Complex Feigenbaum Phenomena of High Type

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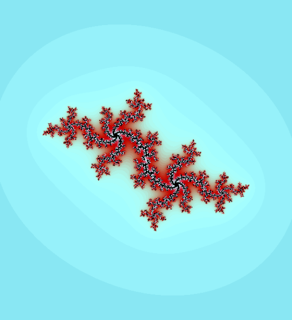
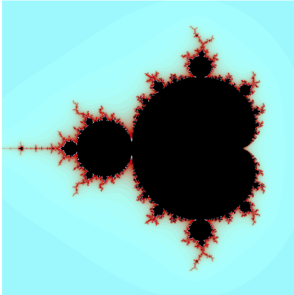
The main object of study is the dynamics of

$$P_c(z) = z^2 + c$$

Julia set $J(P_c) := \{z \in \mathbb{C} \mid \{P_c^n\}_{n=1}^{\infty} \text{ is not equicontinuous at } z\}$
$= \partial\{z \in \mathbb{C} \mid \{P_c^n(z)\}_{n=1}^{\infty} \text{ is bounded}\}$

Mandelbrot set $\mathcal{M} := \{c \in \mathbb{C} \mid J(P_c) \text{ is connected}\}$





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When $J(P_c)$ is locally connected, one can construct a simple topological model for $J(P_c)$ and give a symbolic description of P_c on $J(P_c)$.

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If P_c is at most finitely renormalizable with all periodic points repelling, then $J(P_c)$ is locally connected. (Yoccoz)

Polynomial-like Renormalization

P_c is PL-renormalizable if there is an integer $k > 1$ and simply connected domains $U \Subset V$ such that

- $P_c^k|_U : U \to V = P_c^k(U)$ is a proper branched covering map of degree two
- The little julia set, $\partial\{z \in U \mid P_c^{kn}(z) \text{ remains in } U\}$, is connected.
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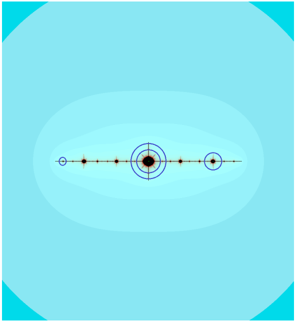

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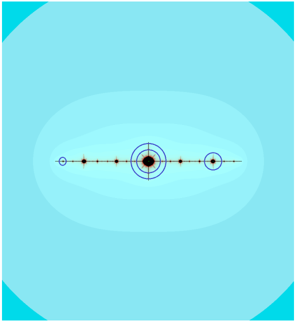
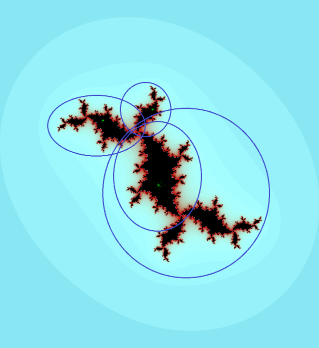
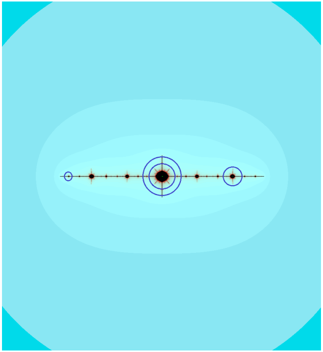
It follows from staightening theorem that $P_c^k|_U$ is topological (hybrid) conjugate to a (unique) quadratic $P_{c'}$.



Primitive and Satellite Renormalization

The little Julia sets for a renormalization are either pairwise disjoint or intersect at exactly one point. We refer to these as primitive and satellite types respectively.





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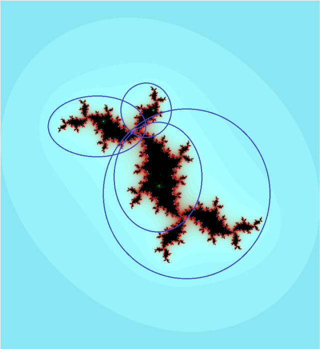

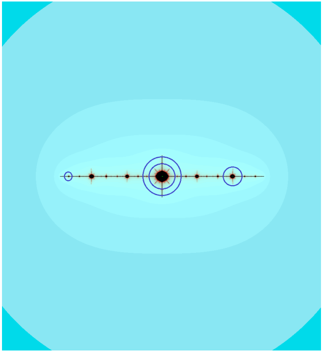

Each satellite renormalization can be combinatorially described by a rational number in $(-1/2, 1/2]$. Every such rational number is realized.

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For real quadratics and several classes of infnitely renormalizable maps of primitive type, a priori bounds exists and J is locally connected (McMullen, Graczyk, Swiatek, Levin, van Strien, Kahn, Lyubich, Yampolsky, Jiang , et al ...)

The Satellite Case

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If $|\frac{p_i}{q_i}|$ converges to zero sufficiently fast, the Julia set is not locally connected. (Douady, Hubbard - '90s)

The Satellite Case

In 2006, Milnor suggested the following criteria as the optimal condition for non-local connectivity of the Julia sets :

$$\sum_{k=1}^{\infty}\left(\frac{p_{k+1}}{q_{k+1}}\right)^{1/q_k} < \infty$$

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In 2009, Levin showed the this condition was sufficient but not necessary. He showed that if

$$q_k \to \infty \text{ and } \limsup_k (\frac{p_{k+1}}{q_{k+1}})^{1/q_k} < 1$$

then $J(P_c)$ is not locally connected.

The Post Critical Set

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If P_c is infinitely renormalizable and $J(P_c)$ is locally connected, then $\mathcal{PC}(P_c)$ is a Cantor set.

High type combinatorics

Any rational number $\frac{p}{q} \in (-1/2, 1/2] \setminus \{0\}$ can be written as

$$\frac{p}{q} = \pm\cfrac{1}{b_1 \pm \cfrac{1}{\ddots \pm \cfrac{1}{b_n}}}$$

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We define the high type combinatorics as :

$$\mathcal{HT}_N := \{\{p_i/q_i\}_{i=1}^{\infty} \mid b_{i,j} > N\}$$

We fix a large $N_0 \in \mathbb{N}$ and for all $\tau \in \mathcal{HT}_{N_0}$ we choose a specific parameter $c(\tau) \in \mathcal{M}$ which is infinitely satellite renormalizable with combinatorics τ.

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The class of maps we consider is :

$$\mathcal{S}_{N_0} := \{P_{c(\tau)} \mid \tau \in \mathcal{HT}_{N_0}\}$$

A Dichotomy

Cheraghi, P. - 2017

For all $\tau \in \mathcal{HT}_{N_0}$, we have one of the following two statements:

- τ satisfies *generalised Herman-Yoccoz condition and* $\mathcal{PC}(P_{c(\tau)})$ *is a Cantor set of points,*
- τ *does not satisfy generalised Herman-Yoccoz condition and* $\mathcal{PC}(P_{c(\tau)})$ *is a hairy Cantor set.*

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Corollary

If $\tau \in \mathcal{HT}_{N_0} \setminus \mathcal{GHY}$, then $J(P_{c(\tau)})$ is not locally connected.

Herman-Yoccoz Condition

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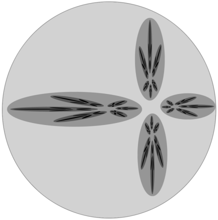
The condition $\mathcal{GHY}$ on sequence of rationals, is very similar to the condition $\mathcal{H}$ on irrational numbers.

Hairy Cantor Set

A *Hairy Cantor set* is a compact set $X \subset \mathbb{R}^2$ such that :



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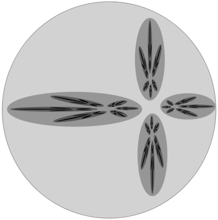

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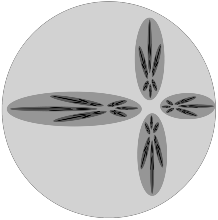

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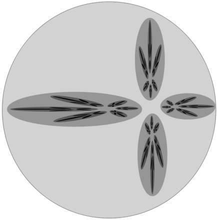

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- Every curve in X can be approximated with other curves in X.
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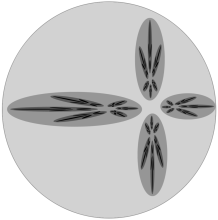

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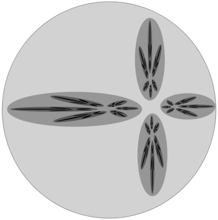

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All hairy Cantor sets are homeomorphic.

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- This is a powerful method that have been used to construct the first examples of quadratic Julia sets with positive area by Buff and Cheritat in 2008.
- Every map in the set $\mathcal{S}_{N_0}$ is in fact both infinitely satellite renormalizable and infinitely near-parabolic renormalizable.

Thank you !

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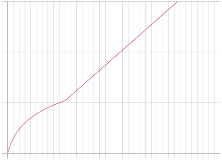

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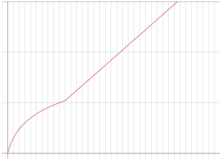

Then $\alpha \in \mathcal{H}$ iff

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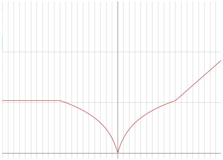

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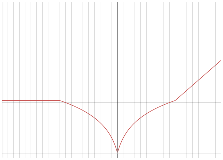

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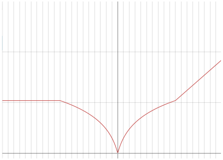

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