Anosov flows / exponential mixing / robustness

Oliver Butterley

ICTP

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Tarbiat Modares University

Anosov flows

$\Phi^t : M \to M$ is *Anosov* if there exists a continuous invariant splitting $T_x M = \mathbb{E}_s(x) \oplus \mathbb{E}_c(x) \oplus \mathbb{E}_u(x)$ and constants $C, \lambda > 0$, such that:

- $\mathbb{E}_c$ is 1D and coincides with the flow direction;
- $\|D\Phi^t \nu\| \leq Ce^{-\lambda t} \|\nu\|$ for each $\nu \in \mathbb{E}_s$, $t \geq 0$;
- $\|D\Phi^{-t} \nu\| \leq Ce^{-\lambda t} \|\nu\|$ for each $\nu \in \mathbb{E}_u$, $t \geq 0$.

For example:

- Geodesic flows on surfaces of negative curvature;
- Hunt-MacKay triple linkage;
- Suspensions over Anosov maps.

Anosov flow $\Longleftrightarrow$ vector field.

Hunt-MacKay triple linkage

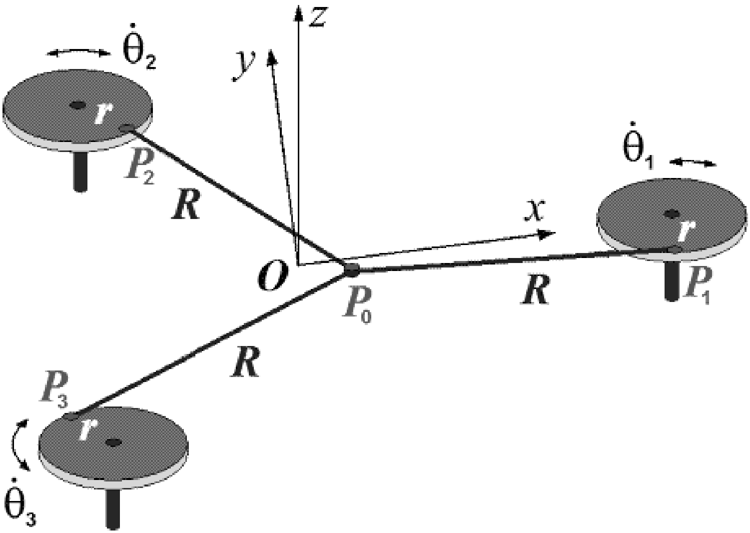


Image: http://www.sgtnd.narod.ru

SRB measures

Each transitive Anosov flow admits a unique SRB measure μ.

Equivalently for Anosov flows (Sinai-Ruelle-Bowen):

- μ has absolutely continuous conditional measures on unstable manifolds.
- There is a full Lebesgue measure subset B such that, for every continuous observable $f : M \to \mathbb{R}$, for every $x \in B$,

$$\frac{1}{t}\int_0^t f(\Phi^t x)\, dt \to \int_M f \; d\mu.$$

Perturbation

Two flows are C^r-*close* if the associated vector fields are C^r close.

Structural stability: A perturbed vector field (small perturbation) also defines an Anosov flow.

Exponential mixing

$\Phi^t : M \to M$ *mixes exponentially* if there exist $C, \gamma > 0$ such that, for all C^1 observables $f, g : M \to \mathbb{R}$ and for all $t \geq 0$,

$$\left| \int_M f \cdot g \circ \Phi^t \, d\mu - \int_M f \, d\mu \cdot \int_M g \, d\mu \right| \leq C \left\| f \right\|_{C^1} \left\| g \right\|_{C^1} e^{-\gamma t}.$$

We will always consider mixing w.r.t. the unique SRB measure.

Bowen-Ruelle conjecture

Every mixing Anosov flow mixes exponentially.

(Un)stable foliations

- (Un)stable bundle is integrable;
- Leaves of the foliation are smooth - (un)stable manifolds;
- (Un)stable bundle of an Anosov flow is Hölder continuous;
- Typically regularity is not better (Hasselblatt-Wilkinson).

Theorem (Dolgopyat)

Suppose that $\Phi^t : M \to M$ is a transitive Anosov flow and that the stable and unstable bundle of an Anosov flow are both C^1. If $\mathbb{E}_s$ and $\mathbb{E}_u$ are not jointly integrable then the flow mixes exponentially.

Joint non-integrability is C^1-open, C^r-dense.
Regularity of at least one bundle would be destroyed by a typical perturbation.

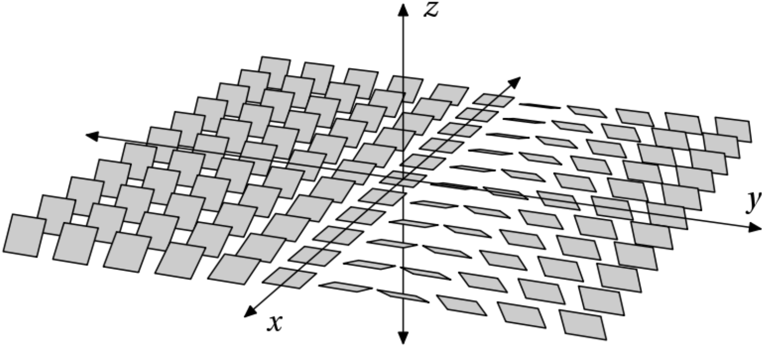
Theorem (Liverani)

Suppose that an Anosov flow preserves a C^2 contact form. Then the flow mixes exponentially.

Regularity of the contact form would be destroyed by a typical perturbation.

Theorem (Tsujii)

There exists a C^3-open, C^r-dense subset of volume preserving 3D Anosov flows which mix exponentially.


z
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Theorem (B-War)

Suppose that $\Phi^t : M \to M$ is a C^{1+} transitive Anosov flow and that the stable bundle is C^{1+}. If $\mathbb{E}_s$ and $\mathbb{E}_u$ are not jointly integrable then the flow mixes exponentially.

Theorem (B-War)

Suppose that $\Phi^t : M \to M$ is a C^{2+} volume-preserving Anosov flow and that $\dim \mathbb{E}_s = 1$ and $\dim \mathbb{E}_u \geq 2$. There exists a C^1-neighbourhood of this flow, such that, for all C^{2+} Anosov flows in the neighbourhood, if $\mathbb{E}_s$ and $\mathbb{E}_u$ are not jointly integrable, then the flow mixes exponentially.

4D volume-preserving case: full solution to the conjecture. (Plante: mixing implies joint non-integrability in codimension one case.)

Bunching (Hirsch-Pugh-Shub)

Suppose that $\Phi^t : M \to M$ is a C^{2+} Anosov flow. If there exists $t, \alpha > 0$ such that

$$\sup_{x \in M} \left\| D\Phi^t\big|_{\mathbb{E}_s}(x) \right\| \left\| D\Phi^t\big|_{\mathbb{E}_{cu}}^{-1}(x) \right\| \left\| D\Phi^t\big|_{\mathbb{E}_{cu}}(x) \right\|^{1+\alpha} < 1,$$

then the stable bundle is $C^{1+\alpha}$.

$(\mathbb{E}_{cu} = \mathbb{E}_u \oplus \mathbb{E}_0)$

- Volume-preserving and codimension one $\implies$ good regularity.
- The bunching estimate is essentially optimal unless other structure implies better regularity (e.g., codimension one or contact structure).

Laplace transform of $\mathcal{L}^t : B \to B$

For all $\Re(z) > 0$ let

$$R(z) := \int_0^\infty e^{-zt} \mathcal{L}^t \, dt.$$

Generator of o.p.s.g.

For all $\mu \in B$ where the limit is defined, let

$$Z\mu := \lim_{t \to \infty} \frac{1}{t} \left(\mathcal{L}^t \mu - \mu \right).$$

Typically $\mathrm{Dom}(Z) \subsetneq B$.
Meromorphic extension of $R(z)$... Control of the poles of $R(z)$...