}

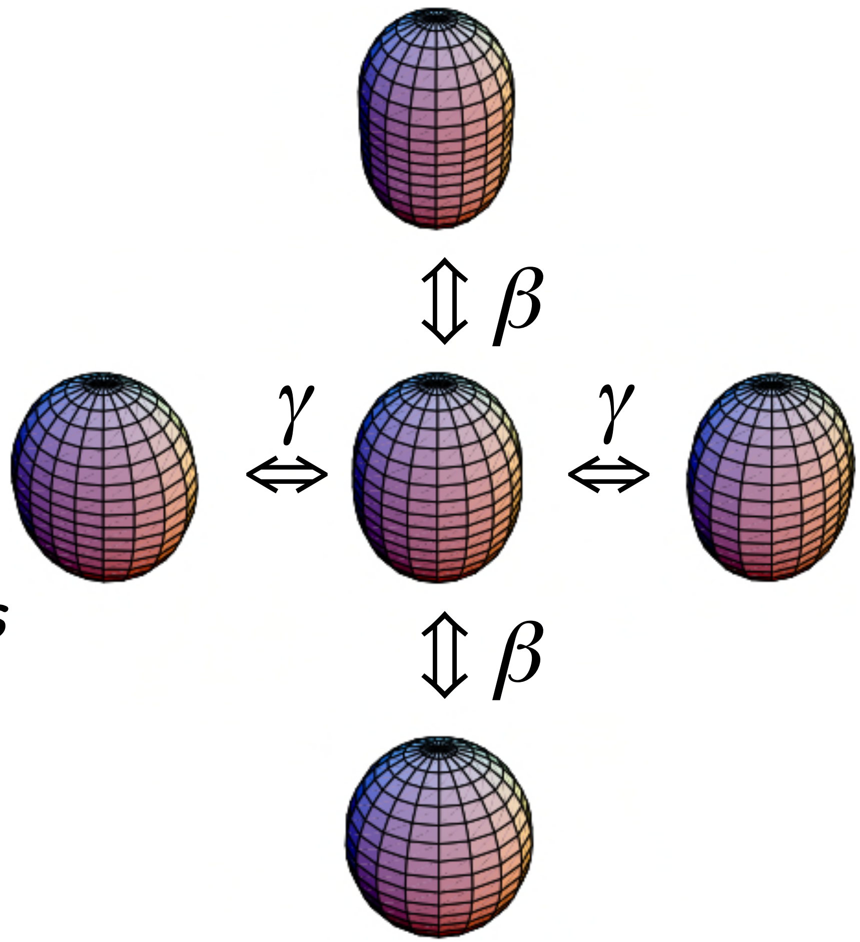


Spheroidal quadrupole vibrations

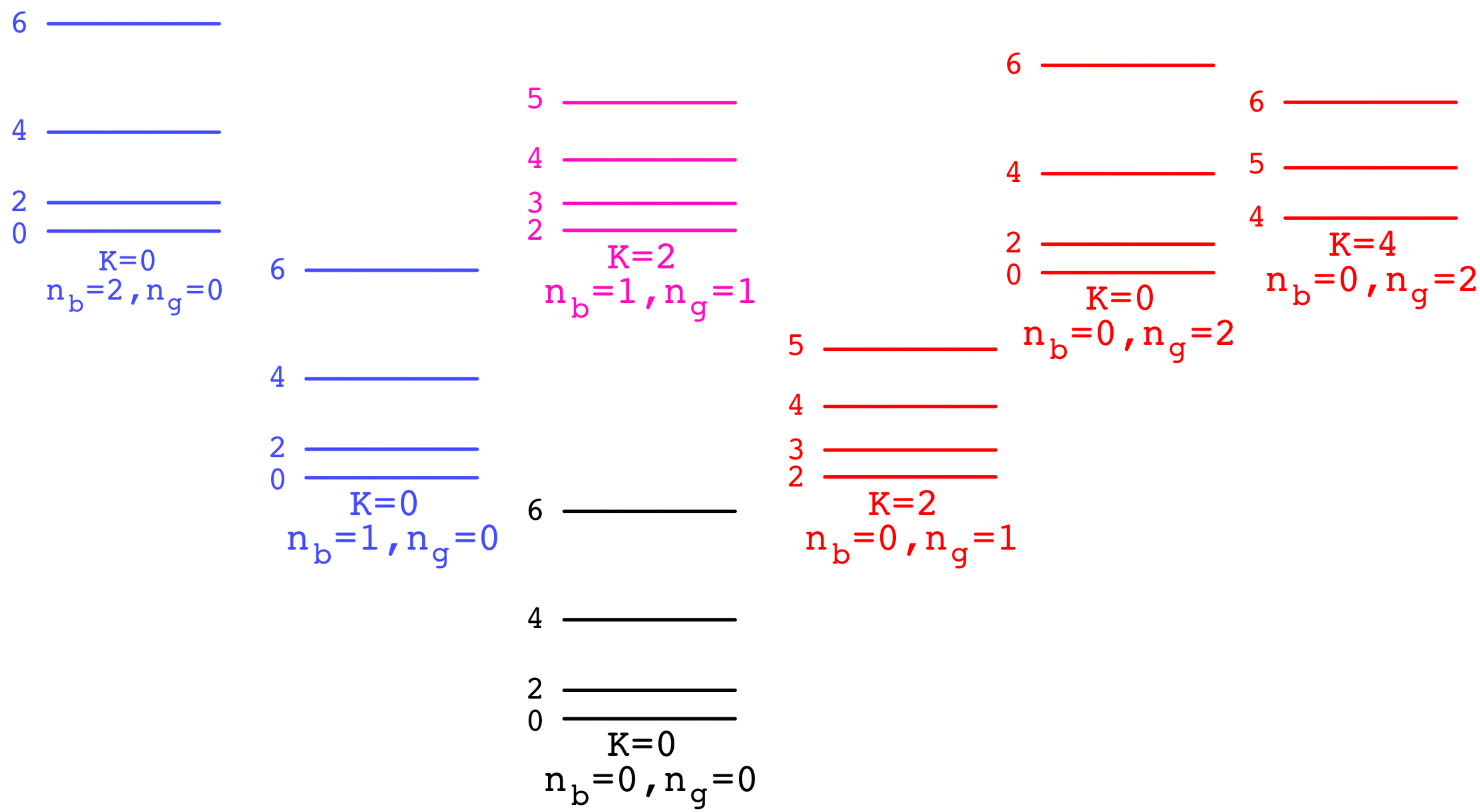
The vibration of a shape with axial symmetry is characterized by $a_{\lambda\nu}$.

Quadrupole oscillations:

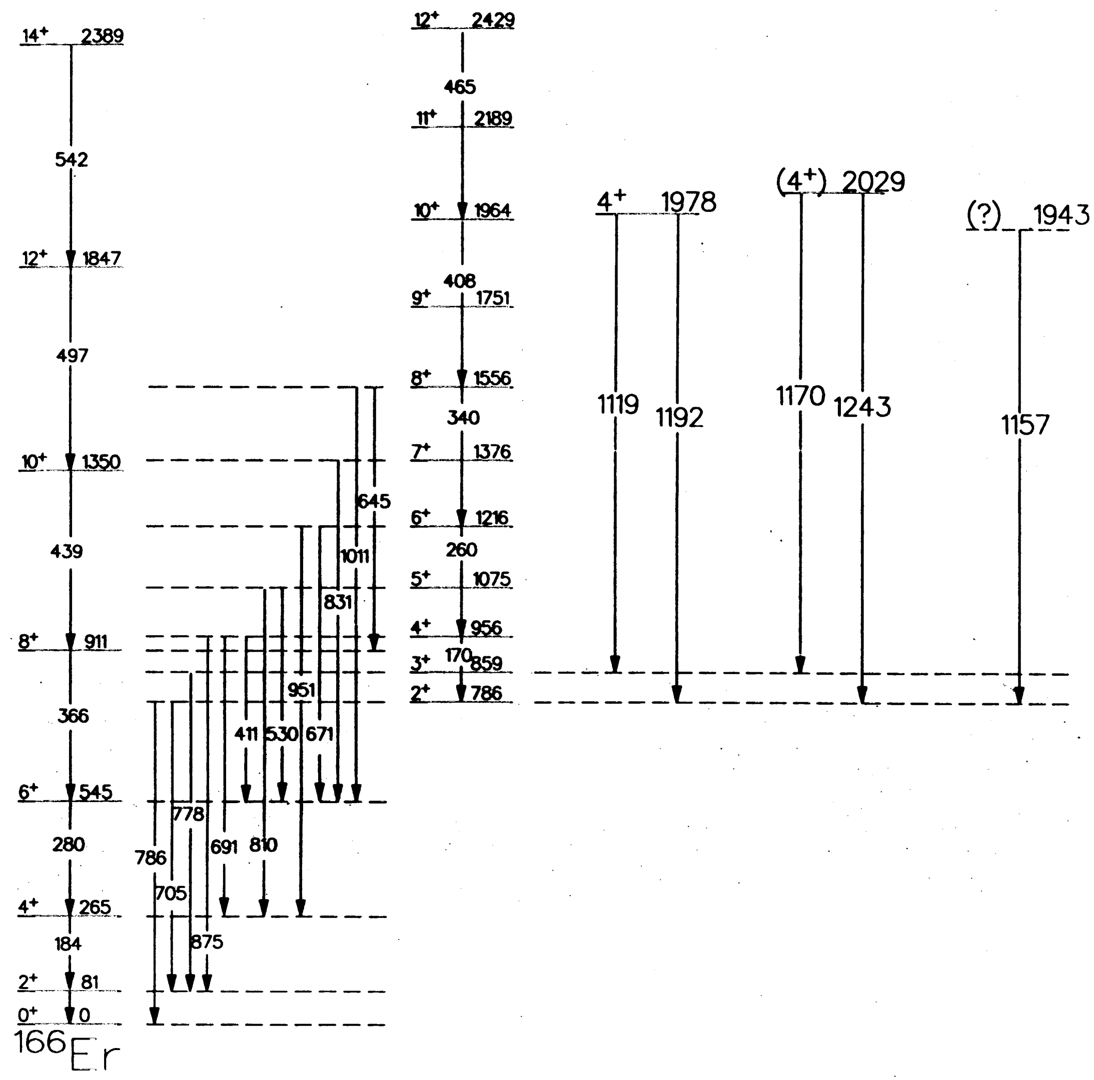
- *ν=0: along the axis of symmetry (β)*
- *ν=±1: spurious rotation*
- *ν=±2: perpendicular to axis of symmetry (γ)*



Spectrum of spheroidal vibrations



Example of ^{166}Er

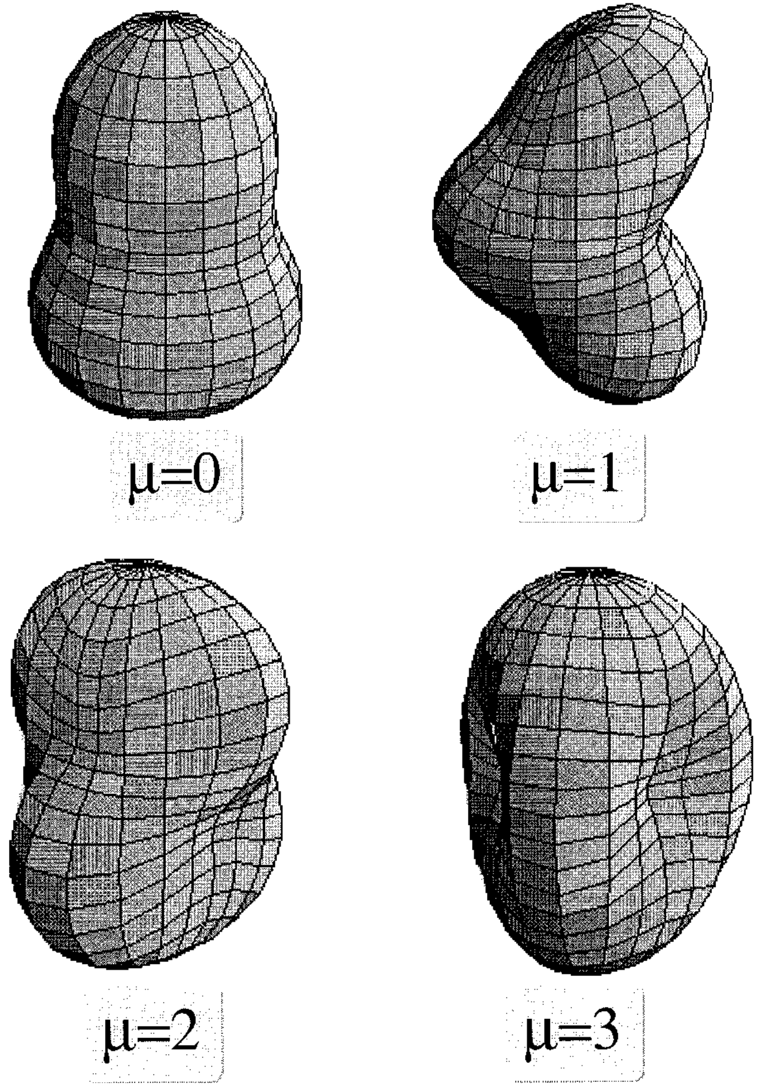


Quadrupole-octupole shapes

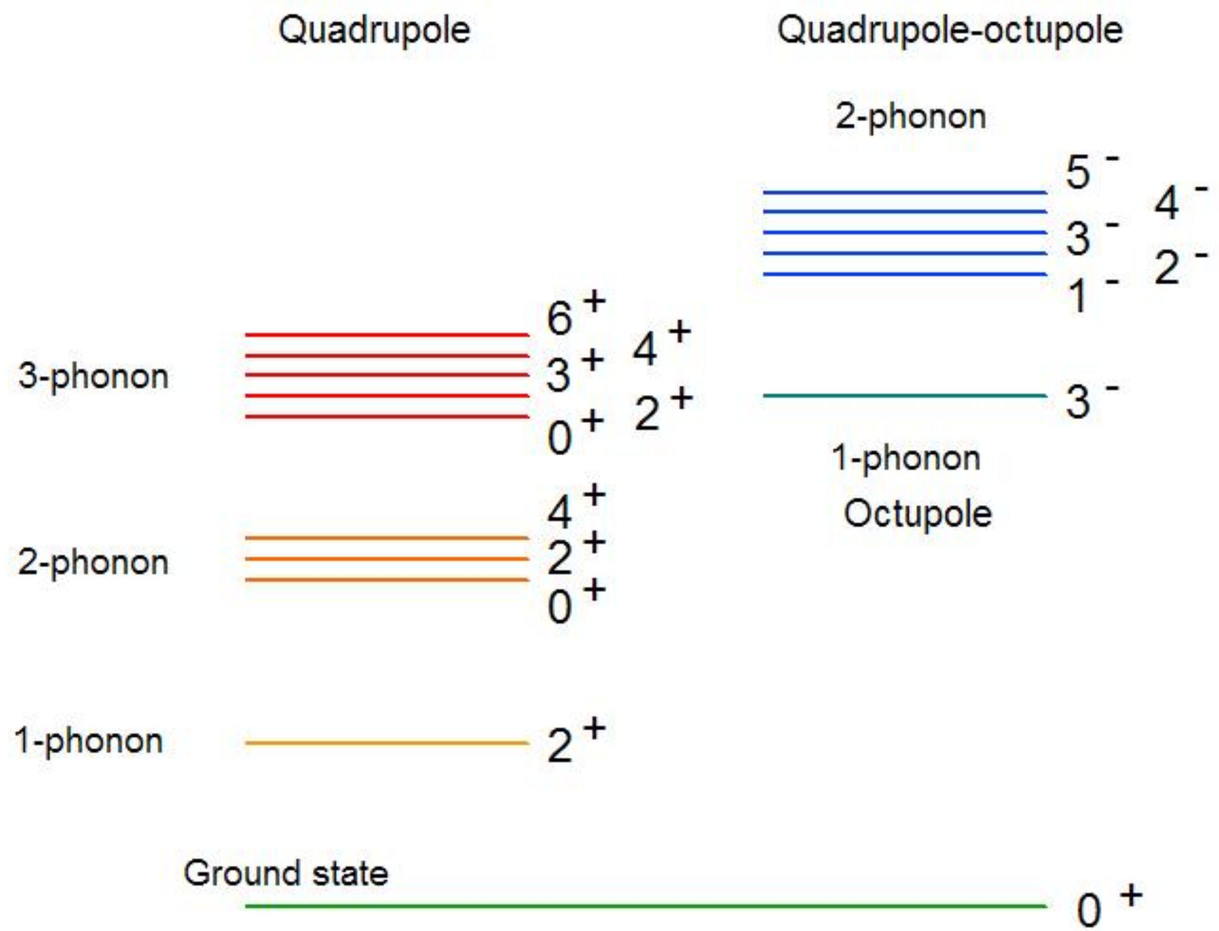
It is difficult to define an intrinsic frame for a pure octupole shape.

Quadrupole-octupole: use quadrupole frame ➔ two quadrupole and seven octupole intrinsic variables $\alpha_{3\mu}$.

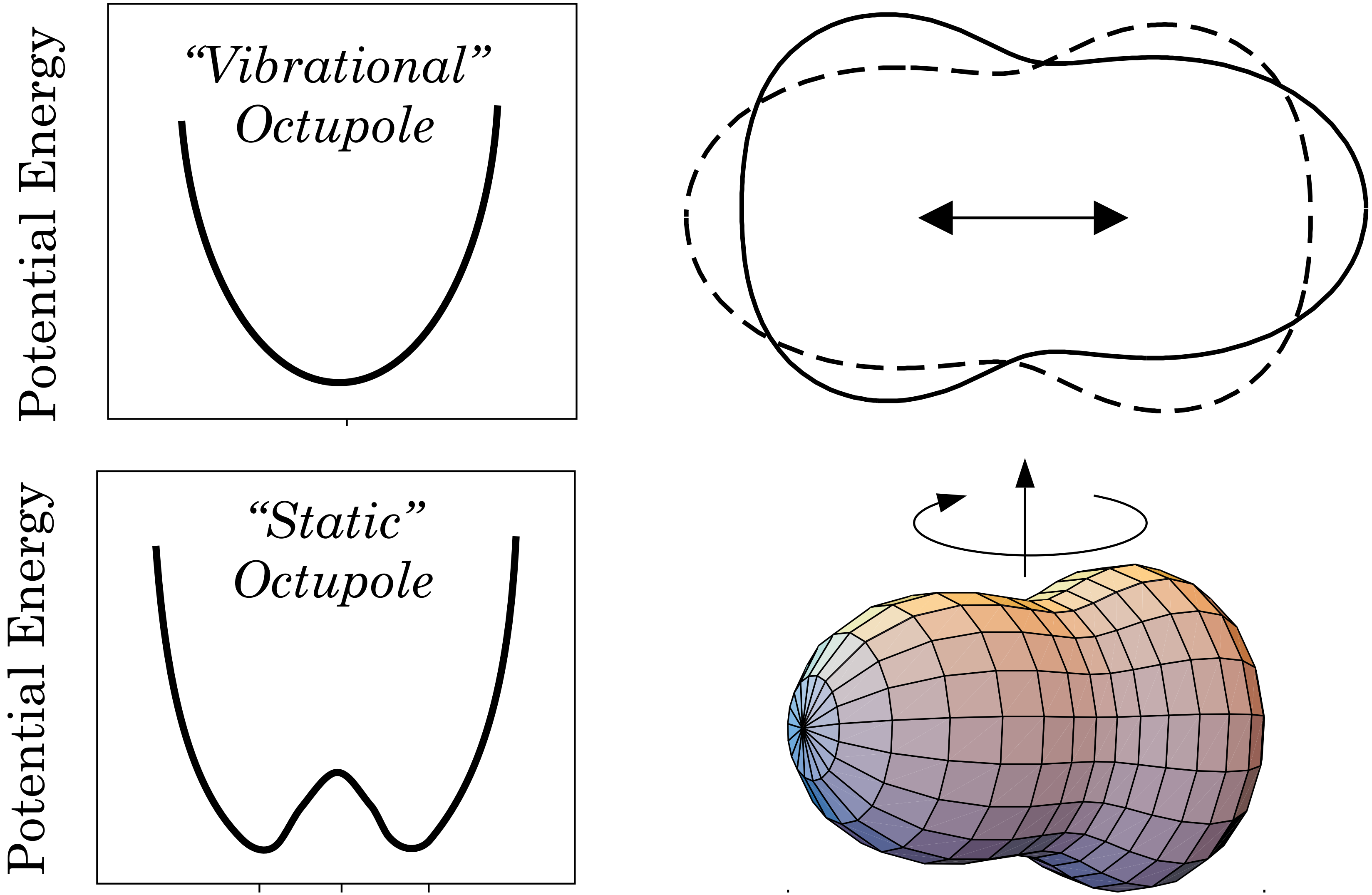
Most important case: β_3=α_{30} (axial symmetry).



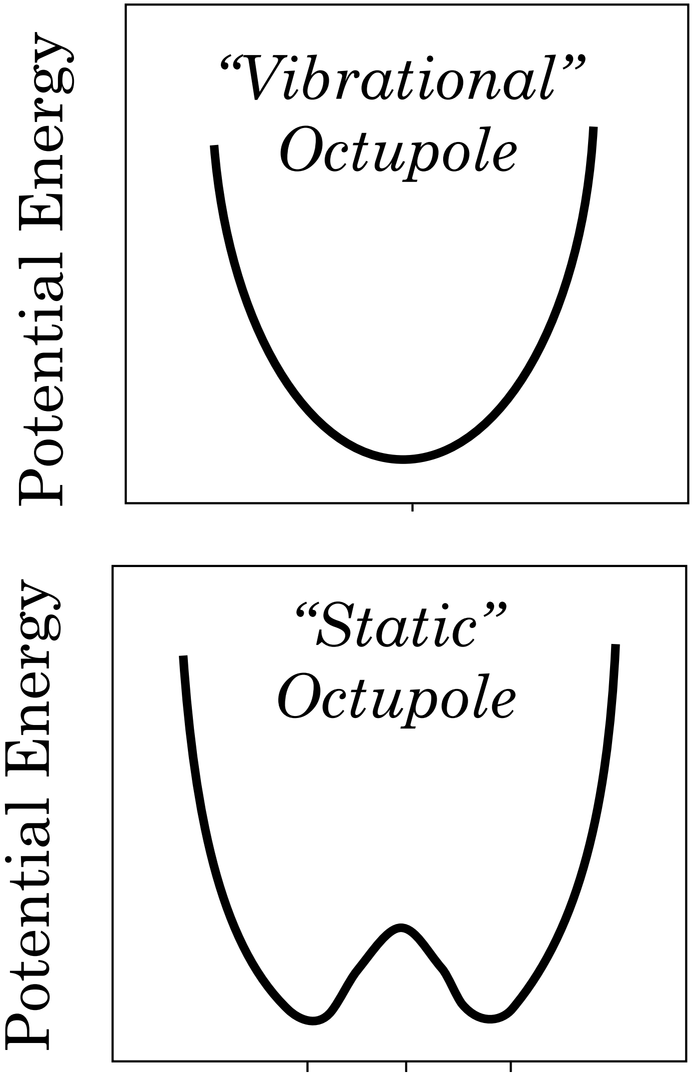
Quadrupole-octupole vibrations

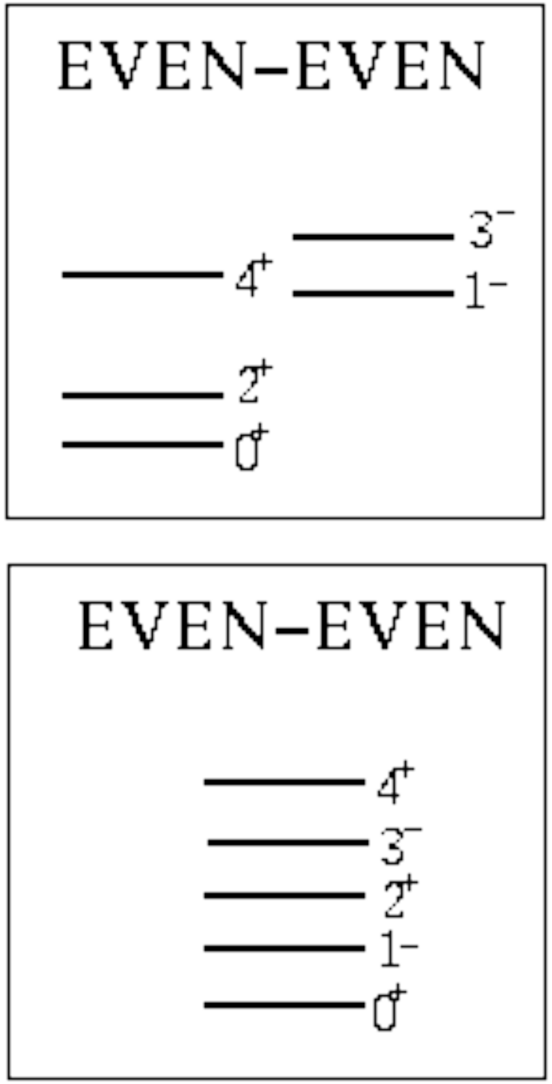


Octupole rotation-vibrations

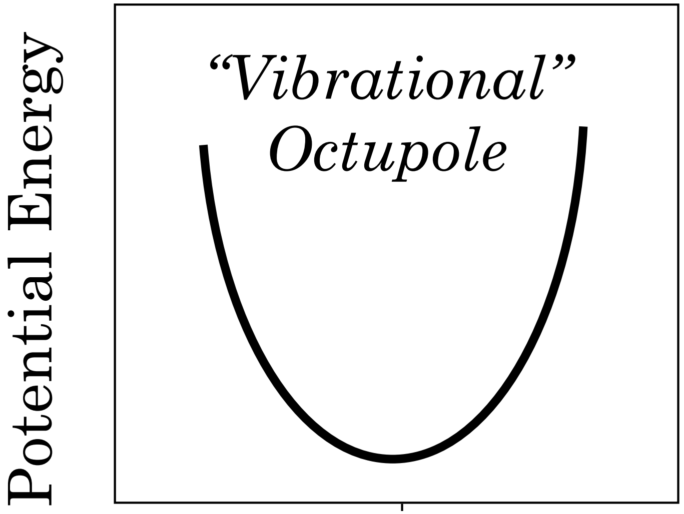


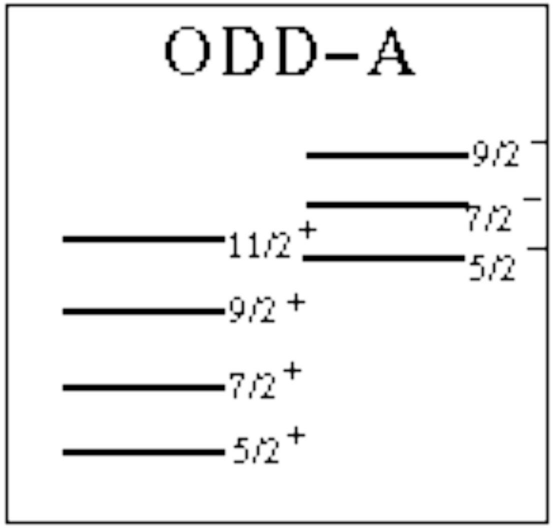
Octupole rotation-vibrations

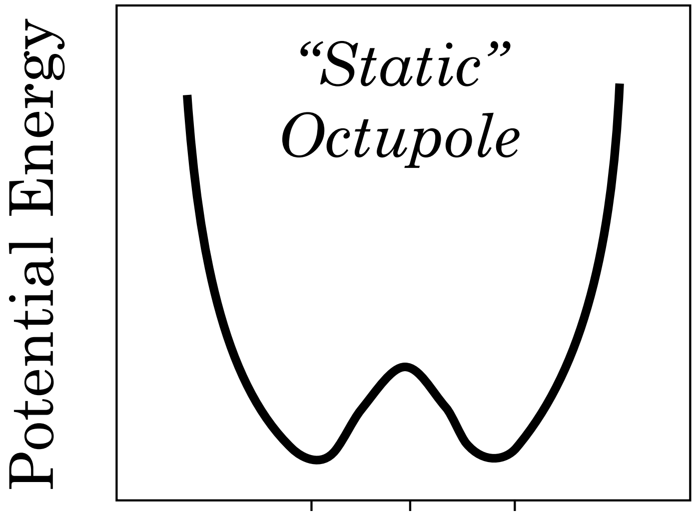


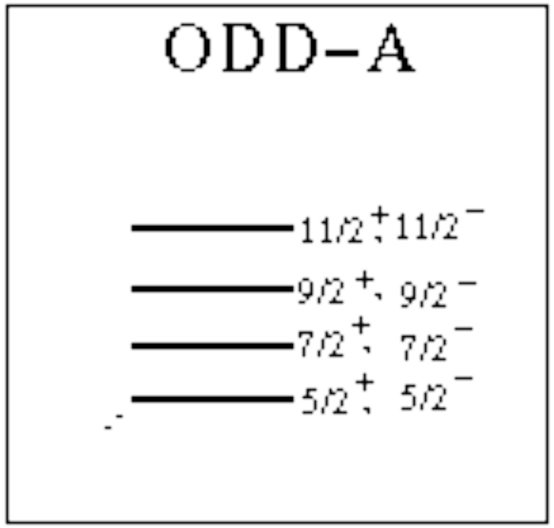


Octupole rotation-vibrations

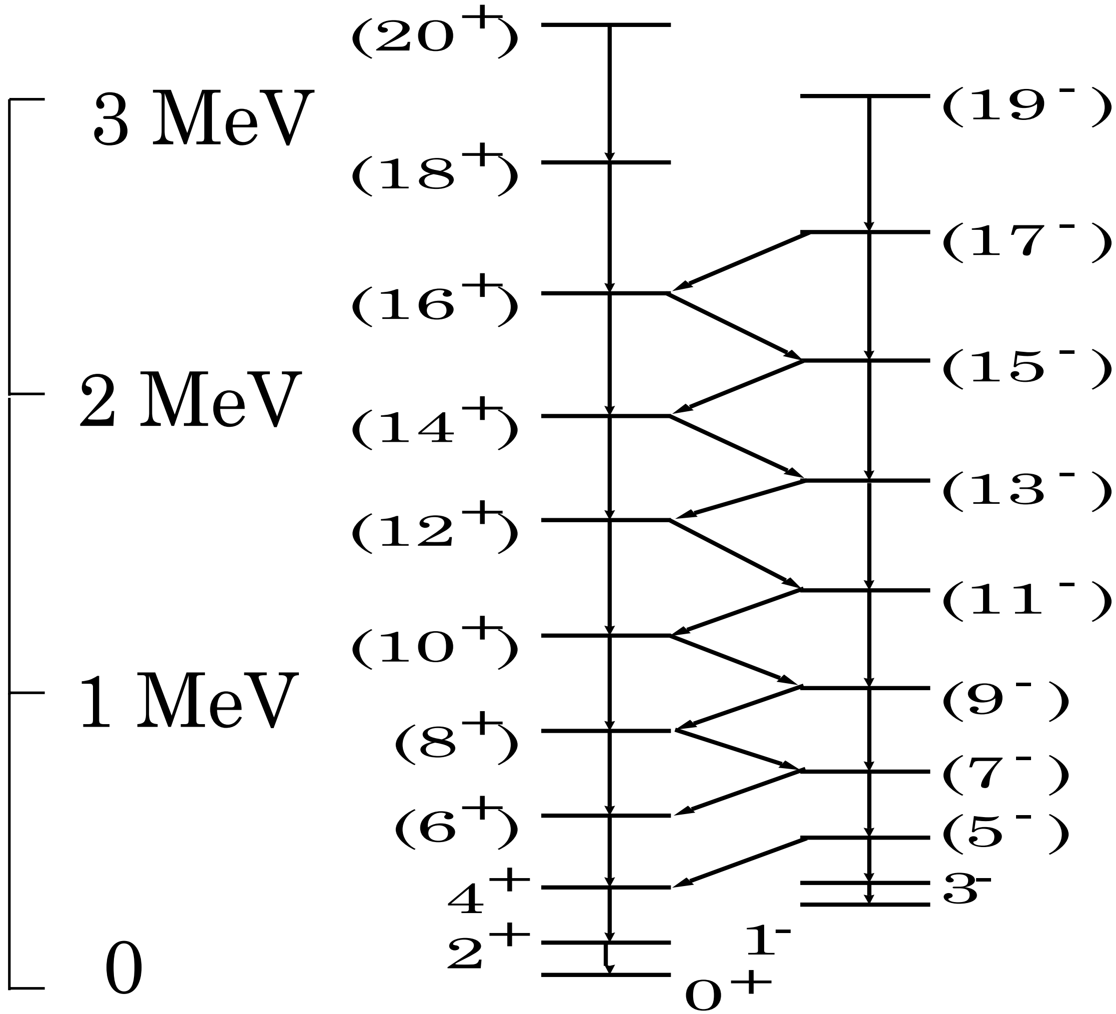








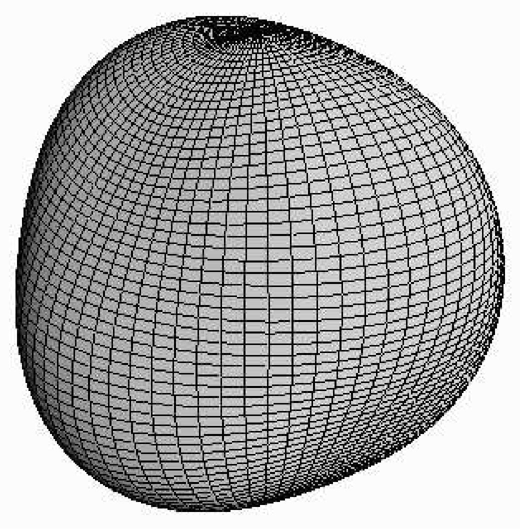
Example: ^{222}Ra



Discrete nuclear symmetries

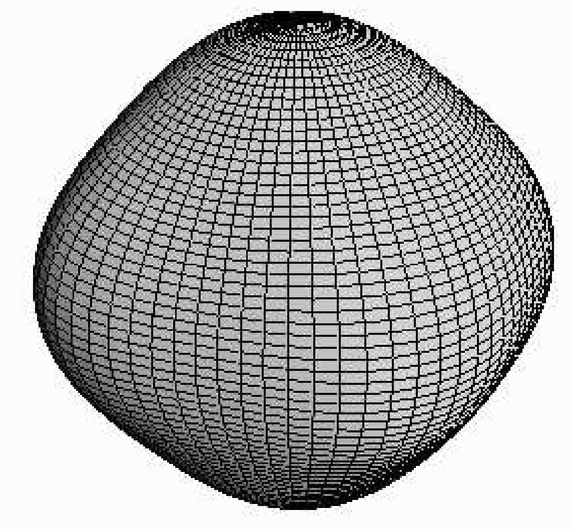
Tetrahedral symmetry:

$$\alpha_{3\pm2} \neq 0$$



Octahedral symmetry:

$$\alpha_{40} = \sqrt{\frac{14}{5}}\alpha_{4\pm2} \neq 0$$



Experimental evidence?

J. Dudek *et al.*, Phys. Rev. Lett. **88** (2002) 252502

Rigid rotor model

Hamiltonian of quantum-mechanical rotor in terms of 'rotational' angular momentum ***R***:

$$\hat{H}_{\text{rot}} = \frac{\hbar^2}{2}\left[\frac{R_1^2}{\mathfrak{I}_1} + \frac{R_2^2}{\mathfrak{I}_2} + \frac{R_3^2}{\mathfrak{I}_3}\right] = \frac{\hbar^2}{2}\sum_{i=1}^{3}\frac{R_i^2}{\mathfrak{I}_i}$$

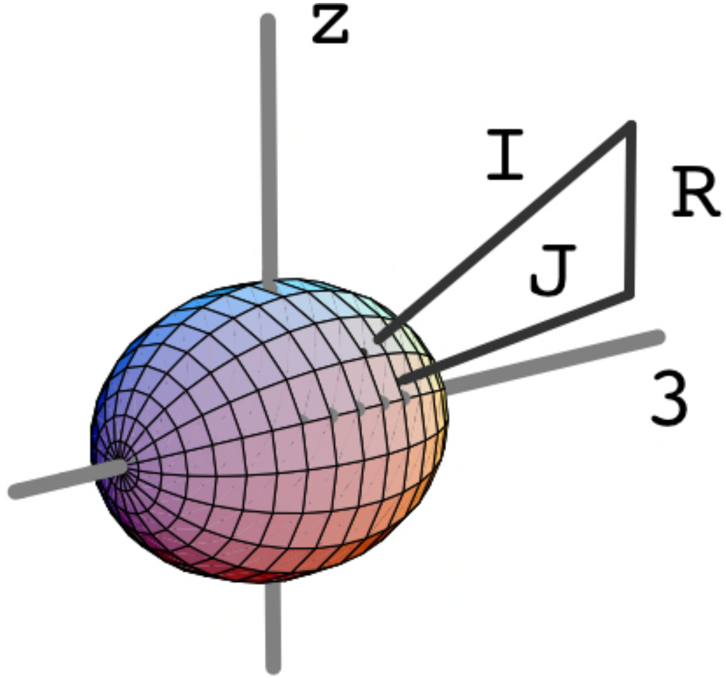
Nuclei have an additional intrinsic part H_{intr} with 'intrinsic' angular momentum ***J***.

The total angular momentum is ***I=R+J***.

Ground-state band of axial rotor

The ground-state spin of even-even nuclei is I=0. Hence K=0 for ground-state band:

$$E_I = \frac{\hbar^2}{2\Im} I(I+1)$$



E2 properties of rotational nuclei

Intra-band E2 transitions:

$$B(\mathrm{E2}; KI_\mathrm{i} \to KI_\mathrm{f}) = \frac{5}{16\pi} \langle I_\mathrm{i} K\, 20 | I_\mathrm{f} K \rangle^2 e^2 Q_0(K)^2$$

E2 moments:

$$Q(KI) = \frac{3K^2 - I(I+1)}{(I+1)(2I+3)} Q_0(K)$$

Q₀(K) is the 'intrinsic' quadrupole moment:

$$e\hat{Q}_0 \equiv \int \rho(\boldsymbol{r}') r^2 (3\cos^2\theta' - 1) d\boldsymbol{r}', \quad Q_0(K) = \langle K | \hat{Q}_0 | K \rangle$$

E2 properties of gs bands

For the ground state (*I*=*K*):

$$Q(I=K)=\frac{I(2I-1)}{(I+1)(2I+3)}Q_0(K)$$

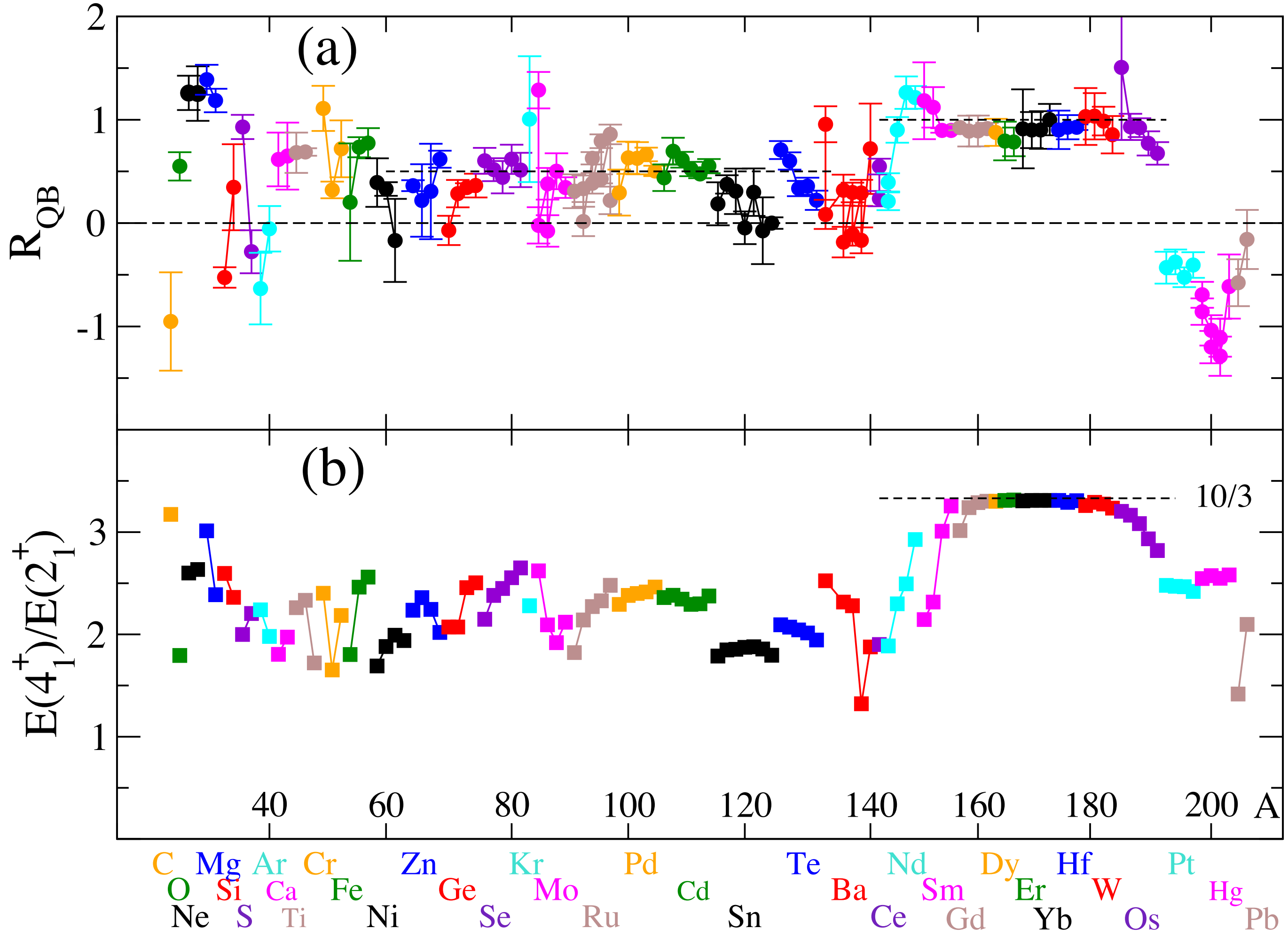
For the gsb in even-even nuclei (*K*=0):

$$B(\mathrm{E2};I\rightarrow I-2)=\frac{15}{32\pi}\frac{I(I-1)}{(2I-1)(2I+1)}e^2Q_0^2$$

$$Q(I)=-\frac{I}{2I+3}Q_0$$

$$\Rightarrow R_{QB}\equiv\frac{\left|eQ(2_1^+)\right|}{\sqrt{B(\mathrm{E2};0_1^+\rightarrow 2_1^+)}}=\frac{8}{7}\sqrt{\frac{\pi}{5}}\approx 0.91$$

Ratio R_{QB}



Y. Y. Sharon *et al.*, to be published

Generalized intensity relations

Mixing of *K* arises from

*Dependence of Q_0 on **I** (stretching)*

Coriolis interaction

Triaxiality

Generalized *intra*- and *inter*-band matrix elements (*eg* E2):

$$\frac{\sqrt{B\left(\mathrm{E2};K_\mathrm{i}I_\mathrm{i}\rightarrow K_\mathrm{f}I_\mathrm{f}\right)}}{\left|\left\langle I_\mathrm{i}K_\mathrm{i}\;2K_\mathrm{f}-K_\mathrm{i}\middle|I_\mathrm{f}K_\mathrm{f}\right\rangle\right|}=M_0+M_1\Delta+M_2\Delta^2+\cdots$$

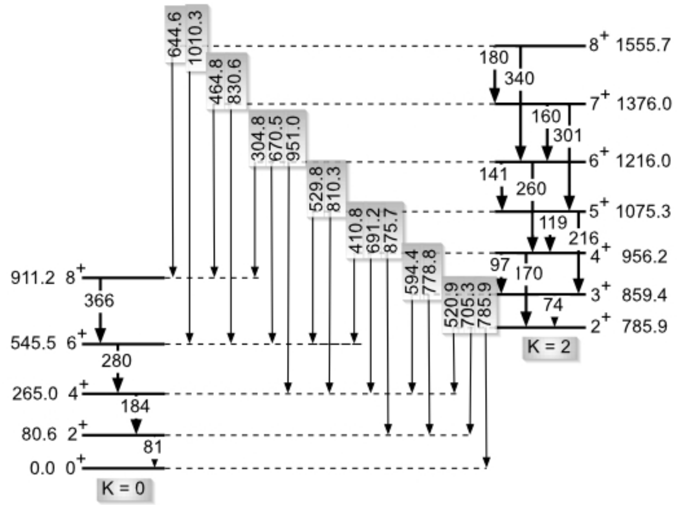
with $\Delta=I_\mathrm{f}\left(I_\mathrm{f}+1\right)-I_\mathrm{i}\left(I_\mathrm{i}+1\right)$

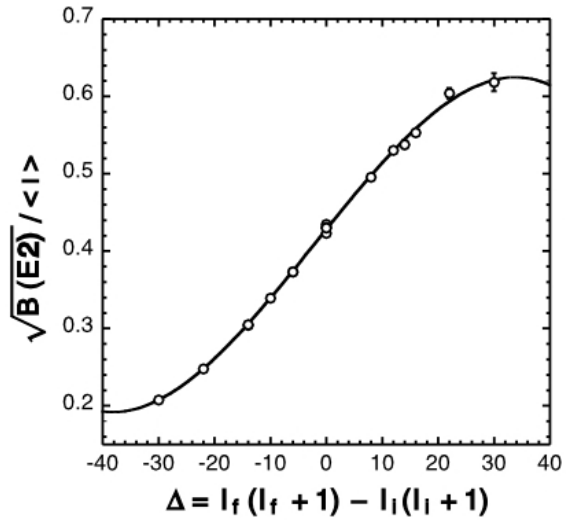
Inter-band E2 transitions

Example of $\gamma \rightarrow$g transitions in ^{166}Er:

$$\frac{\sqrt{B\left(\mathrm{E2}; I_\gamma \rightarrow I_g\right)}}{\left|\left\langle I_\gamma 2\, 2 - 2 \middle| I_g 0 \right\rangle\right|} = M_0 + M_1 \Delta + M_2 \Delta^2 + \cdots$$

$$\Delta = I_g\left(I_g + 1\right) - I_\gamma\left(I_\gamma + 1\right)$$





W.D. Kulp *et al.*, Phys. Rev. C **73** (2006) 014308

Rigid triaxial rotor

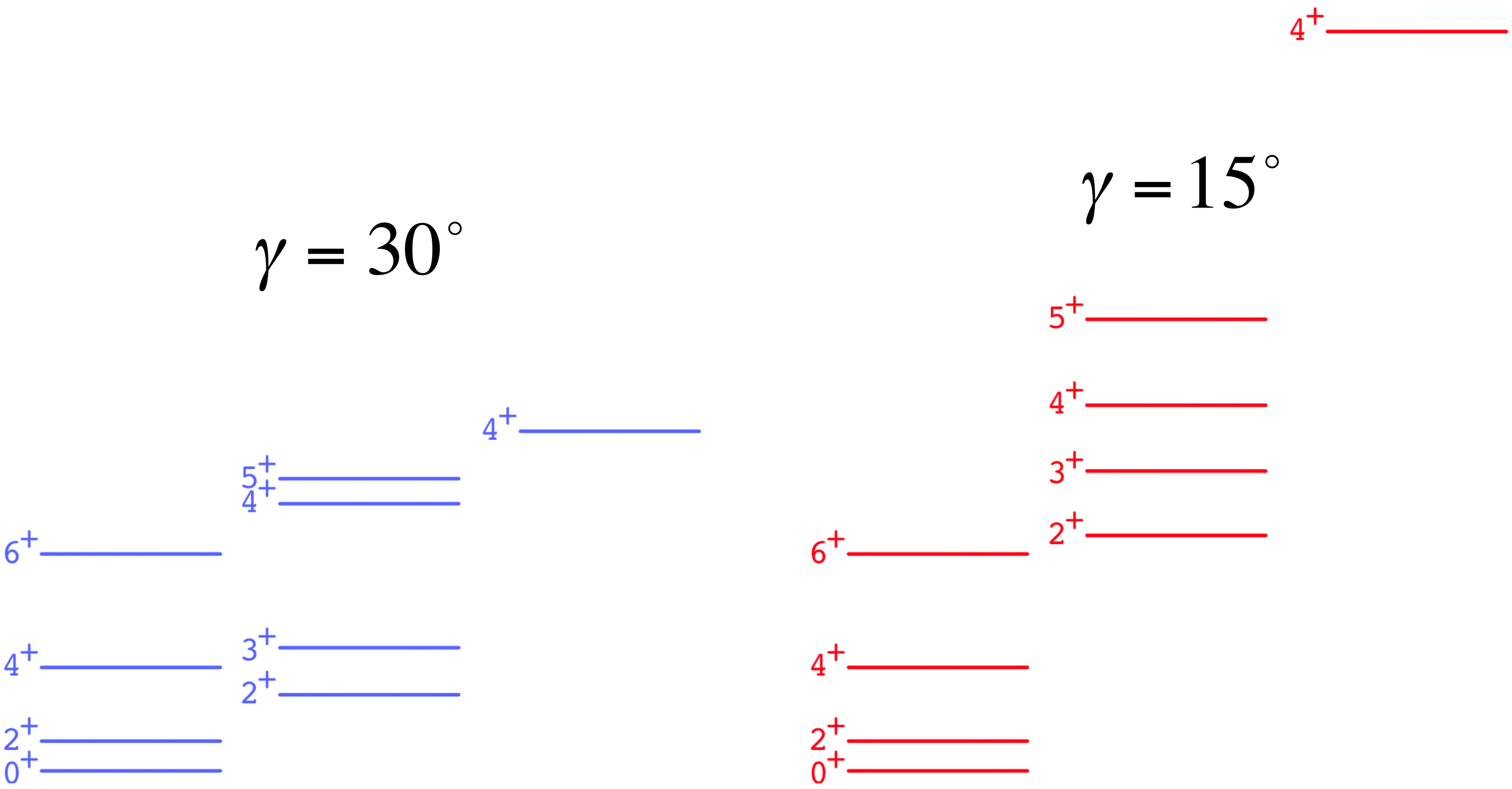
Triaxial rotor hamiltonian $\Im_1 \neq \Im_2 \neq \Im_3$:

$$\hat{H}'_{\text{rot}} = \sum_{i=1}^{3} \frac{\hbar^2}{2\Im_i} I_i^2 = \underbrace{\frac{\hbar^2}{2\Im} I^2 + \frac{\hbar^2}{2\Im_f} I_3^2}_{\hat{H}'_{\text{axial}}} + \underbrace{\frac{\hbar^2}{2\Im_g}\left(I_+^2 + I_-^2\right)}_{\hat{H}'_{\text{mix}}}$$

$$\frac{1}{\Im} = \frac{1}{2}\left(\frac{1}{\Im_1} + \frac{1}{\Im_2}\right), \quad \frac{1}{\Im_f} = \frac{1}{\Im_3} - \frac{1}{\Im}, \quad \frac{1}{\Im_g} = \frac{1}{4}\left(\frac{1}{\Im_1} - \frac{1}{\Im_2}\right)$$

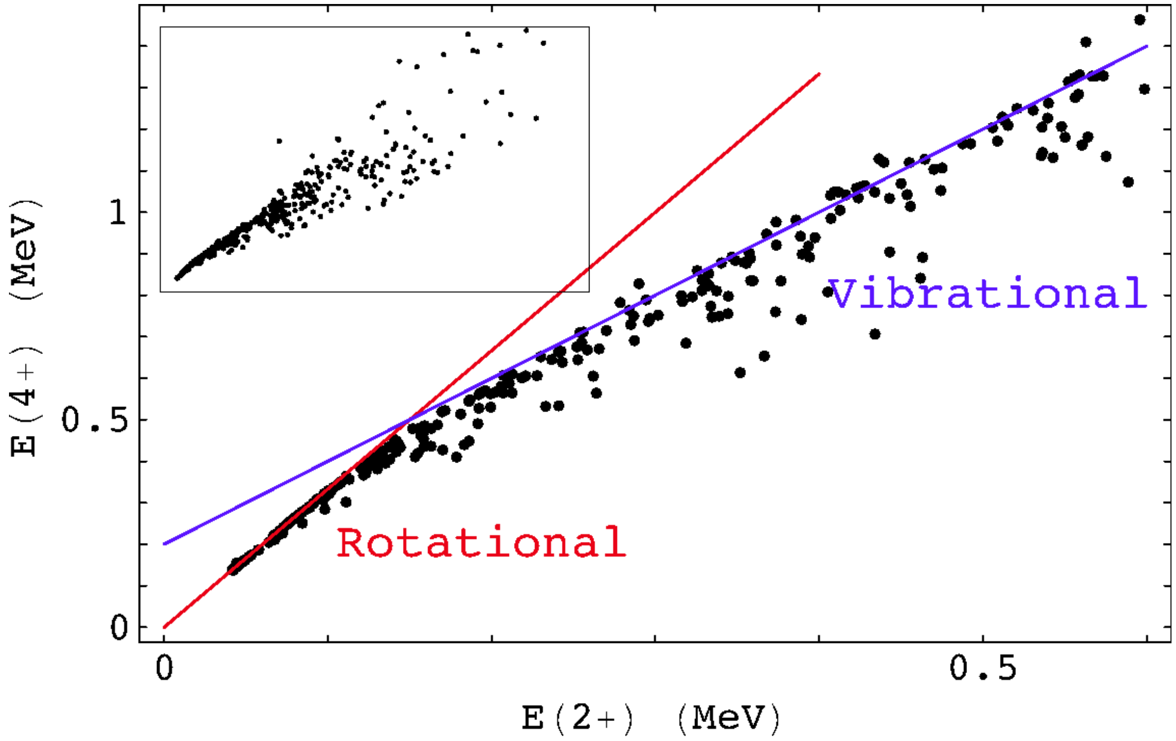
H'_{mix} non-diagonal in axial basis $|KIM\rangle \Rightarrow K$ is *not* a conserved quantum number

Rigid triaxial rotor spectra



Tri-partite classification of nuclei

Empirical evidence for seniority-type, vibrational- and rotational-like nuclei.



N.V. Zamfir *et al.*, Phys. Rev. Lett. **72** (1994) 3480

Interacting boson model

Describe the nucleus as a system of *N* interacting *s* and *d* bosons. Hamiltonian:

$$\hat{H}_{\mathrm{IBM}} = \sum_{i=1}^{6} \varepsilon_i \hat{b}_i^+ \hat{b}_i + \sum_{i_1 i_2 i_3 i_4 = 1}^{6} \upsilon_{i_1 i_2 i_3 i_4} \hat{b}_{i_1}^+ \hat{b}_{i_2}^+ \hat{b}_{i_3} \hat{b}_{i_4}$$

Justification from

Shell model: s and d bosons are associated with S and D fermion (Cooper) pairs.

Geometric model: for large boson number the IBM reduces to a liquid-drop hamiltonian.

A. Arima & F. Iachello, Phys. Rev. Lett. **35** (1975) 1069

Dimensions

Assume Ω available 1-fermion states. Number of *n*-fermion states is $\binom{\Omega}{n} = \frac{\Omega!}{n!(\Omega - n)!}$

Assume Ω available 1-boson states. Number of *n*-boson states is $\binom{\Omega + n - 1}{n} = \frac{(\Omega + n - 1)!}{n!(\Omega - 1)!}$

Example: $^{162}Dy_{96}$ with 14 neutrons (Ω=44) and 16 protons (Ω=32) ($^{132}Sn_{82}$ inert core).

SM dimension: $7 \cdot 10^{19}$

IBM dimension: 15504

Dynamical symmetries

Boson hamiltonian is of the form

$$\hat{H}_{\mathrm{IBM}} = \sum_{i=1}^{6} \varepsilon_i \hat{b}_i^+ \hat{b}_i + \sum_{i_1 i_2 i_3 i_4 = 1}^{6} \upsilon_{i_1 i_2 i_3 i_4} \hat{b}_{i_1}^+ \hat{b}_{i_2}^+ \hat{b}_{i_3} \hat{b}_{i_4}$$

In general not solvable analytically.

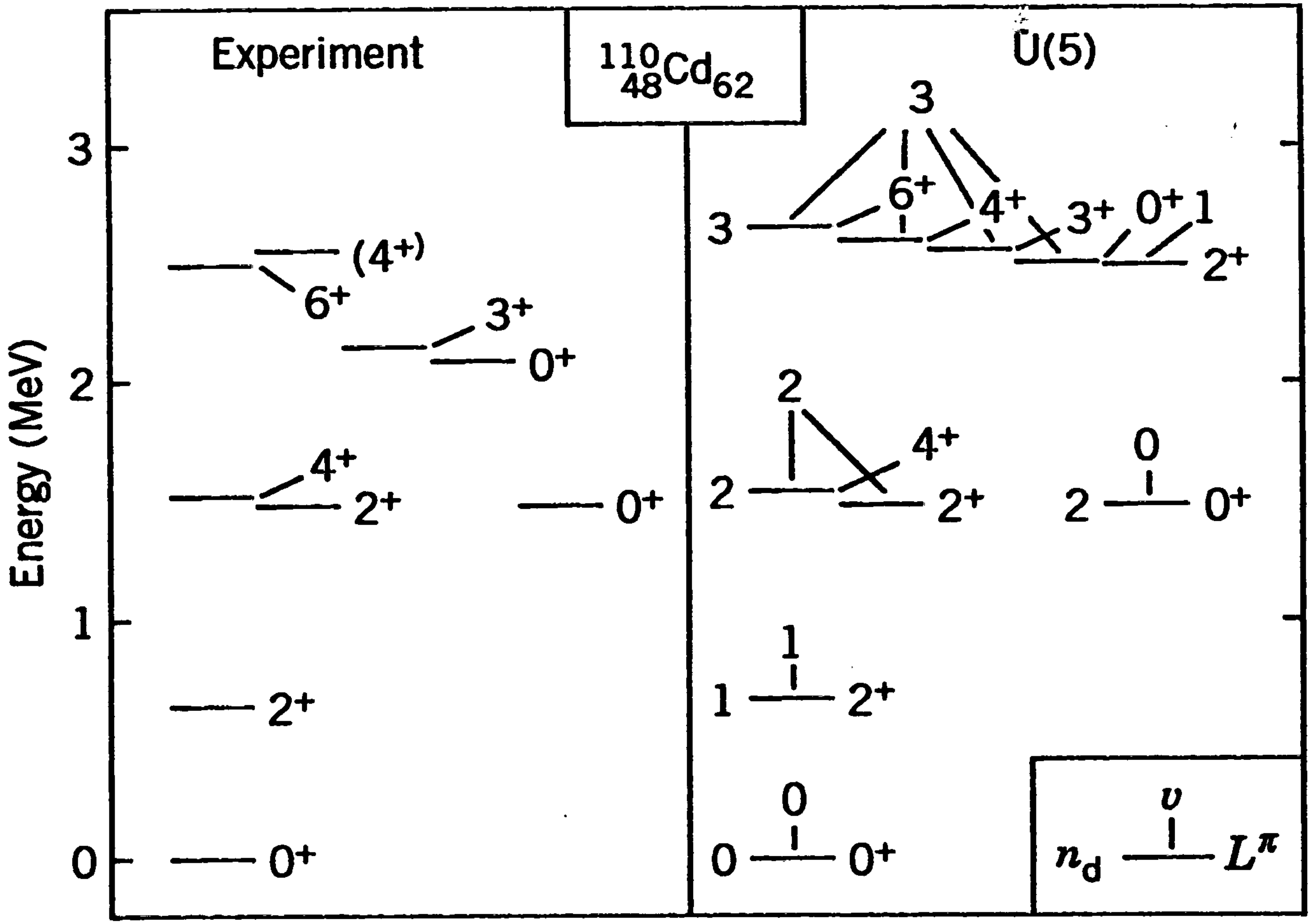
Three solvable cases with SO(3) symmetry:

$$\mathrm{U}(6) \supset \mathrm{U}(5) \supset \mathrm{SO}(5) \supset \mathrm{SO}(3)$$

$$\mathrm{U}(6) \supset \mathrm{SU}(3) \supset \mathrm{SO}(3)$$

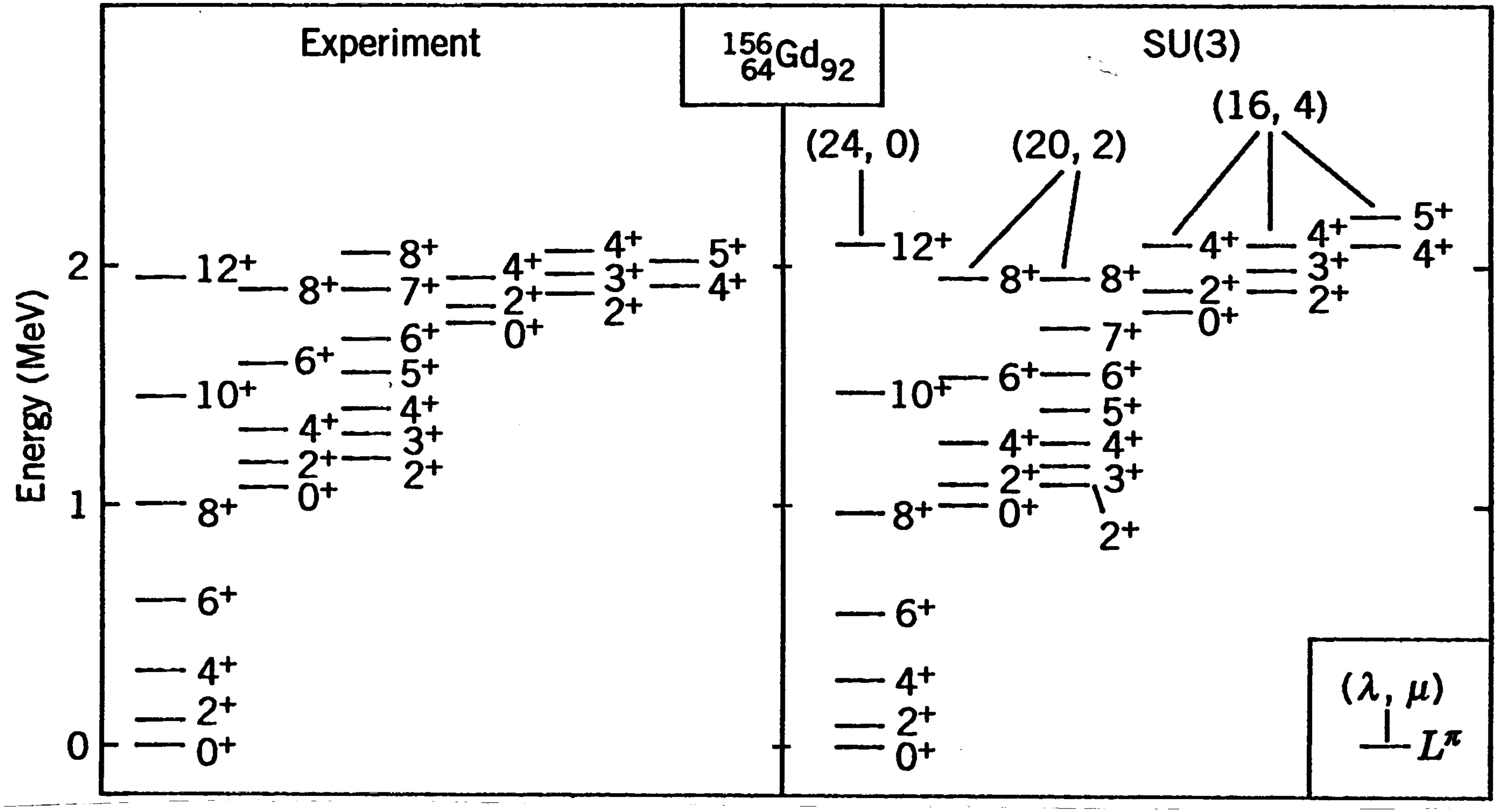
$$\mathrm{U}(6) \supset \mathrm{SO}(6) \supset \mathrm{SO}(5) \supset \mathrm{SO}(3)$$

U(5) vibrational limit: $^{110}Cd_{62}$



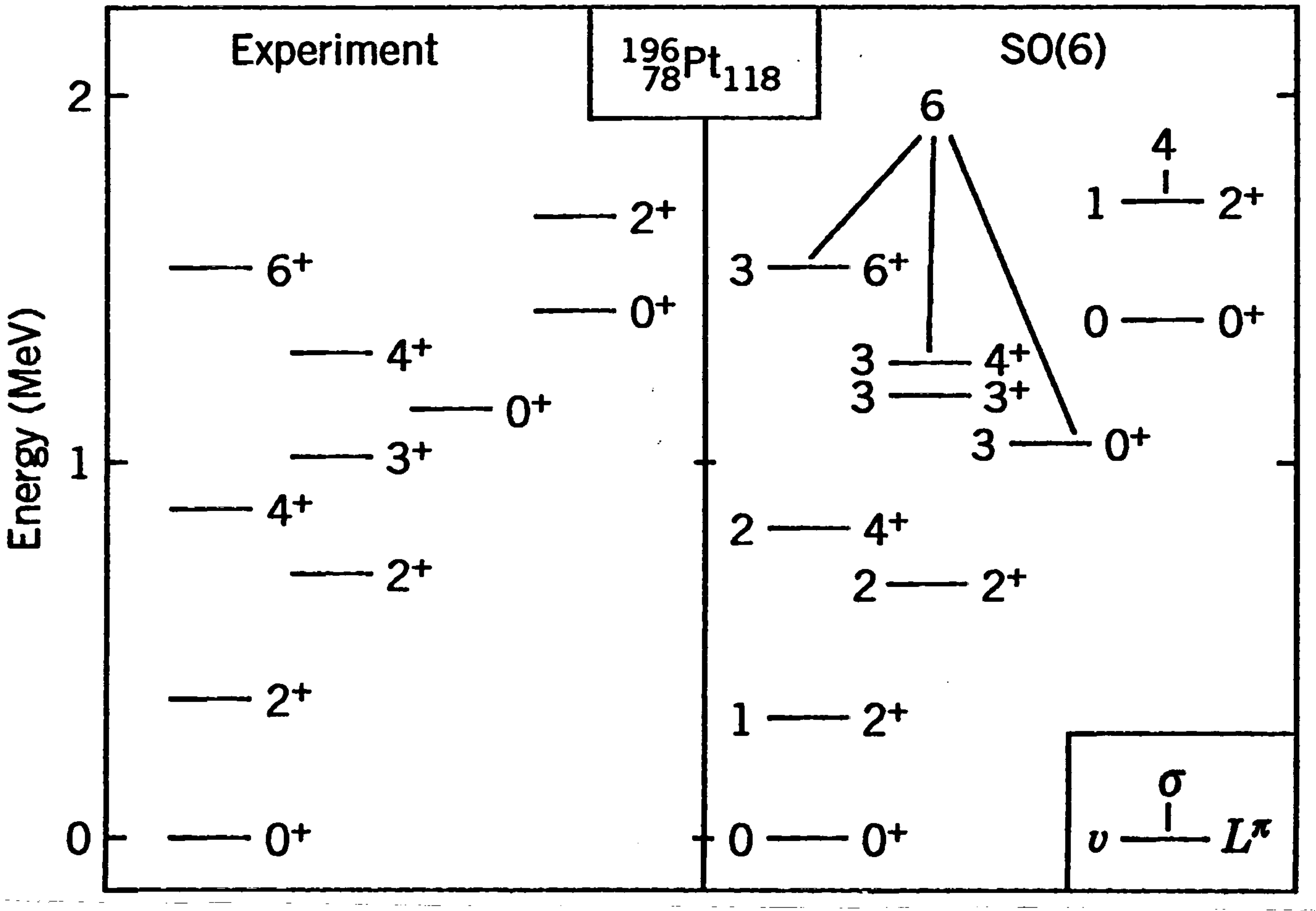
A. Arima & F. Iachello, Ann. Phys. (NY) **99** (1976) 253

SU(3) rotational limit: $^{156}Gd_{92}$



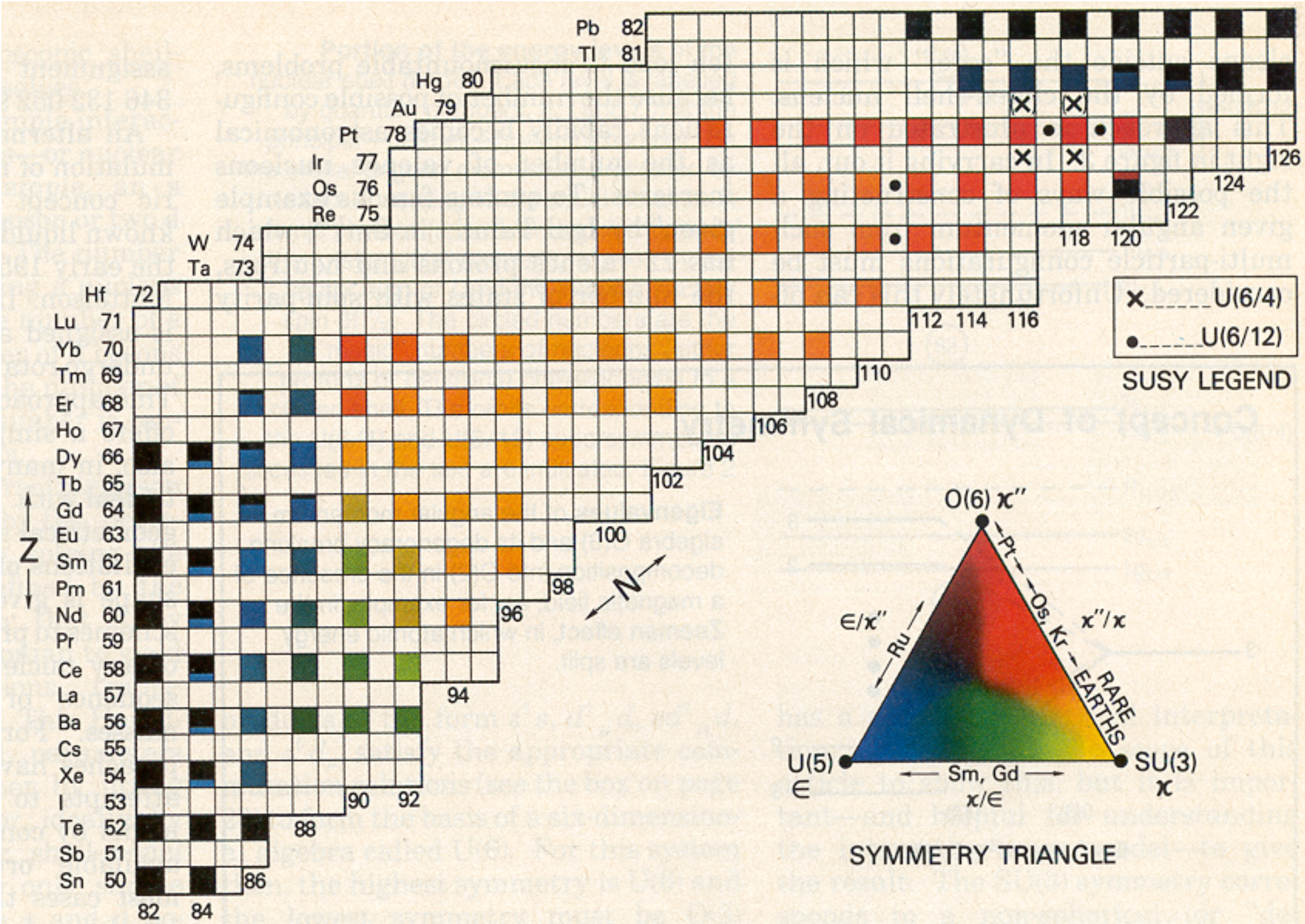
A. Arima & F. Iachello, Ann. Phys. (NY) 111 (1978) 201

SO(6) γ-unstable limit: $^{196}Pt_{118}$



A. Arima & F. Iachello, Ann. Phys. (NY) **123** (1979) 468

Applications of IBM



Classical limit of IBM

For large boson number *N*, a *coherent* (or *intrinsic*) state is an approximate eigenstate,

$$\hat{H}_{\mathrm{IBM}}\left|N;\alpha_\mu\right\rangle \approx E\left|N;\alpha_\mu\right\rangle, \qquad \left|N;\alpha_\mu\right\rangle \propto \left(s^+ + \sum_\mu \alpha_\mu d_\mu^+\right)^N \left|\mathrm{o}\right\rangle$$

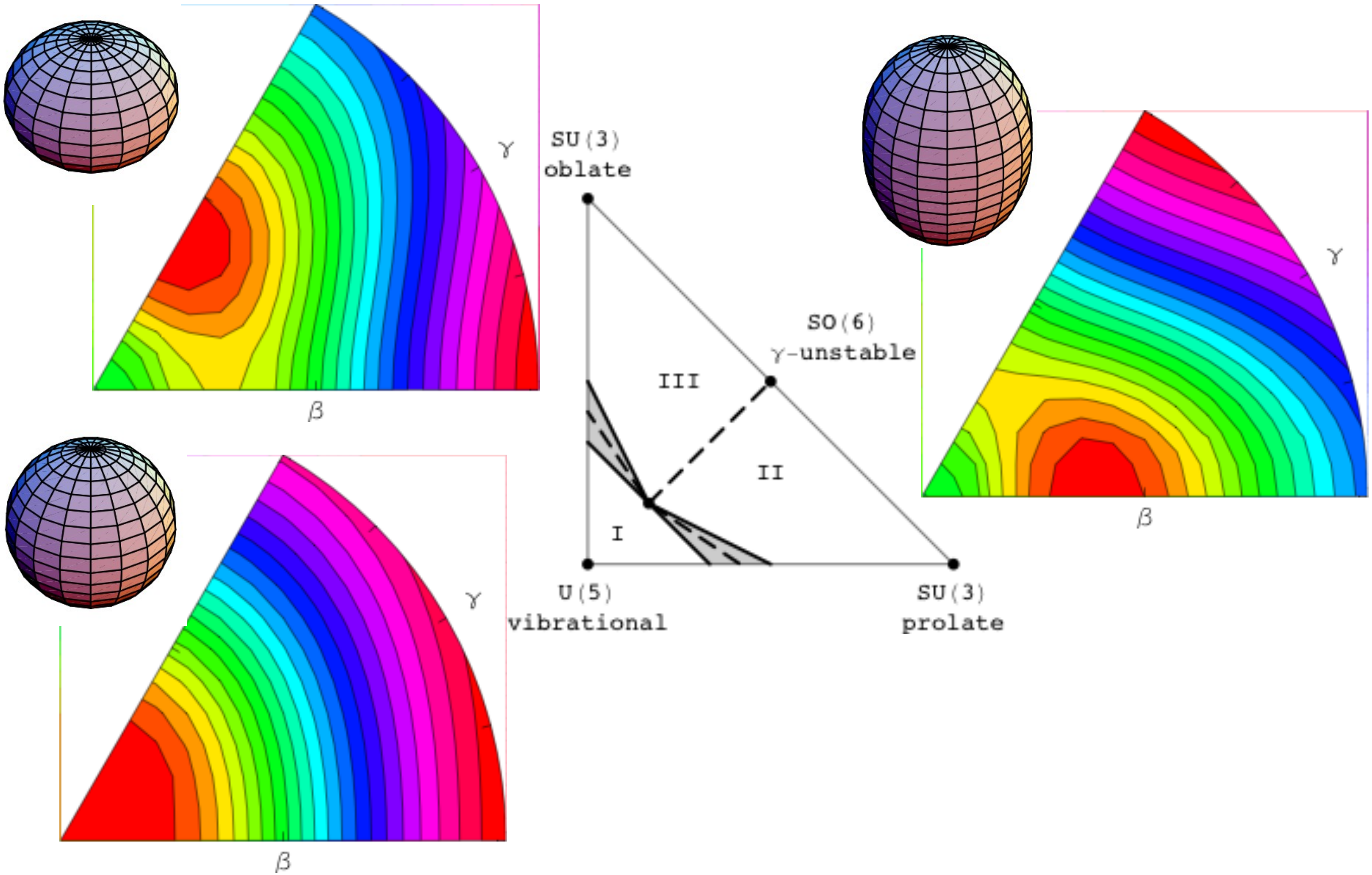
The real parameters α_μ are related to the three Euler angles and shape variables β and γ.

Any IBM hamiltonian yields energy surface:

$$\left\langle N;\alpha_\mu\right|\hat{H}_{\mathrm{IBM}}\left|N;\alpha_\mu\right\rangle = \left\langle N;\beta\gamma\right|\hat{H}_{\mathrm{IBM}}\left|N;\beta\gamma\right\rangle \equiv V(\beta,\gamma)$$

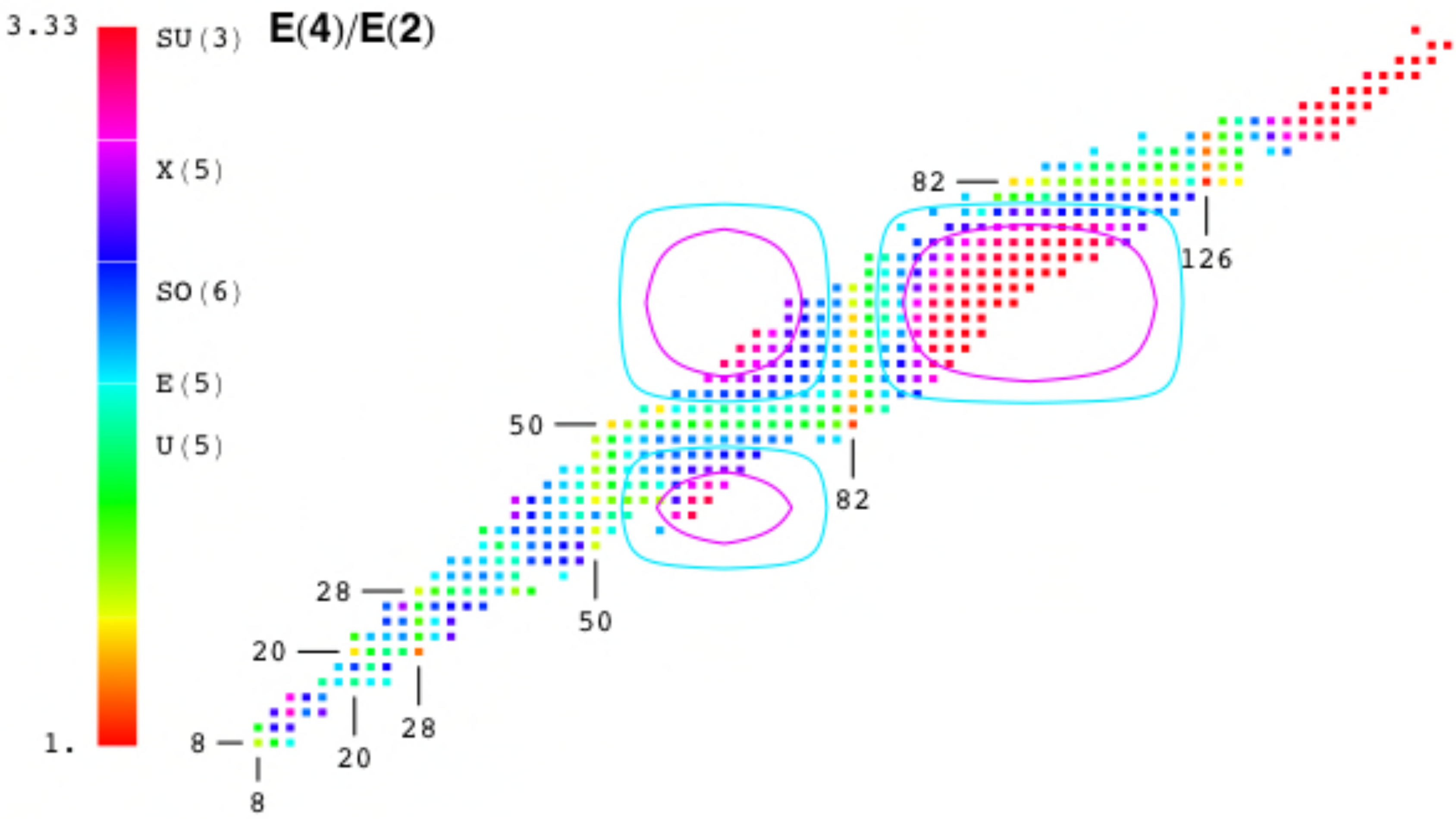
J. N. Ginocchio & M. W. Kirson, Phys. Rev. Lett. 44 (1980) 1744
A. E. L. Dieperink *et al.*, Phys. Rev. Lett. **44** (1980) 1747
A. Bohr & B. R. Mottelson, Phys. Scripta 22 (1980) 468

Phase diagram of IBM



J. Jolie *et al.* , Phys. Rev. Lett. **87** (2001) 162501

The ratio R_{42}



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A. Bohr and B.R. Mottelson, *Nuclear Structure. II Nuclear Deformations* (Benjamin, 1975).

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