On cohomological theory on dynamical zeta functions

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Cohomological theory: the case of Anosov diffeomorphism

Let $f : M \to M$ be an Anosov diffeomorphism. By Lefchetz fixed point formula, we can count its periodic orbits:

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\begin{aligned}
(-1)^{\dim E_u} \cdot \#\mathrm{Fix}(f^n) &= \sum_{j=0}^{\dim M} (-1)^j \mathrm{Tr}((f^n)^* : H^j(M) \to H^j(M)) \\
&= \sum_{i=1}^{I} \pm \rho_i^n
\end{aligned}
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Hence the Artin-Mazur zeta function is written

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\zeta(z) :&= \exp\left(-\sum_{n=1}^{\infty} \frac{z^n \#\mathrm{Fix}(f^n)}{n} \right) \\
&= \prod_i \exp\left(\mp \sum_{n=1}^{\infty} \frac{z^n \rho_i^n}{n} \right) = \prod_i \exp\left(\pm \log(1 - \rho_i z) \right) \\
&= \prod_i (1 - \rho_i z)^{\pm 1}
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Theorem (Smale,1967)

For Anosov diffeomorphism $f : M \to M$, the Artin-Mazur zeta function

$$\zeta(z) = \exp\left(-\sum_{n=1}^{\infty} \frac{z^n \#\mathrm{Fix}(f^n)}{n}\right)$$

is a rational function and its zeros and poles $\{1/\rho_i\}$ come from the action of f on the cohomology space (and satisfies the symmetry as a consequence from Poincare duality).

Question

Can we extend this to the case of Anosov flows?

Is this a good question?

- The action of the flow on cohomology space is trivial.
- How we count the periodic orbits? (Periods are not topological.)

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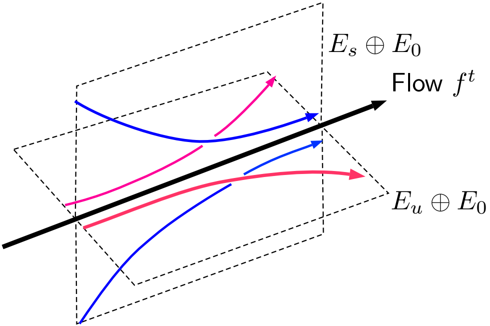
Anosov flow

Definition (Anosov flow)

A flow $f^t : M \to M$ is an *Anosov flow* if $\exists$ Df^t-invariant C^0-splitting

$$TM = E_0 \oplus E_s \oplus E_u$$

such that $E_0 = \langle V := \partial_t f^t \rangle$ and $Df^t|_{E_u}$ (resp. $Df^t|_{E_s}$) is exponentially expanding (resp. contracting).

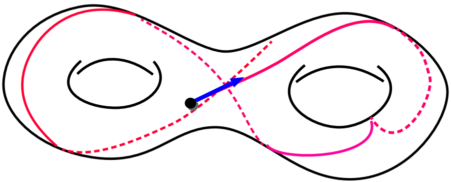


Geodesic flows on negatively curved manifolds

The geodesic flow on a closed Riemann manifold N is a flow

$$f^t : M \to M, \qquad M := T_1 N : \text{unit tangent bundle of } N$$

which describes the motion of free particle (of unit speed) on N.



Fact

If the sectional curvature of N is negative everywhere, its geodesic flow f^t is a (contact) Anosov flow and exhibit strongly chaotic behavior of trajectories.

Remark. We will mainly consider this kind of flow.

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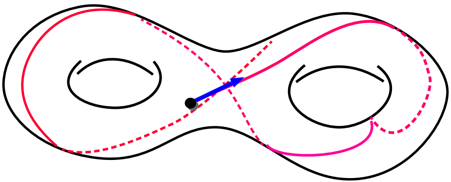

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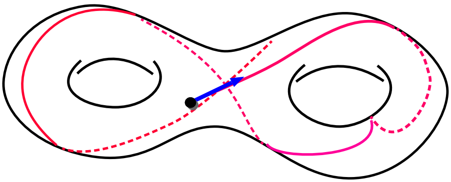

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Transfer operators

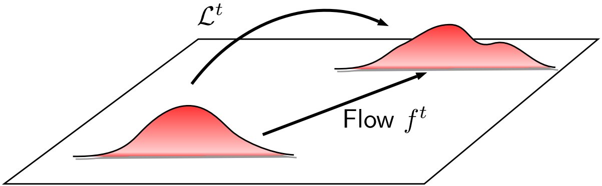
Studying chaotic dynamical systems, it is useful to consider a "cloud" of initial conditions and observe its evolution by the flow.

Definition (Ruelle transfer operator)

Let $f^t : M \to M$ be a flow on a closed manifold M. The Ruelle transfer operator is a one-parameter group of operators of the form

$$\mathcal{L}^t : C^\infty(M) \to C^\infty(M), \quad \mathcal{L}^t u = (g^t \cdot u) \circ f^{-t}$$

where g^t is a C^∞ function on M. Its generator is $X = -V + \partial_t g^t|_{t=0}$.

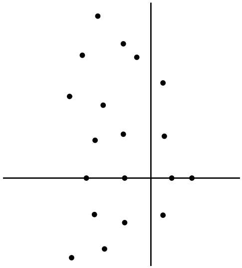


Ruelle-Pollicott resonances

A recent discovery in ergodic theory of smooth dynamical systems is that the transfer operators associated to hyperbolic dynamical systems exhibit "discrete spectrum".

Theorem (Butterley-Liverani, Faure-Sjostrand)

For an Anosov flow (or more general uniformly hyperbolic flows), the generator X of Ruelle transfer operators L^t have "discrete spectrum" $\{\rho_i\}$, called Ruelle-Pollicott resonances.



Remark

To observe the discrete spectrum in $\Re(s) > -C$, we need to consider "anisotropic Sobolev spaces"

$$C^\infty \subset C^r \subset H^r \subset (C^r)' \subset \mathcal{D}'$$

with some $r = r(C)$ adapted to hyperbolic structure of flow. The space H^r necessarily consists of "distributions."

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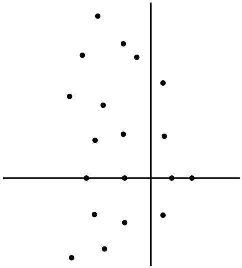

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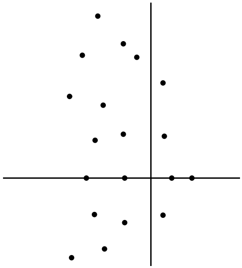

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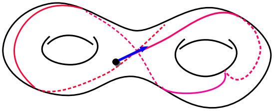
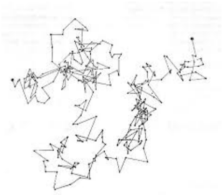
Chaotic dynamics vs. Diffusion process (on a cpt mfd)





	Hyperbolic flow	Brownian Motion
System	Deterministic	Probabilistic
	ODE	Brownian motion
Mechanism	Expansion by flow	Random external force
Action on density	Transfer operator $\mathcal{L}^t$	Heat semi-group $H^t = e^{\Delta t}$
Generator	$X = V + \partial_t g^t\|_{t=0}$	Laplacian Δ
Spectrum	R-P resonance	Discrete Eigenvalues of Δ

Atiyah-Bott trace of L^t

The transfer operator

$$L^t u(y) = g^t(f^{-t}(y)) \cdot u(f^{-t}(y)) = \int g^t(x)\delta(x - f^{-t}(y))u(x)dx$$

is expressed as an integral operator

$$L^t u(y) = \int K(y,x)u(x)dx \quad \text{with } K^t(y,x) = g^t(x)\delta(x - f^{-t}(y)).$$

The Atiyah-Bott trace (or flat trace) of $\mathcal{L}^t$ is defined as

$$\begin{aligned}\mathrm{Tr}^\flat \mathcal{L}^t &= \int K^t(x,x)dx = \int g^t(x) \cdot \delta(\mathrm{Id} - f^{-t}(x))dx \\ &= \sum_{\gamma \in PO} \sum_{n=1}^{\infty} \frac{|\gamma| \cdot g_\gamma^n \cdot \delta(t - n|\gamma|)}{|\det(\mathrm{Id} - D_\gamma^{-n})|}\end{aligned}$$

where PO be the set of (prime) periodic orbits, $|\gamma|$ is the prime period of γ, $g_\gamma := g^{|\gamma|}(x)$ for a point x on γ, D_γ is the differential of the Poincare map.

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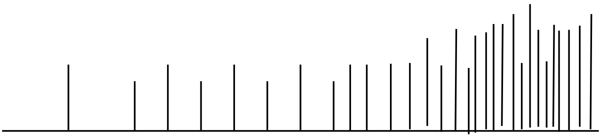
Dynamical trace formula

If we confuse the Atiyah-Bott trace with the usual one, we expect that the distribution of periods of periodic orbits (counted with some weight) is given by the spectrum of the generator of $\mathcal{L}^t$:

$$\mathrm{Tr}^\flat \mathcal{L}^t = \sum_{\gamma \in PO} \sum_{n=1}^{\infty} \frac{|\gamma| \cdot g_\gamma^n \cdot \delta(t - n|\gamma|)}{|\det(\mathrm{Id} - D_\gamma^{-n})|} \quad \sim \quad \sum_i e^{\rho_i t}$$

This is indeed true if we interpret the right-hand side appropriately. (cf. the recent work of Dyatlov-Zworski). The formula is a bit magical!



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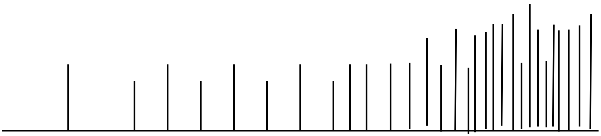

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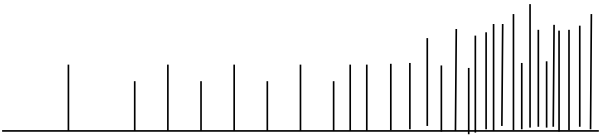

Dynamical Fredholm determinant

The dynamical Fredholm determinant of L^t is defined

$$d(s) := \exp\left(- \sum_{\gamma \in PO} \sum_{n=1}^{\infty} \frac{1}{n} \frac{g_\gamma^n \cdot e^{-sn|\gamma|}}{|\det(\mathrm{Id} - D_\gamma^{-n})|} \right)$$

$$= \exp\left(- \int_{0+}^{\infty} \frac{e^{-st}}{t} \mathrm{Tr}^\flat \mathcal{L}^t dt \right) = \exp\left(- \mathrm{Tr}^\flat \int_{0+}^{\infty} \frac{e^{-(s-X)t}}{t} dt \right)$$

$$\sim \exp(\mathrm{Tr} \log(s - X)) \sim \det(s - X) \sim \prod_i (s - \rho_i)$$

Excersise: Try to justify the last line (not too seriously).

Theorem

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Theorem (Smale's conjecture, Giuletti-Liverani-Pollicott 2013)

The dynamical zeta function extends to a meromorphic function on $\mathbb{C}$.

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Up to technical argument, the point is that $\zeta(s)$ is expressed as

$$\zeta(s) = \prod_{j=0}^{\dim M - 1} d_j(s)^{(-1)^j}$$

where $d_j(s)$ is the dyn. Fred. det. of the (vector-valued) transfer operator

$$\mathcal{L}_j^t : \Gamma^\infty((T^*M^\perp)^{\wedge j}) \to \Gamma^\infty((T^*M^\perp)^{\wedge j})$$

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Gutzwiller-Voros zeta function

From now on, we consider geodesic flow f^t on negatively curved manifold.

The Gutzwiller-Voros zeta function

$$\zeta_{sc}(s) = \exp\left(-\sum_{\gamma \in PO} \sum_{n=1}^{\infty} \frac{1}{n} \frac{e^{-sn|\gamma|}}{\sqrt{|\det(\mathrm{Id} - D_\gamma^{-n})|}} \right)$$

introduced by physicists in "semi-classical theory of quantum chaos".
It is expressed as an alternating product

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of the dyn. Fred. det. $d_j^u(s; \mathcal{L}_j^t)$ of the (vector-valued) transfer operators

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given as the natural action of f^t. ($\mathrm{Det}^u = (E_u)^{\wedge d}$)

Gutzwiller-Voros zeta function

From now on, we consider geodesic flow f^t on negatively curved manifold.

The Gutzwiller-Voros zeta function

$$\zeta_{sc}(s) = \exp\left(- \sum_{\gamma \in PO} \sum_{n=1}^{\infty} \frac{1}{n} \frac{e^{-sn|\gamma|}}{\sqrt{|\det(\mathrm{Id} - D_\gamma^{-n})|}} \right)$$

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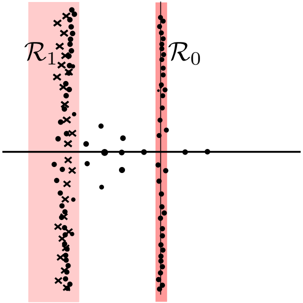
Analytic properties of Gutzwiller-Voros zeta function

Theorem (Faure-T, 2017)

The Gutzwiller-Voros zeta function $\zeta_{sc}(s)$ *extends to a meromorphic function on* $\mathbb{C}$*. For any* $\varepsilon > 0$*, the zeros are contained in the union of*

$$\mathcal{R}_0 = \{|\Re(s)| < \varepsilon\}, \qquad \mathcal{R}_1 = \{\Re(s) < -\chi + \varepsilon\}$$

and the poles are contained in $\mathcal{R}_1$*, but for finitely many exceptions. The zeros in* $\mathcal{R}_0$ *satisfies an analogue of Weyl law: Density* $\sim |\mathrm{Im} s|^{\dim N - 1}$



Cohomological theory

Idea by V. Guillemin (1977)

The zeros and poles of $\zeta_{sc}(s)$ will come from the action of the flow on "leafwise cohomology space".

Recall that we have $\zeta_{sc}(s) = \prod_{j=0}^{\dim E_u} d(s; \mathcal{L}_j^t)^{(-1)^j}$ with

$$\mathcal{L}_j^t : \Gamma^\infty(V_j) \to \Gamma^\infty(V_j) \quad \text{where } V_j := |\mathrm{Det}^u|^{1/2} \otimes (E_u^*)^{\wedge j}$$

We have the following commutative diagram:

$$(\star) \qquad \begin{array}{ccccccc} \Gamma^\infty(V_0) & \xrightarrow{\delta_0^u} & \Gamma^\infty(V_1) & \xrightarrow{\delta_1^u} & \cdots & \xrightarrow{\delta_{d-1}^u} & \Gamma^\infty(V_d) \\ \downarrow{\scriptstyle \mathcal{L}_0^t} & & \downarrow{\scriptstyle \mathcal{L}_1^t} & & & & \downarrow{\scriptstyle \mathcal{L}_d^t} \\ \Gamma^\infty(V_0) & \xrightarrow{\delta_0^u} & \Gamma^\infty(V_1) & \xrightarrow{\delta_1^u} & \cdots & \xrightarrow{\delta_{d-1}^u} & \Gamma^\infty(V_d) \end{array}$$

where δ_k^u denotes exterior derivative along unstable leafs.

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Rigorous justification of cohomological theory

Theorem (T, 2018)

There are (scales of) Hilbert spaces $\Lambda_j \supset \Gamma^\infty(V_j)$, *obtained as the completion of* $\Gamma^\infty(V_j)$ *w.r.t. some norm, such that the last diagram* $(\star)$ *extends to*

$$\begin{array}{ccccccc} \Lambda_0 & \xrightarrow{\delta_0^u} & \Lambda_1 & \xrightarrow{\delta_1^u} & \cdots & \xrightarrow{\delta_{d-1}^u} & \Lambda_D \\ \downarrow{\scriptstyle \mathcal{L}_0^t} & & \downarrow{\scriptstyle \mathcal{L}_1^t} & & & & \downarrow{\scriptstyle \mathcal{L}_d^t} \\ \Lambda_0 & \xrightarrow{\delta_0^u} & \Lambda_1 & \xrightarrow{\delta_1^u} & \cdots & \xrightarrow{\delta_{d-1}^u} & \Lambda_D \end{array}$$

and the generator of $\mathcal{L}_j^t : \Lambda_j \to \Lambda_j$ *exhibits discrete spectrum. In particular,*

$$\mathrm{Div}\zeta_{sc} = \sum_{j=0}^{D} (-1)^j \cdot \mathrm{Spec}(\mathbb{A}_j)$$

where $\mathbb{A}_j$ *is the generator of the action on the leaf-wise cohomology space*

$$\mathbb{L}_j^t = \mathcal{L}_j^t : \mathbb{H}^j \to \mathbb{H}^j, \quad \mathbb{H}^j := \ker(\delta_j^u)/\overline{\mathrm{Im}(\delta_{j-1}^u)}$$

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Final remarks

Remark

The zeros of $\zeta_{sc}(s)$ in the neighborhood $\mathcal{R}_0$ of the imaginary axis comes from the discrete spectrum of the action on the bottom cohomology class: $\mathcal{L}_0^t : \mathbb{H}^0 = \ker(\delta_0^u) \to \mathbb{H}^0$. This is the "geometric quantization" of f^t, taking the unstable foliation as "polarization".

Remark

For the moment, we have no result about the spectrum of the generator of $\mathbb{L}_j^t : \mathbb{H}^j \to \mathbb{H}^j$ for $0 < j \leq d$. We expect that they gives only small number of zeros and poles.

Now how about the case of Smale's dynamical zeta function?