The Gauss map, Lagrange-Markov spectrum and Cookie cutters

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Overview

Often simple ideas from dynamical systems can be used to prove interesting results in other areas of mathematics.

Question

What sort of problems might we like to understand?

- We will first consider a simple dynamical system (The Gauss map)
- We will relate this to results in number theory (Diophantine approximation)
- We will formulate a particular problem (on the Lagrange-Markov spectra)
- We will analyze this using cookie cutters (related to the Gauss map)

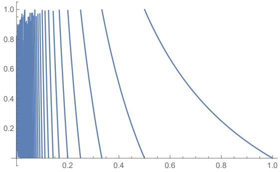
All of these objects will be defined (hopefully).

The Gauss map

The *Gauss map* $G : [0,1) \to [0,1)$ is a classical map of the interval $[0,1)$ to itself defined by

$$G(x) = \begin{cases} \frac{1}{x} - \left[\frac{1}{x}\right] & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

We can plot the graph:



It is monotone and expanding on each interval $\frac{1}{k+1} < x < \frac{1}{k}$ $(k \in \mathbb{N})$.

Gauss

Johann Carl Friedrich Gauss (1777-1855) was a famous german mathematican whose smiling face was on german banknotes from 1991-2002.

The "Gauss transformation" was central to his work on continued fractions. On 30 January, 1812 he famously wrote to the mathematician Laplace describing properties of G for which he said he had found by a "very simple reasoning". However, (as usual in these stories) no details were given nor ever found in his notes.

Continued fractions and the Gauss map

As for any dynamical system, given $x \in (0,1]$ we can consider its orbit

$$x, G(x), G(G(x)), \cdots, G^n(x), \cdots$$

If $\frac{1}{k_1+1} < x < \frac{1}{k_1}$ $(k_1 \in \mathbb{N})$ then we can write

$$x = \frac{1}{k_1 + G(x)}.$$

Continuing this for the iterates $G^n = G \circ \cdots \circ G$ $(n \in \mathbb{N})$:

Assume that for each $n \geq 1$ we have $\frac{1}{k_n+1} < G^{n-1}(x) < \frac{1}{k_n}$ $(k_n \in \mathbb{N})$ then we can write

$$x = \frac{1}{k_1 + \frac{1}{k_2 + \frac{1}{k_3 + \cdots}}} =: [k_1, k_2, k_3, \cdots]$$

This is the *continued fraction expansion* of x.

Gauss map and diophantine approximation

Given an irrational number $x \in (0,1]$, the first part of the orbit $G^n(x)$, $0 \le n \le N$ gives a rational number

$$\frac{p_N}{q_N} := \frac{1}{k_1 + \frac{1}{k_2 + \cdots \frac{1}{k_N}}}$$

It is easy to show that

$$\left| x - \frac{p_N}{q_N} \right| \le \frac{1}{q_N^2}$$

which shows the following classical result:

Theorem (Dirichlet, 1840)

There infinitely many rational numbers $\frac{p}{q}$ $(p, q \in \mathbb{Z},\ q \neq 0)$ satisfying

$$\left| x - \frac{p}{q} \right| \le \frac{1}{q^2}$$

Dirichlet

Johann Peter Gustav Lejeune Dirichlet (1805 1859) was a german mathematician working in number theory and analysis.

After Gauss died in 1855, it was Dirichlet who took his chair in Gottingen. Dirichlet was married Rebecka Mendelssohn the sister of the famous german composer Mendelssohn.

Better approximations: Lagrange spectrum

Question

How can we improve on Dirichet's theorem?

For different irrational α we can choose the largest values $c(\alpha) > 1$ such that

$$\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{c(\alpha)q^2}$$

still has infinitely many solutions with $\frac{p}{q} \in \mathbb{Q}$ i.e.,

$$c(\alpha) = \inf\{|q|.|q\alpha - p| : p, q \in \mathbb{Z}, q \neq 0\}.$$

For example, $c\left(\frac{1+\sqrt{5}}{2}\right) = \sqrt{5}$ and $c\left(\sqrt{2}-1\right) = \sqrt{8}$.

Definition

One defines the Lagrange spectrum $\mathcal{L}$ to be the collection of all these constants $\mathcal{L} \subset \mathbb{R}$, i.e.,

$$\mathcal{L} = \{c(\alpha) : \alpha \in \mathbb{R} - \mathbb{Q}\}.$$

The Lagrange spectrum $\mathcal{L}$ and continued fractions

There is a more dynamical characterization of the Lagrange spectrum $\mathcal{L}$.

Let $\Sigma = \mathbb{N}^{\mathbb{Z}}$ be the space of bi-infinite sequences of natural numbers and define:

- the shift map $\sigma : \Sigma \to \Sigma$ which moves sequences one place to the left, i.e., $(\sigma \underline{x})_n = x_{n+1}$ where $\underline{x} = (x_n) \in \Sigma$; and
- a function $\lambda : \Sigma \to \mathbb{R}^+$ by

$$\lambda(\underline{x}) = x_0 + [x_1, x_2, x_3, \cdots] + [x_{-1}, x_{-2}, \cdots].$$

where

$$\underline{x} = (\ldots, x_{-j}, \cdots, x_{-1}, x_0, x_1, \cdots, x_j, \cdots) \in \Sigma$$

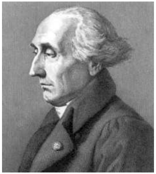
Lemma (Pisot, 1931)

We can rewrite

$$\mathcal{L} = \sup_{i \in \mathbb{Z}} \left\{ \lambda(\sigma^i \underline{x}) : \underline{x} \in \Sigma \right\}.$$

Lagrange

Joseph-Louis Lagrange (1736 - 1813) was an italian mathematician who worked in Paris. Appointed a senator in France in 1799, he was involved in the annexation of his native Piedmont to France - thus obtaining french citizenship in the process.



He was the first professor of analysis at the Ecole Polytechnique in Paris. According to Fourier (a student at the time) his teaching wasn't very good:

"his voice is very feeble ... he has a very marked Italian accent and pronounces the s like z ...The students, of whom the majority are incapable of appreciating him, give him little welcome"

Of course, he didn't actually define the Lagrange spectrum .

The Markov spectrum $\mathcal{M}$ and continued fractions

Recall that one can write

$$\mathcal{L} = \sup_{i \in \mathbb{Z}} \{\lambda(\sigma^i \underline{a}) : \underline{x} \in \Sigma\}.$$

where $\lambda : \Sigma \to \mathbb{R}^+$ was defined by by

$$\lambda(\underline{x}) = x_0 + [x_1, x_2, x_3, \cdots] + [x_{-1}, x_{-2}, \cdots].$$

where $\underline{x} = (\ldots, x_{-j}, \cdots, x_{-1}, x_0, x_1, \cdots, x_j, \cdots) \in \Sigma = \mathbb{N}^{\mathbb{Z}}$.
If we consider the limit supremum in place of the supremum then we get what is called the Markov spectrum:

Definition

The Markov spectrum $\mathcal{M} \subset \mathbb{R}$ is defined by

$$\mathcal{M} = \left\{ \limsup_{i \in \mathbb{Z}} \{\lambda(\sigma^i \underline{x})\} : \underline{x} \in \Sigma \right\}.$$

Markov spectra

Question

Who would care about the set $\mathcal{M}$?

It also has a simple interpretation in terms of number theory.
Consider those binary quadratic forms

$$Q(x, y) = ax^2 + bxy + cy^2 \ (a, b, c \in \mathbb{R})$$

with $a, b, c \in \mathbb{R}$ and discriminant $b^2 - 4ac = 1$.
Associate

$$\alpha(Q) = \inf \left\{ \frac{1}{|Q(n, m)|} : (n, m) \in \mathbb{Z}^2 \setminus (0, 0) \right\}.$$

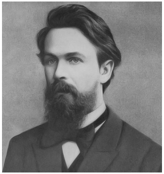
Definition (Markov, 1879; Pisot, 1921)

The *Markov spectrum* is also given by

$$\mathcal{M} = \left\{ \frac{1}{\alpha(Q)} : Q(x, y) = ax^2 + bxy + cy^2 \text{ and } b^2 - 4ac = 1 \right\}$$

Markov

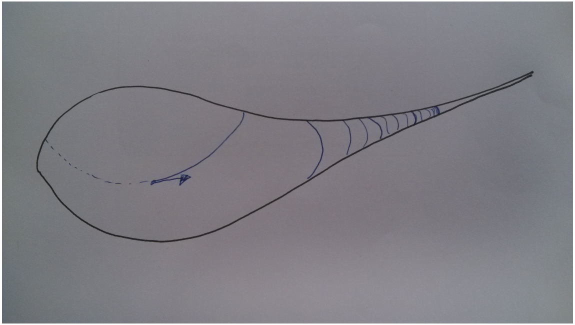
Andrey Andreyevich Markov (1856 1922) was a famous russian mathematician from St. Petersburg.



He is particularly well know for his work on *Markov chains*. However, in two papers in 1879 and 1880 (the same year as his Master's thesis) he published two fundamental papers on Number Theory.
Markov was a man of strong opinions. He was an aethiest and was ex-communicated from Russian Orthodox Church (at his own request).
In 1917 he was sent to Zaraisk, a small country town, where he taught mathematics in the secondary school without salary (at his own request).

Aside: Geodesic flows and the spectra

Although we will not pursue it, there is another dynamical interpretation of both the Lagrange and Markov spectra. This is based on the classical connection between the geodesic flow on the modular surface and continued fractions.



In particular, these spectra can be understood in terms of geodesics which make excursions of bounded heights into the cusp.

Comparing the sets $\mathcal{L}$ and $\mathcal{M}$

Surprisingly (or not) the sets $\mathcal{L}, \mathcal{M} \subset \mathbb{R}_+$ are very similar.

- Below 3 they they take the same (countable) set of values, i.e.,

$$\mathcal{L} \cap (0,3] = \mathcal{M} \cap (0,3] = \{\sqrt{5}, \sqrt{8}, \sqrt{221}/5 \cdots\}$$

- Hall (1947) showed that above 4.5 they are both a half-line, i.e.,

$$\mathcal{L} \cap [4.5, +\infty) = \mathcal{M} \cap [4.5, +\infty) = [4.5, +\infty)$$

However, they are actually different:

- Tornheim (1955) showed $\mathcal{L} \subset \mathcal{M}$;
- Freiman (1968) showed $\mathcal{L} \neq \mathcal{M}$;

Question

How big is the set $\mathcal{M} \setminus \mathcal{L}$?

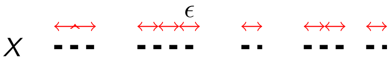
In fact, it has zero Lebesgue measure. Therefore, we might like to know how large a set of zero measure it might be.

Dimension of sets

We want to consider the dimension of certain dynamically defined sets $X \subset \mathbb{R}$.

In some of our examples the Hausdorff Dimension will be the same as the Box Dimension, so we can cheat and recall its (simpler) definition instead.

For each $\epsilon > 0$ we let $N(\epsilon)$ be the smallest number of intervals of length ϵ needed to cover X.



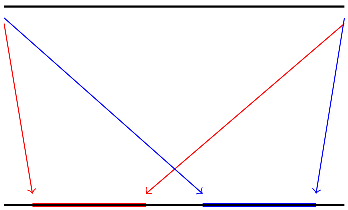
Definition

We define the dimension by: $\dim(X) = \limsup_{\epsilon \to 0} \frac{\log N(\epsilon)}{\log(1/\epsilon)}$

A simple construction: Cookie cutters

Consider a cookie cutter (or iterated function scheme) given by $T_1, T_2, \cdots, T_k : [0,1] \to [0,1]$ where

1. Each T_i is a C^ω contraction.
2. The images are disjoint (i.e., $T_i[0,1] \cap T_j[0,1] = \emptyset$ for $i \neq j$).

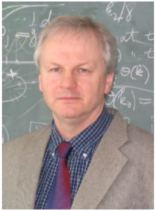


The limit set Λ is the Cantor set of limit points

$$\Lambda = \left\{ \lim_{n \to +\infty} T_{i_1} T_{i_2} \cdots T_{i_n}(x_0) : i_1, i_2, \cdots \{1, \cdots, k\} \right\} \text{ for any } x_0 \in [0,1].$$

Cookie cutters

The first occurrence of the term "cookie cutter" (according to MathSciNet) appears in the paper by Thomas Bohr and David Rand, *The entropy function for characteristic exponents. Phys. D 25 (1987), no. 1-3, 387-398*


David Rand is (still) my colleague at Warwick.
In their paper they attribute the term to Dennis Sullivan.

Warm-up exercise : Middle third Cantor set

Let us begin with a trivial example.
Consider the contractions $T_1, T_2 : [0,1] \to [0,1]$ defined by

$$T_1(x) = \frac{x}{3} \text{ and } T_2(x) = \frac{x}{3} + \frac{2}{3}$$

The limit set Λ is the usual middle third Cantor set, i.e.,

$$\Lambda = \left\{ \sum_{n=1}^{\infty} \frac{j_n}{3^n} : j_1, j_2, j_3, \cdots \in \{0,2\} \right\}.$$

In this special case it is easy to compute the dimension $\dim(\Lambda)$.
It is easy to see from the definitions that

$$\dim(\Lambda) = \frac{\log 2}{\log 3} = 0.630929753571457 43710 \cdots$$

which we can easily compute to whatever reasonable accuracy we choose.

Markov and Lagrange Spectra revisited

We now recall recent work of Matheus and Moreira on the difference $\mathcal{M} \setminus \mathcal{L}$ between the Markov and Lagrange spectra.
They showed that this zero measure set has non-zero Hausdorff Dimension:

Theorem (Matheus-Moreira)

There is a lower bound

$$\dim_H(\mathcal{M} \setminus \mathcal{L}) > 0.343\ldots$$

This leads to two natural questions

Question

What has this to do with cookie cutters?
Can we improve this bound?

$\mathcal{M} \setminus \mathcal{L}$ lower bound

The approach of Matheus and Moreira is to observe that

$$\dim_H(\mathcal{M} \setminus \mathcal{L}) \geq \dim(\Lambda)$$

where Λ is the limit set for the cookie cutter associated to

$$T_1(x) = \frac{1}{1+x} \text{ and } T_2(x) = \frac{1}{2 + \frac{1}{2+x}}.$$

We can improve this bound:

Theorem (Jenkinson-P.)

$\dim_H(\mathcal{M} \setminus \mathcal{L}) \geq \dim(\Lambda) = 0.355 \cdots.$

Perhaps the improvement from $0.343\ldots$ to $0.355\ldots$ doesn't seem so impressive?

Question

Can we do better with limit sets for other iterated function schemes?

$\mathcal{L} \setminus \mathcal{M}$ and iterated function schemes

From the paper of Matheus and Moreira one can also see

$$\dim_H(\mathcal{M} \setminus \mathcal{L}) \geq \dim(\Lambda)$$

where Λ is now the limit set for the five contractions

$$T_1(x) = \frac{1}{1 + \frac{1}{1+\frac{1}{1+x}}}, \quad T_2(x) = \frac{1}{1 + \frac{1}{1+\frac{1}{2+x}}} \quad T_3(x) = \frac{1}{2 + \frac{1}{1+\frac{1}{1+x}}}$$

$$T_4(x) = \frac{1}{2 + \frac{1}{2+\frac{1}{1+x}}} \quad T_5(x) = \frac{1}{2 + \frac{1}{2+\frac{1}{2+x}}}$$

This leads to a better lower bound:

Theorem (Jenkinson-P.)

$$\dim_H(\mathcal{M} \setminus \mathcal{L}) \geq \dim(\Lambda) = 0.418\ldots$$

This is at least an improvement on the lower bounds of $0.343\cdots$ and 0.355

Matheus, Moreira and Jenkinson



Carlos Matheus and friend; Carlos “Gugu” Moreria and Oliver Jenkinson.

The question that remains is:

Question

How do we get (rigorous) estimates on the limit sets of cookie cutters?

Dynamical determinants

We begin with some notation.

- For $n \geq 1$, let $\underline{i} = (i_1, \cdots, i_n) \in \{1,2,3,4,5\}^n$ and $|\underline{i}| = n$; and
- Let $x_{\underline{i}} = T_{\underline{i}}(x_{\underline{i}})$ be the fixed point for

$$T_{\underline{i}} = T_{i_1} \circ \cdots \circ T_{i_n} : [0,1] \to [0,1].$$

We can associate a function $D(z,t)$ of two variables z and t, called a *dynamical determinant* formally defined by

$$D(z,t) := \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{|\underline{i}|=n} \frac{|(T_{\underline{i}})'(x_{\underline{i}})|^t}{1-(T_{\underline{i}})'(x_{\underline{i}})} \right)$$

where $z \in \mathbb{C}$ and $t \in \mathbb{R}$.

Lemma (after Bowen, Posthumous paper 1978)

Let $z = 1$. Then $t = \dim(\Lambda)$ solves $D(1,t) = 0$.

Estimates using determinants

Question

How can we use the determinant to actually estimate the dimension?

We need to replace the function(s) $z \mapsto D(z,t)$ by more computer friendly polynomials $P_N(z)$.
Choose $N \geq 1$. Let us write the series expansion

$$D(z,t) = 1 + \sum_{n=1}^{\infty} a_n(t) z^n = \underbrace{1 + \sum_{n=1}^{N} a_n(t) z^n}_{=:P_N(z)} + \underbrace{\sum_{n=N+1}^{\infty} a_n(t) z^n}_{=:\epsilon_N(z)}$$

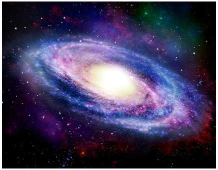
where $P_N(z)$ is a polynomial in z and $\epsilon_N(z)$ is the associated "error ".

Ideally, we would like to choose N large to make ϵ_N small.

Computing $P_N(z)$

But we can only choose N as large as our computer (and our own patience) allows.

For $N = 11$: Takes a week to compute $P_{11}(z)$
For $N = 14$: Takes a year to compute $P_{14}(z)$
For $N = 30$: Takes at least the age of the universe to compute $P_{30}(z)$ (i.e., 13 Billion years, according to Planck and Wikipedia)



We can choose $N = 11$ (one week being the limit of my patience)

Question

How good an error does this give?

The lower bound on $\dim_H(\mathcal{M} \setminus \mathcal{L})$ is accurate to 13 decimal places.

The End

Thank you for your attention