Equilibrium stability for non-uniformly hyperbolic maps

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TMU-ICTP School and Conference in
Dynamical Systems and Ergodic Theory

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May 6, 2018

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Equilibrium states

Let $T : X \to X$ be continuous and X a compact metric space. Consider

- $\mathbb{P}(X)$ the space of probability measures on the Borel sets of X endowed with the *weak* topology:* $(\mu_n)_n$ converges to μ if
$$\int \phi d\mu_n \longrightarrow \int \phi d\mu, \quad \text{for all } \phi : X \to \mathbb{R} \text{ continuous;}$$
- $\mathbb{P}_T(X) \subset \mathbb{P}(X)$ the set of T-invariant probability measures;
- $h_\eta(T)$ the entropy of $\eta \in \mathbb{P}_T(X)$.

Given a continuous *potential* $\phi : X \to \mathbb{R}$, we say that $\mu \in \mathbb{P}_T(X)$ is an *equilibrium state* for (T, ϕ) if

$$h_\mu(T) + \int \phi \, d\mu = \sup_{\eta \in \mathbb{P}_T(X)} \left\{ h_\eta(T) + \int \phi \, d\eta \right\}.$$

Remark

- Measures of maximal entropy are equilibrium states for $\phi \equiv 0$.
- SRB measures are equilibrium states for $\phi = \log |\det DT|$.

The classical theory of equilibrium states for *discrete-time* dynamical systems goes back to the 1970's, with many contributions on their existence and uniqueness/finiteness:

- Shifts with finitely many symbols / uniformly hyperbolic systems:
    - *Bowen, Ruelle, Sinai, Walters*

And not so classical...

- Shifts with infinitely many symbols:
    - *Buzzi, Ledrappier, Lima, Sarig...*
- One-dimensional systems:
    - *Bruin, Demers, Hofbauer, Iommi, Keller, Pesin, Senti, Todd...*
- Non-uniformly hyperbolic systems:
    - *Arbieto, Climenhaga, Fisher, Leplaideur, Matheus, Oliveira, Ramos, Rios, Siqueira, Thompson, Varandas, Viana,...*

Once the existence is established, a natural question arises:

Under which conditions do these equilibrium states depend continuously on the dynamics/potential?

Equilibrium stability

Let M be a compact Riemannian manifold. Consider

- $\mathcal{F}$ a set of C^1 local diffeomorphisms $f : M \to M$ endowed with the C^1 topology;
- $C^\alpha(M)$ the space of Hölder continuous potentials $\phi : M \to \mathbb{R}$ endowed with the C^α norm;
- $\mathcal{F} \times C^\alpha(M)$ endowed with the product topology;
- $\mathcal{H} \subset \mathcal{F} \times C^\alpha(M)$ such that each $(f, \phi) \in \mathcal{H}$ has a unique equilibrium state $\mu_{f,\phi}$.

We say that $\mathcal{H}$ is equilibrium stable if the function

$$\mathcal{H} \ni (f, \phi) \longmapsto \mu_{f,\phi} \in \mathbb{P}_f(M)$$

is continuous.

Uniformly expanding maps

A C^1 local diffeomorphism $f : M \to M$ is *uniformly expanding* if for some choice of a Riemannian metric in M there is $\sigma < 1$ such that for all $x \in M$

$$\|Df(x)^{-1}\| \leq \sigma.$$

Example

Consider $T^d = \mathbb{R}^d/\mathbb{Z}^d$ and $f : T^d \to T^d$ given by the quotient of a linear map having a diagonal matrix with integer eigenvalues $\lambda_1, \ldots, \lambda_d \geq 2$.

Let $\mathcal{E}$ be the set of C^1 uniformly expanding maps.

Theorem (Bowen)

Each $(f, \phi) \in \mathcal{E} \times C^\alpha(M)$ has a unique equilibrium state $\mu_{f,\phi}$.

$\mathcal{E} \times C^\alpha(M)$ is equilibrium stable.

Hyperbolic potentials

Given $c > 0$, define $\Sigma_c(f)$ as the set of points $x \in M$ where

$$\liminf_{n\to+\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \|Df(f^j(x))^{-1}\| \leq -c.$$

We say that a continuous $\phi : M \to \mathbb{R}$ is *c-hyperbolic* if the *topological pressure* of ϕ is equal to the *relative pressure* of ϕ on the set $\Sigma_c(f)$. Let

$$\mathcal{H}_c = \{(f,\phi) \in \mathcal{F} \times C^\alpha(M) : \ \phi \text{ is } c\text{-hyperbolic for } f\}$$

Theorem (Ramos-Viana)

Each $(f,\phi) \in \mathcal{H}_c$ has finitely many equilibrium states. If $\{f^{-n}(x)\}_{n\geq 0}$ is dense in M for all $x \in M$, then $(f,\phi) \in \mathcal{H}_c$ has a unique equilibrium state.

$$\mathcal{H}_c^* = \left\{(f,\phi) \in \mathcal{H}_c : \{f^{-n}(x)\}_{n\geq 0} \text{ is dense in } M \text{ for all } x \in M\right\}.$$

Main Theorem (A.-Ramos-Siqueira)

$\mathcal{H}_c^*$ is equilibrium stable.

Example: nonuniformly expanding maps

Let $f : M \to M$ be a C^1 local diffeomorphism for which there are $\delta > 0$ small, $\sigma < 1$ and $A \subset M$ a (bad) domain of injectivity of f such that:

(1) $\|Df^{-1}(x)\| < 1 + \delta$, *for every* $x \in A$;
(2) $\|Df^{-1}(x)\| < \sigma$, *for every* $x \in M \setminus A$.

There is $c > 0$ such that if a Hölder continuous $\phi : M \to \mathbb{R}$ has *small variation*, i.e.

$$\sup \phi - \inf \phi < \log \deg(f),$$

then ϕ is c-hyperbolic. Consider $\mathcal{F}$ the class of C^1 local diffeomorphisms satisfying (1)-(2) and

$$\mathcal{H} = \{(f, \phi) : f \in \mathcal{F} \text{ and } \phi : M \to \mathbb{R} \text{ Hölder with small variation}\}.$$

It follows from our Main Theorem that $\mathcal{H}$ is equilibrium stable.

Metric entropy

Let $T : X \to X$ be a measurable transformation preserving some probability measure μ. We define the entropy of a partition[1] $\mathcal{P}$ as

$$H_\mu(\mathcal{P}) = \sum_{P\in\mathcal{P}} \mu(P)\log\mu(P),$$

(with the convention that $0\log 0 = 0$). Consider for each $n \geq 0$

$$\mathcal{P}^n = \bigvee_{k=0}^{n} T^{-k}\mathcal{P} = \Big\{P_0 \cap T^{-1}(P_1) \cap \cdots \cap T^{-n}(P_n) : P_0, \dots, P_n \in \mathcal{P}\Big\},$$

define the entropy of (T, μ) with respect to $\mathcal{P}$

$$h_\mu(T, \mathcal{P}) = \lim_{n\to\infty} \frac{1}{n} H_\mu(\mathcal{P}^n) = \inf_n \frac{1}{n} H_\mu(\mathcal{P}^n),$$

and the entropy of (T, μ)

$$h_\mu(T) = \sup_{\mathcal{P}} h_\mu(f, \mathcal{P}).$$

[1] A countable family of pairwise disjoint sets whose union has full μ measure.

Topological pressure

Let X be a compact space, $T: X \to X$ and $\phi: X \to \mathbb{R}$ be continuous. Given $\delta > 0$, $x \in X$ and $n \in \mathbb{N}$, consider the *dynamic ball*

$$B_n(x,\delta) = \left\{y \in X : \ \operatorname{dist}(T^j(x), T^j(y)) < \delta, \ \text{for } \ 0 \le j \le n\right\}.$$

Define

$$S_n\phi(x) = \phi(x) + \phi(T(x)) + \cdots + \phi(T^{n-1}(x))$$

and

$$S_{n,\delta}\phi(x) = \sup_{y \in B_n(x,\delta)} S_n\phi(y).$$

Consider for each $N \in \mathbb{N}$

$$\mathcal{F}_N = \{B_n(x,\delta);\ x \in X \text{ and } n \ge N\}.$$

Given $\Lambda \subset X$, let $\mathcal{F}_N(\Lambda)$ be the set of at most countably many elements in $\mathcal{F}_N$ which cover Λ. Define for a T-invariant set $\Lambda \subset X$, $\gamma > 0$ and $N \in \mathbb{N}$

$$m_T(\phi,\Lambda,\delta,\gamma,N) = \inf_{\mathcal{U} \in \mathcal{F}_N(\Lambda)} \left\{ \sum_{B_n(x,\delta) \in \mathcal{U}} e^{-\gamma n + S_{n,\delta}\phi(x)} \right\}.$$

Define

$$m_T(\phi, \Lambda, \delta, \gamma) = \lim_{N\to+\infty} m_T(\phi, \Lambda, \delta, \gamma, N),$$

and

$$P_T(\phi, \Lambda, \delta) = \inf \{\gamma > 0 \mid m_T(\phi, \Lambda, \delta, \gamma) = 0\}.$$

The *relative pressure* of ϕ on Λ is

$$P_T(\phi, \Lambda) = \lim_{\delta\to 0} P_T(\phi, \Lambda, \delta).$$

We call $P_T(\phi) := P_T(\phi, X)$ the *topological pressure of ϕ*. It satisfies

$$P_T(\phi) = \sup \{P_T(\phi, \Lambda),\ P_f(\phi, X \setminus \Lambda)\}.$$

Theorem (Walters)

$$P_T(\phi) = \sup_{\eta\in\mathbb{P}_T(X)} \left\{ h_\eta(T) + \int \phi \, d\eta \right\}.$$

Cones

Let E be a Banach space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

A closed convex set $\{0\} \neq \mathcal{C} \subset E$ is called a *cone* if both

- $\forall \lambda \geq 0 : \lambda\mathcal{C} \subset \mathcal{C}$; and
- $\mathcal{C} \cap (-\mathcal{C}) = \{0\}$.

A cone $\mathcal{C}$ defines a partial order in E through the relation

$$x \leq y \iff y - x \in \mathcal{C}.$$

The cone is called *normal* if

$$\exists\, \gamma \in \mathbb{R} : 0 \leq x \leq y \implies \|x\| \leq \gamma\|y\|.$$

Example

Consider $E = C^0(M)$ with the usual sup norm $\|\quad\|_0$ and

$$\mathcal{C} = \{\varphi \in C^0(M) : \varphi \geq 0\}.$$

$\mathcal{C}$ is a normal cone in E (with $\gamma = 1$).

A bounded linear operator $T : E \to E$ is *positive* if

$$T(\mathcal{C}) \subset \mathcal{C}.$$

Note that if T is positive, then $T(x) \leq T(y)$ whenever $x \leq y$.
The dual space of E is

$$E^* = \{x^* : E \to \mathbb{K} \mid x^* \text{ is linear and bounded}\},$$

and the dual operator $T^* : E^* \to E^*$ is defined for each $x^* \in E^*$ by

$$T^*(x^*) = x^* \circ T.$$

Lemma (Mazur)

Let E be a Banach space partially ordered by a normal cone $\mathcal{C}$ with non-empty interior and $T : E \to E$ a positive bounded operator. Then the spectral radius of T is an eigenvalue of T^*.

The *spectral radius* of T is

$$\lambda_T = \lim_{n\to\infty} \sqrt[n]{\| T^n \|}.$$

Transfer operator

Define for $(f,\phi)\in\mathcal{H}_c$ the *transfer operator*

$$\mathcal{L}_{f,\phi}: C^0(M)\longrightarrow C^0(M)$$

which associates to each $\varphi: M\to\mathbb{R}$ the continuous function

$$\mathcal{L}_{f,\phi}(\varphi)\colon M\longrightarrow\mathbb{R}$$

defined by

$$\mathcal{L}_{f,\phi}\varphi\,(x)=\sum_{y\in f^{-1}(x)} e^{\phi(y)}\varphi(y).$$

Considering $C^0(M)$ ordered by the cone of non-negative functions, we have that $\mathcal{L}_{f,\phi}$ is a positive bounded linear operator.
By Riesz-Markov Theorem, we may think of its *dual operator*

$$\mathcal{L}^*_{f,\phi}:\mathbb{P}(M)\to\mathbb{P}(M).$$

For every $\varphi \in C^0(M)$ and every $\eta \in \mathbb{P}(M)$ we have

$$\int \varphi \, d\mathcal{L}_{f,\phi}^{*}\eta = \int \mathcal{L}_{f,\phi}(\varphi) \, d\eta.$$

For each $n \in \mathbb{N}$ we have

$$\mathcal{L}_{f,\phi}^{n}\varphi(x) = \sum_{y \in f^{-n}(x)} e^{S_n\phi(y)}\varphi(y).$$

Moreover

$$\|\mathcal{L}_{f,\phi}^{n}\| = \|\mathcal{L}_{f,\phi}^{n}1\|, \quad \forall n \geq 1.$$

Using this, we can easily see that the spectral radius $\lambda_{f,\phi}$ of $\mathcal{L}_{f,\phi}$ satisfies

$$\deg(f)e^{\inf \phi} \leq \lambda_{f,\phi} \leq \deg(f)e^{\sup \phi}.$$

Reference measure

Consider $C^0(M)$ ordered by the cone $\mathcal{C}$ of non-negative functions. Consider $\lambda_{f,\phi}$ the spectral radius of $\mathcal{L}_{f,\phi}$. By Mazur Lemma we have

Lemma

There exists a probability measure $\nu_{f,\phi}$ satisfying $\mathcal{L}^*_{f,\phi}\nu_{f,\phi} = \lambda_{f,\phi}\nu_{f,\phi}$.

Lemma (Ramos-Viana)

If $(f,\phi) \in \mathcal{H}_c$, then $\lambda_{f,\phi} = e^{P_f(\phi)}$.

Next goal: $\lambda_{f,\phi}$ is the only real eigenvalue of $\mathcal{L}^*_{f,\phi}$ for $(f,\phi) \in \mathcal{H}^*_c$.

Hyperbolic times

We say that n is a *hyperbolic time* for x if

$$\frac{1}{k}\sum_{j=n-k}^{n-1}\log\|Df(f^j(x))^{-1}\| \le -\frac{c}{2}, \quad \text{for all } 1\le k<n.$$

Lemma (Pliss)

Each $x\in\Sigma_c(f)$ has infinitely many hyperbolic times.

Let H_n be the set of points for which n is a hyperbolic time.

Lemma (A.-Bonatti-Viana)

There is $\delta_1>0$ such that if $x\in H_n$ and $\varepsilon\le\delta_1$, then the dynamic ball $B_n(x,\varepsilon)$ is mapped diffeomorphically onto $B(f^n(x),\varepsilon)$. Moreover, for all $y,z\in B_n(x,\varepsilon)$ and all $1\le k\le n$ we have

$$d(f^{n-k}(y),f^{n-k}(z))\le e^{-ck/4}d(f^n(y),f^n(z)).$$

Eigenmeasures

Assume that $\mathcal{L}^*_{f,\phi}\nu = \lambda\nu$ for some $\lambda \in \mathbb{R}$ and $\nu \in \mathbb{P}(M)$.

Lemma (Existence of Jacobian)

If $f^k|_A$ is injective, then

$$\nu(f^k(A)) = \int_A \lambda^k e^{-S_k\phi} d\nu.$$

Lemma (Gibbs property)

For each $\varepsilon \leq \delta_1$ there exists $C = C(\varepsilon) > 0$ such that if n is a hyperbolic time for $x \in supp(\nu)$, then for all $y \in B_n(x,\varepsilon)$

$$C^{-1} \leq \frac{\nu(B_n(x,\varepsilon))}{\exp(S_n\phi(y) - n\log\lambda)} \leq C.$$

Assume now that $(f, \phi) \in \mathcal{H}_c^*$.

Proposition

$\lambda_{f,\phi} = e^{P_f(\phi)}$ *is the only real eigenvalue of* $\mathcal{L}_{f,\phi}^*$.

Lemma

$\operatorname{supp}(\nu) = M$.

Given any open set $U \subset M$, we have $M \subset \bigcup_{k\in\mathbb{N}} f^k(U)$. Decompose U into subsets $V_i(k) \subset U$ such that $f^k|_{V_i(k)}$ is injective. We have

$$1 = \nu(M) \leq \sum_k \nu(f^k(U)) \leq \sum_k \sum_i \int_{V_i(k)} \lambda^k e^{-S_k\phi(x)} d\nu$$
$$\leq \sum_k \lambda^k \sum_i \sup_{x\in V_i(k)} (e^{S_k\phi(x)})\nu(V_i(k)).$$

Hence, there exists some $V_i(k) \subset U$ such that $\nu(U) \geq \nu(V_i(k)) > 0$.

Proof of Proposition

For $\varepsilon > 0$ small and $N \in \mathbb{N}$ we have

$$\Sigma_c(f) \subset \bigcup_{n \geq N} \bigcup_{x \in H_n} B_n(x, \varepsilon).$$

Besicovitch Lemma gives a subcovering $\mathcal{U}$ with bounded overlaps.
For any $\gamma > \log \lambda$, by Gibbs property there is $\widetilde{C} = \widetilde{C}(\varepsilon) > 0$ s.t.

$$\sum_{B_n(x,\varepsilon) \in \mathcal{U}} e^{-\gamma n + S_{n,\varepsilon}\phi(x)} \leq \widetilde{C} \sum_{n \geq N} e^{-(\gamma - \log \lambda)n} \leq \widetilde{C} e^{-(\gamma - \log \lambda)N}$$

Taking limit in N we obtain

$$m_f(\phi, \Sigma_c(f), \varepsilon, \gamma) = \lim_{N \to +\infty} m_f(\phi, \Sigma_c(f), \varepsilon, N, \gamma) = 0,$$

which then gives $P_f(\phi, \Sigma_c(f)) \leq \log \lambda$. Since ϕ is c-hyperbolic

$$\log \lambda \leq \log \lambda_{f,\phi} = P_f(\phi) = P_f(\phi, \Sigma_c(f)) \leq \log \lambda.$$

Remark

$\mathcal{H}_c^*$ can be replaced by $\mathcal{H}_c$ if $\Sigma_c(f) \subset \operatorname{supp}(\nu)$ for any eigenmeasure ν.

Continuity of transfer operators

Consider in $C^\alpha(M)$ the seminorm

$$|\varphi|_\alpha = \sup_{x\neq y} \frac{|\varphi(x)-\varphi(y)|}{\mathrm{dist}(x,y)^\alpha}$$

and the norm

$$\|\varphi\|_\alpha = \|\varphi\|_0 + |\varphi|_\alpha.$$

As we are considering f and ϕ Hölder, we can easily see that

$$\mathcal{L}_{f,\phi}(C^\alpha(M)) \subset C^\alpha(M).$$

Let $\mathcal{B}(C^\alpha(M))$ be the space of bounded linear maps from $C^\alpha(M)$ to $C^\alpha(M)$, with $\|\ \ \|_\alpha$ in the first space and $\|\ \ \|_0$ in the second one. Define

$$\Gamma : \mathcal{H}_c^* \longrightarrow \mathcal{B}(C^\alpha(M))$$

assigning to each $(f,\phi) \in \mathcal{H}_c^*$ the restriction of $\mathcal{L}_{f,\phi}$ to $C^\alpha(M)$.

Lemma

Γ *is continuous.*

Let $(f_n, \phi_n)_n$ be any sequence in $\mathcal{H}_c^*$ converging to $(f, \phi) \in \mathcal{H}_c^*$.
For each $x \in M$ and $i = 1, \ldots, \deg(f)$ consider y_i such $f(y_i) = x$.
Since $\deg(f_n) = \deg(f)$ for large n, for each $i = 1, \ldots, \deg(f)$ there is a unique $y_{i,n}$ close to y_i such that $f(y_{i,n}) = x$. Moreover,

$$y_{i,n} \to y_i, \qquad \text{as } n \to \infty.$$

We have

$$\begin{aligned}
\|\mathcal{L}_{f,\phi} - \mathcal{L}_{f_n,\phi_n}\| &= \sup_{\|\psi\|_\alpha \le 1} \|\mathcal{L}_{f,\phi}(\psi) - \mathcal{L}_{f_n,\phi_n}(\psi)\|_0 \\
&\le \sup_{\|\psi\|_\alpha \le 1} \sup_{x \in M} \sum_{i=1}^{\deg(f)} |\psi(y_i)| |e^{\phi(y_i)} - e^{\phi_n(y_{i,n})}| \\
&\quad + \sup_{\|\psi\|_\alpha \le 1} \sup_{x \in M} \sum_{i=1}^{\deg(f)} |e^{\phi_n(y_{i,n})}| |\psi(y_i) - \psi(y_{i,n})|.
\end{aligned}$$

Since $(y_{i,n})_n$ converges to y_i and $(\phi_n)_n$ converges to ϕ, each term in the last inequality converges to zero.

Continuity of the pressure

Theorem

The function $\mathcal{H}_c^* \ni (f, \phi) \longmapsto P_f(\phi) \in \mathbb{R}$ *is continuous.*

Given any sequence $(f_n, \phi_n)_n$ in $\mathcal{H}_c^*$ converging to $(f, \phi) \in \mathcal{H}_c^*$, let

- $\lambda_n = e^{P_{f_n}(\phi_n)}$ the spectral radius of $\mathcal{L}_{f_n,\phi_n}$,
- $\lambda = e^{P_f(\phi)}$ be the spectral radius of $\mathcal{L}_{f,\phi}$.

For all n we have

$$\deg(f_n)e^{\inf \phi_n} \leq \lambda_n \leq \deg(f_n)e^{\sup \phi_n}.$$

The convergence of (f_n, ϕ_n) to (f, ϕ) gives that $(\lambda_n)_n$ is bounded, thus having some accumulation point $\bar{\lambda} \in \mathbb{R}$. As $\lambda_{f,\phi} = e^{P_f(\phi)}$ is the only real eigenvalue of $\mathcal{L}_{f,\phi}^*$, to prove the Theorem above we are left to

Next goal: $\bar{\lambda}$ is an eigenvalue for $\mathcal{L}_{f,\phi}^*$.

Taking subsequences, we may assume that there is $\nu \in \mathbb{P}(M)$ such that

$$\nu_n \xrightarrow{w*} \nu \quad \text{and} \quad \lambda_n \longrightarrow \bar{\lambda}.$$

We need to see that $\mathcal{L}^*_{f,\phi}(\nu) = \bar{\lambda}\nu$. Since $C^\alpha(M)$ is dense in $C^0(M)$, it is enough to show that

$$\mathcal{L}^*_{f,\phi}(\nu)(\psi) = \bar{\lambda}\nu(\psi), \quad \forall\, \psi \in C^\alpha(M).$$

For all $\psi \in C^\alpha(M)$ we have

$$\begin{aligned}
\mathcal{L}^*_{f,\phi}(\nu)(\psi) &= \nu\left(\mathcal{L}_{f,\phi}(\psi)\right) && \text{(by definition)}\\
&= \nu\left(\lim_{n\to\infty} \mathcal{L}_{f_n,\phi_n}(\psi)\right) && \text{(}\Gamma\text{ is continuous)}\\
&= \lim_{n\to\infty} \nu\left(\mathcal{L}_{f_n,\phi_n}(\psi)\right) && (\nu\text{ is continuous})\\
&= \lim_{n\to\infty} \nu_n\left(\mathcal{L}_{f_n,\phi_n}(\psi)\right) && (\nu_n \to \nu)\\
&= \lim_{n\to\infty} \mathcal{L}^*_{f_n,\phi_n}(\nu_n)(\psi) && \text{(by definition)}\\
&= \lim_{n\to+\infty} \lambda_n \nu_n(\psi) && (\nu_n\text{ eigenmeasure})\\
&= \bar{\lambda}\nu(\psi).
\end{aligned}$$

Convergence to equilibrium

Considering again a sequence $(f_n, \phi_n)_n$ in $\mathcal{H}_c$ converging to $(f, \phi) \in \mathcal{H}_c$, let now

- μ_n be the equilibrium state for (f_n, ϕ_n);
- μ_0 be a weak* accumulation point of $(\mu_n)_n$.

To prove our Main Theorem...

Next goal: μ_0 is an equilibrium state for (f, ϕ).

Lemma

μ_0 is an f-invariant measure.

Since each μ_n is f_n-invariant we have for any $\varphi : M \to \mathbb{R}$ continuous

$$\int \varphi \circ f_n \, d\mu_n = \int \varphi \, d\mu_n \longrightarrow \int \varphi \, d\mu_0, \quad \text{as } n \to +\infty.$$

Hence, to verify the f-invariance of μ_0 it suffices to prove that

$$\int \varphi \circ f_n \, d\mu_n \longrightarrow \int \varphi \circ f \, d\mu_0, \quad \text{as } n \to +\infty.$$

For each $n \in \mathbb{N}$ we may write

$$\begin{aligned} \left| \int \varphi \circ f_n \, d\mu_n - \int \varphi \circ f \, d\mu_0 \right| \;&\leq\; \left| \int \varphi \circ f_n \, d\mu_n - \int \varphi \circ f \, d\mu_n \right| \\ &+\; \left| \int \varphi \circ f \, d\mu_n - \int \varphi \circ f \, d\mu_0 \right|. \end{aligned}$$

Variational Principle

We have

$$\begin{aligned} P_f(\phi) &= \lim_{n\to+\infty} P_{f_n}(\phi_n) && \text{(previous Theorem)} \\ &= \lim_{n\to+\infty} \left(h_{\mu_n}(f_n) + \int \phi_n \, d\mu_n \right) && \text{(Walters Thm. + } \mu_n \text{ eq. state)} \\ &= \lim_{n\to+\infty} h_{\mu_n}(f_n) + \int \phi \, d\mu_0 && (\phi_n \to \phi \text{ and } \mu_n \to \mu) \end{aligned}$$

Final goal: $\lim_{n\to+\infty} h_{\mu_n}(f_n) \leq h_{\mu_0}(f)$.

Generating partitions

Given a partition $\mathcal{P}$ and $x \in M$, consider $P^n(x)$ the element in $\vee_{k=0}^{n} f^{-k}\mathcal{P}$ containing the point x. Notice that

$$P^{n+1}(x) \subset P^n(x), \quad \text{for all } n \geq 0. \qquad (*)$$

A partition $\mathcal{P}$ with finite entropy such that for μ almost every $x \in M$ we have $\operatorname{diam} P^n(x) \to 0$ is called a *generating partition* for (f, μ).

Lemma (Araújo)

Let $\mu_n \xrightarrow{w*} \mu_0$ and $\mathcal{P}$ be a generating partition for all (f_n, μ_n) such that $\mu_0(\partial\mathcal{P}) = 0$. Then $\limsup_{n\to\infty} h_{\mu_n}(f_n) \leq h_{\mu_0}(f)$.

By $(*)$, if we find a subsequence of times $(n_k)_k$ such that $\operatorname{diam} P^{n_k}(x) \to 0$ when $k \to \infty$, then $\mathcal{P}$ is a generating partition.

Final step

- Consider $\delta_1 > 0$ given by A.-Bonatti-Viana Lemma.
- Let $\mathcal{P}$ be a finite partition of M with $\operatorname{diam}(\partial P) < \delta_1$ and $\mu_0(\partial \mathcal{P}) = 0$.
- As each $x \in \Sigma_c(f)$ has infinitely many hyperbolic times we have (almost) finished the proof.

Proposition (Ramos-Viana)

If μ is an equilibrium state for $f \in \mathcal{H}_c$, then $\mu(\Sigma_c(f)) = 1$.

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Thank you!