

# Lecture 1

## AdS<sub>d</sub> black holes in d ≥ 4

We are interested in BHs that can be embedded in string theory or M-theory and are asymptotic to AdS<sub>d</sub> vacua with a known field theory dual. There are plenty that can be embedded in the maximally supersymmetric backgrounds:

$$\begin{array}{ll} \text{AdS}_4 \times S_7 & \Rightarrow \text{ABJM theory in 3d} \\ \text{AdS}_5 \times S_5 & \Rightarrow N=4 \text{ SYM in 4d} \\ \text{AdS}_7 \times S_4 & \Rightarrow (2,0) \text{ theory in 6d} \end{array}$$

They are all characterized by a set of charges and angular momentum. For example

$$\begin{array}{ll} \text{AdS}_5 \times S_5 & \begin{array}{l} SO(2,4) \times SO(6) \\ \text{U}(1)^2 \subset SO(4) \quad \text{U}(1)^3 \subset SO(6) \\ (J_1, J_2) \text{ spin} \quad (R_1, R_2, R_3) \text{ charges} \end{array} \end{array}$$

we have two spins and three charges. (Obviously, supersymmetry imposes a constraint among them:  $f(J_i, R_k) = 0$  (but see Martevicute-Santos 1806.01249)).

In this game, AdS<sub>4</sub> is special because we can also introduce magnetic charges

$$\begin{array}{ll} \text{AdS}_4 \times S_7 & \begin{array}{l} SO(2,3) \times SO(8) \\ U(1) \subset SO(3) \quad U(1)^4 \subset SO(8) \\ J \quad (R_1, R_2, R_3, R_4) \text{ electric} \\ \quad (m_1, m_2, m_3, m_4) \text{ magnetic} \end{array} \end{array}$$

and have dyonic black holes with one angular momentum in AdS<sub>4</sub> and four electric and four magnetic charges in S<sub>7</sub>. Again supersymmetry imposes constraints. In

In this case, there is a linear constraint among magnetic charges,  $\sum m_i = \text{constant}$ , and a constraint among all other charges  $f(m_i, R_i, j) = 0$ .

There is a big difference between black holes with magnetic charges (which only exist in 4d) and the others that is well explained by holography. Consider the effective theory on AdS<sub>d</sub> (often a gauged supergravity or a consistent truncation – most of our BH have been found in d dimensions and then uplifted to 10 or 11 d)

$$\sqrt{g} (R + g_{\Lambda\Sigma} F_{\mu\nu}^\Lambda F^{\mu\nu\Sigma} + \dots )$$

where the vector fields  $F_{\mu\nu}^\Lambda$  correspond to the isometry of the internal manifold, which we break for simplicity to the Cartan subgroup (for example, in AdS<sub>4</sub>,  $F_{\mu\nu}^\Lambda$  correspond to  $U(1)^4 \subset SO(8)$ , isometry of  $S^7$ ). According to the rules of holography, gauge fields in the bulk correspond to global symmetries on the boundary CFT (in AdS<sub>4</sub>,  $SO(8)$  becomes the R-symmetry of ABJM, in AdS<sub>5</sub>  $SO(6)$  becomes the R-symmetry of  $N=4$  SYM and so on). The BH are asymptotic to AdS<sub>d</sub> in global coordinates, so that the metric and fields behave, for large  $r$ , as

$$\left\{ \begin{array}{l} ds \sim \frac{dr^2}{r^2} + r^2 dS_{M_{d-1}} + \dots \\ A^\Lambda \sim a^\Lambda(r) r^\alpha + \dots \end{array} \right. (*)$$

where  $M_{d-1} = \mathbb{R} \times S^{d-2}$  (time direction times a sphere). For more exotic BH, the boundary can be replaced by  $R \times M_{d-2}$ , where  $M_{d-2}$  can be a Riemann surface for  $d=4$  or a more general manifold. Holography tells us that we need to interpret (\*) as the dual of our CFT<sub>d-1</sub> on the curved manifold

$M_{d-1}$  : what's about  $A^\Lambda$ ? Recall that, for any field  $\phi(v, x)$  in  $AdS_d$ , if we have an expansion of the form

$$\phi(v, x) = v^{d_1} \underbrace{\phi_0(x)}_{\text{non-normalizable}} + v^{d_2} \underbrace{\phi_1(x)}_{\text{normalizable}} + \dots$$

of the second order equations of motion, we interpret  $\phi_0$  as a deformation of the dual CFT with the operator associated to  $\phi$  (call it  $O$ )

$$h_{\text{CFT}} \rightarrow h_{\text{CFT}} + O\phi$$

and  $\phi_1$  as a vac for  $O$ :  $\langle O \rangle \neq 0$ . So, if  $v$  is a solution,  $\phi_0(x) \neq 0$ , we are deforming the CFT; if  $\phi_0 = 0$  and  $\phi_1 \neq 0$  we are considering the CFT in the state with vac  $\langle O \rangle \neq 0$ . There are cases where  $\phi_0$  and  $\phi_1$  are both normalizable. In this case there are two possible quantizations of the theory and you need to "choose" who plays the role of  $\phi$ .

- For electrically charged rotating BH,  $A^\Lambda$  falls off at large  $r$  as a normalizable mode. The same is true of the metric deformation induced by rotation: these BH correspond to a spinful charged set of states of the CFT on  $M_{d-1} = R \times S_{d-2}$  (or  $R \times M_{d-2}$ )

- For magnetically charged BH, in  $AdS_5$ ,  $A^\Lambda$  approach a constant value for large  $r$  corresponding to a magnetic charge on the boundary  $M_3 = R \times S^2$  (or  $R \times \Sigma_g$ )

$$\boxed{\frac{1}{2\pi i} \int_{S^2} F^\Lambda = p^\Lambda \in \mathbb{Z}}$$

$\nwarrow$  (or  $\Sigma_g$ )

Explicitly,  $A^\Lambda = -\frac{p^\Lambda}{r^2} \cos \theta d\phi$  on  $S^2$  (or generalizations on  $\Sigma_g$ ). So (in the natural quantization of the theory) the CFT is deformed by a magnetic background for the global symmetries

$$h_{\text{CFT}} + A^\Lambda J_\Lambda$$

(a monopole background gauge field has been turned on)

To understand better what is going on, it is useful to look at how supersymmetry is preserved. The gravitino variation is schematically

$$S\psi_\mu = \partial_\mu \epsilon + \frac{1}{4} \overset{\text{ab}}{\underset{\uparrow}{w_\mu}} \overset{\text{ab}}{\Gamma} \epsilon + \overset{\text{R}}{\underset{\uparrow}{A_\mu}} \epsilon = \partial_\mu \epsilon = 0$$

spin connection =  
unit monopole on  $S^2(\mathbb{Z}_3)$

linear combination  
of  $A^A$  selecting ... a(0)  
R-symmetry

and it is satisfied (I'm cheating a bit here - this is technically true for static BH but the logic is general) by cancelling the spin connection with a background for the R-symmetry. This also tells us that the combination of the charges  $p^A$  is fixed to the precise value that cancels the spin connection. We will be more precise later. If we restrict the variation to the boundary we see that the same mechanism is at work in the CFT: on  $\mathbb{R} \times S^2$  (or  $\mathbb{R} \times \mathbb{R}^2$ ), supersymmetry is preserved by a constant spinor  $\epsilon$ . The magnetic background for the R-symmetry field  $A_\mu^R$  compensates the spin connection. This is called a topological twist, that we will discuss more in Lecture 2. The conclusion is that the magnetically and electrically charged rotating BHs in  $AdS_4$  are dual to a set of charged spinful states in the CFT on  $\mathbb{R} \times S^2$  (or  $\mathbb{R} \times \mathbb{R}^2$ ) topologically twisted by the presence of a magnetic background for a  $U(1)$  R-symmetry and possibly deformed by a magnetic background for all other symmetries.

Notice that, for electrically charged and rotating BH, the Killing spinors  $\epsilon(x)$  are non-trivial functions of the coordinates.

Now, it is clear what we should do to compute the entropy of these BHs using field theory:

- Enumerate all the states with electric charges  $Q_i$  and spin  $J_k$  in the CFT on  $\mathbb{R} \times S^{d-2}$  (or  $\mathbb{R} \times M_{d-2}$ ) that preserve the same supersymmetry of the BH when there are no magnetic charges, or in the topologically twisted CFT when there are magnetic charges  $n_i$  (as we will see the  $n_i$  parametrizes the possible inequivalent twists)

Enumerating states is equivalent to knowing the grand-canonical partition function

$$Z(\Delta_i, w_k) = \sum_{J_k, Q_i} C(J_k, Q_i) e^{i(\Delta_i Q_i + w_k J_k)} = \text{tr} [e^{i(\Delta_i R_i + w_k S_k)}]$$

superconformal states

where  $C(J_k, Q_i)$  is the number of states that preserve the same supersymmetry of the BH with charge  $Q_i$  and spin  $J_i$  and  $R_k$  and  $S_k$  are the charge and spin operators. If  $Z(\Delta_i, w_k)$  is known, the number of states (whose logarithm is the entropy) can be obtained by a Fourier expansion

$$e^{S(J_k, Q_i)} = C(J_k, Q_i) = \int \frac{d\Delta_i}{2\pi} \frac{dw_k}{2\pi} Z(\Delta_i, w_k) e^{-i(\Delta_i Q_i + w_k J_k)}$$

In the limit of large charges this can be evaluated by a saddle point approximation

$$S(J_k, Q_i) = \log Z(\Delta, w) - i(\Delta_i Q_i + w_k J_k) \Big|_{\bar{\Delta}, \bar{w}}$$

where  $\bar{\Delta}$  and  $\bar{w}$  are obtained as solution of

$$\partial_{\Delta, w} (\log Z(\Delta, w) - i(\Delta_i Q_i + w_k J_k)) = 0$$

As expected, the entropy is just the Legendre transform of the logarithm of the partition function.

The problem is that  $Z(\Delta, \omega)$  is hard to compute. What we can compute using localization is the supersymmetric trace (an index).

$$\begin{aligned} Z_{\text{susy}}(\Delta, \omega) &= \text{tr} \cdot (-1)^F e^{i(\Delta_i R_i + i \omega_k T_k)} \\ &= \text{tr} (-1)^F e^{-\beta H} e^{i(\Delta_i R_i + \omega_k T_k)} \quad (\text{only } H=0 \\ &= \sum_{S^1 \times M_{d-2}}^{\text{susy}} (\Delta, \omega) \end{aligned}$$

where in the last step we identify the partition function at temperature  $T = 1/\beta$  with the euclidean path integral with euclidean time compactified as a circle of radius  $\beta$ .

In general  $Z_{\text{susy}}(\Delta, \omega) \neq Z(\Delta, \omega)$ . The entropy counts all the states, an index counts bosonic states with a plus and fermionic states with a minus. But, if we are lucky there are no cancellations between bosonic and fermionic ground states at large charges. And indeed

- For magnetically charged BHs the index correctly reproduce the BH entropy
- For all other BH there is cancellation and the problem is still open.

In the rest of the lectures we will consider the case of magnetically charged BH in AdS<sub>5</sub>. These BH can be static and we focus on this case. In particular, we will have in mind BHs embeddable in AdS<sub>5</sub> × S<sup>7</sup>. We can perform a consistent truncation to 4d obtaining an N=2 gauged supergravity with 3 vector multiplets

$$(g_{\mu\nu}, A_{\text{graviphoton}}^0, \dots) \oplus (A_\mu^i, \bar{z}^i, \dots)_{i=1,2,3}$$

for a total of four vectors  $A_\mu^i$ ,  $i=0,1,2,3$ , for  $U(1)^4 \subset SO(8)$ . The prepotential is

$$F = -2i \sqrt{x_0 x_1 x_2 x_3} \quad z^i = x^i/x^0$$

and the FI terms are  $\mathcal{G}_1 = g$  and  $\mathcal{G}^1 = 0$  (only electric in this frame). There is a general family of dyonic static BH (Katznadar; Halmagyi '14, '15 generalized the original example by Cacciatori-Klemm '08 and Dall'Agata-Gnecchi; Hristov-Vandoren '10)

$$ds^2 = -e^{2V(r)} dt^2 + e^{-2V(r)} (dr^2 + V(r)^2 d\bar{s}_g)$$

with charges

$$\int_{\bar{s}_g} F^1 = \text{vol}(\bar{s}_g) p^1 \quad \int_{\bar{s}_g} G_1 = \text{vol}(\bar{s}_g) q_1 \\ (\text{magnetic}) \quad (\text{electric}) \quad (G_1 = 8\pi G_N \frac{\partial \mathcal{H}}{\delta F^1})$$

Supersymmetry imposes 2 constraints:

$$\bullet \quad g_{\mu} p^1 - g^1 q_1 \equiv -\kappa \quad \kappa = \int_0^1 \frac{s^2}{\bar{s}_g s} \Rightarrow \sum p^1 = -\frac{\kappa}{g}$$

This is the twisting condition

$$\delta W_{\mu A} = \partial_\mu E_A + \frac{1}{4} W^{ab} T_{ab} E_A + i g_A A_\mu^i S_{AB} \epsilon^{BC} \epsilon_C \equiv \partial_\mu E_A \\ \boxed{\text{cancels spin connection}}$$

and leaves three independent magnetic charges

- a non-linear constraint  $f(p^i, q_i)$  that can be written in terms of the quartic invariant of supergravity [Halmagyi]. This leaves three independent electric charges

The charges are quantized:

$$\eta g_1 p^1 \in \mathbb{Z}$$

$$\frac{\eta}{4G_N g_1} g_1 \in \mathbb{Z}$$

where  $\eta = 2|g-1|$  for  $g \neq 1$  and  $\eta = 1$  for  $g = 1$  and I normalize  $\text{Vol}(\Sigma_g) = 2\pi\eta$

The entropy of the BH can be obtained by an attractor mechanism that follows from BPS equations at the horizon. You need to extremize [Dall'agata-Ghezzi '10] the quantity ( $F_1 = \frac{\delta F}{\delta x^1}$ )

$$\tilde{I} = -i \frac{g_1 x^1 - p^1 F_1}{g_1 x^1 - g^0 F_1} \Rightarrow x_*^1 \text{ at the horizon}$$

with respect to the scalar fields. The entropy is given by

$$S_{BH}(p^1, q_1) = \frac{\text{Vol}(\Sigma_g)}{4G_N} \tilde{I}(x_*^1)$$

The  $x_*^1$  are in general complex. The implicit condition that  $S_{BH}$  on  $x_*^1$  is real is part of the BPS requirements and is equivalent to the non-linear constraint among charges  $f(p^1, q_1) = 0$ . The explicit expression for the entropy is quite complicated. For  $p^1 = p^2 = p^3 = p$ ,  $p^0 = -1/g - 3p$  one can write (with zero electric charges)

$$S \cong \sqrt{-1 + 6p - 6p^2 + (-1 + 2p)^{3/2} \sqrt{6p - 1}}$$

To finish, let's notice that the static BH's are asymptotic to  $AdS_4$  for large  $r$

$$ds^2 \sim \frac{dr^2}{r^2} + r^2(-dt^2 + ds_{\Sigma_g})$$

and to  $AdS_2 \times \Sigma_g$  at the horizon ( $r = r_c$ )

$$ds^2 \sim \frac{dr^2}{(r-r_c)^2} - (r-r_c)^2 dt^2 + ds_{\Sigma_g}$$

This suggests a RG flow from  $AdS_4$  to  $AdS_2$  and the existence of a IR CFT<sub>1</sub>, IR limit of the twisted compactification of ABJM on  $\Sigma_g$ . If it is so, the entropy should correspond to the "regularized" number of ground states of the CFT<sub>1</sub>. We will come back to this point later.



## Lecture 2

### Topological twist on $\Sigma g$

Magnetically charged BH in  $AdS_5$  are dual to topologically twisted CFT<sub>3</sub> on  $\Sigma g$  and their entropy should be obtained as the Legendre transform of the topologically twisted index

$$Z(\Delta) = \text{tr} (-1)^F e^{-\beta H} e^{i\Delta J}$$

where  $H$  is the Hamiltonian of the topologically twisted theory and  $\Delta$  are chemical potentials for the symmetries of the theory.  $Z(\Delta)$  can be computed in many different ways

- Bethe vacua : Nekrasov-Shatashvili '09 '14

Ohta-Yashida '12

- Localization : Benini-AZ '15 '16

Corset-Gremmehsi-Kim-Willet '15 '16 '17

We look at the problem through localization, which also leads to the other approach. [Benini-AZ 1503.03698 - 1605.06120]

Consider an  $N=2$  theory in 3d. There are two multiplets

$$\text{vector} = (A_p, \lambda, \lambda^+, \sigma, D)$$

$$\text{chiral} = (\phi, \psi, F)$$

$\lambda, \psi$  Dirac spinors

which are the dimensional reduction from  $N=1$  in 4d. Assume that the theory has an R-symmetry

$$\lambda \rightarrow e^{-i\alpha} \lambda \quad \phi \rightarrow e^{i\eta_p \alpha} \phi \quad \psi \rightarrow e^{i\eta_p \alpha} \psi$$

with  $\eta_p$  integer. On  $S^2 \times S^1$  we can turn on a background for the R-symmetry

$$ds^2 = R^2 (d\theta^2 + \sin^2 \theta d\phi^2) + \beta^2 dt^2$$

$$A_p^R = \frac{1}{2} \cos \theta d\phi$$

In order to preserve supersymmetry, we need to solve the Killing spinor eqs. In the Festuccia-Seiberg approach we couple to gravity and freeze metric and other fields and we look for solutions of the gravitino variation

$$\delta \psi_p = D_\mu \varepsilon = \partial_\mu \varepsilon + \frac{1}{4} w_\mu^{ab} \gamma_{ab} \varepsilon + i A_\mu^R \varepsilon = 0$$

which are very simpler if we just turn on  $A_\mu^R$ . With  $\gamma^a = \sigma^a$  and  $\varepsilon$  of charge -1, this equation is solved by  $\varepsilon = (\begin{smallmatrix} \varepsilon^+ \\ 0 \end{smallmatrix})$  ( $\gamma_3 \varepsilon = \varepsilon$ ) if  $\varepsilon^+$  is constant

$$D_\mu \varepsilon = \partial_\mu \varepsilon + \frac{i}{2} w_\mu^{12} \delta_3 \varepsilon + i A_\mu^R \varepsilon = \partial_\mu \varepsilon^+ = 0 \Rightarrow \varepsilon^+ \text{ constant}$$

$$w^{12} = -\cos \theta d\phi \quad A_\mu = \frac{1}{2} \cos \theta d\phi$$

This is just no topological twist in the sense of Witten: turning on  $A_\mu^R$  is like twisting the spinor bundle and  $\varepsilon$  becomes effectively a scalar on  $S^3$  and supersymmetry works as in flat space. The same trick works on  $\mathbb{R}^4$ : just take  $A_\mu = -2w_\mu$

Using Noether's method, or Seiberg-Festuccia one; or add elaborations on twisting we can write the Lagrangian

$$\mathcal{L} = \frac{1}{g^2} \text{tr} \left[ \frac{1}{4} F_{MN} F^{MN} + \frac{1}{2} D_\mu \delta D^\mu \delta + \frac{D^2}{2} - \frac{i}{2} \lambda^+ \gamma^\mu D_\mu \lambda - \frac{i}{2} \lambda^+ [\delta, \lambda] \right] \xleftarrow{\text{LHM}}$$

$$- \frac{ik}{4\pi} \text{tr} \left[ \varepsilon^{\mu\nu\rho} (A_\mu \partial_\nu A_\rho - \frac{2i}{3} A_\mu \lambda_\nu A_\rho) + \lambda^+ \lambda + 2D\delta \right] \xleftarrow{\text{LCS}}$$

$$+ \frac{1}{e^2} \text{tr} \left[ D_\mu \phi^+ D^\mu \phi + \phi^+ (s^2 + iD - V_\phi F_R^R) \phi + FF + i\psi^+ (\gamma^\mu D_\mu - \delta) \psi - i\psi^+ \lambda \phi + i\phi^+ \bar{\psi} \right]$$

where, as usual, in euclidean signature a field and its complex conjugate become independent variables. Notice that  $\delta$  in the vector multiplet behaves as a real mass for the chiral multiplet. SUSY transformations are:

$$\left\{ \begin{array}{l} Q A_\mu = \frac{i}{2} \lambda^+ \gamma_\mu \varepsilon \\ Q \lambda = \frac{1}{2} \gamma^{\mu\nu} F_{\mu\nu} \varepsilon - D\varepsilon + i \gamma^\mu D_\mu \delta \varepsilon \\ Q \lambda^+ = 0 \\ Q \delta = -\frac{1}{2} \lambda^+ \varepsilon \\ Q D = -\frac{i}{2} D_\mu \lambda^+ \gamma^\mu \varepsilon + \frac{i}{2} [\lambda^+, \delta] \end{array} \right.$$

$$\left\{ \begin{array}{l} \tilde{Q} A_\mu = \frac{i}{2} \tilde{\varepsilon}^+ \gamma_\mu \lambda \\ \tilde{Q} \lambda = 0 \\ \tilde{Q} \lambda^+ = -\frac{i}{2} \tilde{\varepsilon}^+ \gamma^{\mu\nu} \tilde{F}_{\mu\nu} + \tilde{\varepsilon}^+ D + i \tilde{\varepsilon}^+ \gamma^\mu D_\mu \delta \\ \tilde{Q} \delta = -\frac{i}{2} \tilde{\varepsilon}^+ \lambda \\ \tilde{Q} D = \frac{i}{2} \tilde{\varepsilon}^+ \gamma^\mu D_\mu \lambda + \frac{i}{2} [\delta, \tilde{\varepsilon}^+] \end{array} \right.$$

The important point is that  $L_{\text{HJM}}$  and  $L_{\text{matter}}$  are exact

$$\tilde{\varepsilon}^+ L_{\text{HJM}} = Q \tilde{Q} \text{tr} \left[ \frac{\lambda \lambda^+}{2} + D\delta \right]$$

$$\tilde{\varepsilon}^+ L_{\text{matter}} = Q \tilde{Q} \text{tr} [\psi^+ \psi + 2i \phi^+ \phi]$$

so that they are of the form  $(Q + \tilde{Q})(\cdot)$ , since  $Q^2 = \tilde{Q}^2 = 0$ . Also a superpotential term would be exact. The only non exact term is  $L_{\text{CS}}$ .

Localization works by sending  $g, e \rightarrow 0$  and performing the saddle point approximation.

The path integral  $Z_{S^2 \times S^1}$  becomes (Benini - AZ '15)

$$\frac{1}{|W|} \sum_m \int_C Z_{\text{int}}(v, m) dv$$

↑  
order Weye group  
sum over gauge magnetic fluxes on  $S^2$

meromorphic integrand  
specific integration contour

Let's analyse all these ingredients.

- A) The saddle points are actually the solutions of the gaugino variation

$$\delta \lambda = \left( \frac{1}{2} F_{\mu\nu} \gamma^{\mu\nu} - D \right) \varepsilon + i \gamma^\mu D_\mu \sigma \varepsilon \quad \varepsilon = \begin{pmatrix} \varepsilon^+ \\ 0 \end{pmatrix}$$

and are labeled by mutually commuting gauge zero-modes

$$G = \text{constant} \quad A_t = \text{constant} \quad \left\{ \text{combine them into } U = A_t + i \beta \right.$$

$$i F_{12} = D \quad \Rightarrow \quad \frac{1}{2\pi} \int_{S^2} F = m \in \text{coroot lattice} \quad \left( e^{2\pi i m} = I_6 \right)$$

We can always diagonalize them: they live in the Cartan subalgebra and are defined up to the Weye group. The Weye elmo  $A_t$  is periodic:  $A_t \cong A_t + 2\pi$  const (physical object being the holonomy  $e^{iA_t}$ ) so  $v$  lives naturally on a cylinder. It is convenient to exponentiate and use  $x = e^{iv}$

Example,  $U(1)$ :  $v \in S^1 \times \mathbb{R} \rightarrow x \in \mathbb{C}^*$

$$\begin{array}{ccc} \text{---} & & \text{---} \\ \bullet & & \bullet \\ \delta = \infty & & \delta = -\infty \end{array} \qquad \qquad \begin{array}{c} \uparrow \\ \text{---} \\ \bullet \end{array}$$

- B) The integrand is  $Z_{\text{int}} = Z_{\text{ee}} Z_{\text{vector}} \sum_{\text{chiral}} \left( \det_{ab} \frac{\partial^2 \log Z_{\text{ee}} Z_{\text{vector}}^{\text{1-loop}}}{\partial v_a \partial v_b} \right)^g$

$Z_{\text{ee}}$  only from CS  $\rightarrow$

chiral in representation  $\rho$  of  $G$

$\chi_f = R$  charge  
must be integer!

$$Z_{\text{ee}} = x^{km} = \prod_{i=1}^{\text{rank } G} x_i^{k m_i}$$

$$Z_{\text{vector}}^{\text{1-loop}} = \prod_a (1 - x^a)^{1-g} (idv)^{\text{rank } G}$$

$$Z_{\text{chiral}}^{\text{1-loop}} = \prod_{p \in R} \left( \frac{x^{p_2}}{1 - x^p} \right)^{\rho(p)} + (g-1)(v_f - 1)$$

The determinant coming from integration of  $g$  fermionic modes on  $S^2$ .

- c) The contour is dictated by susy. It is highly non-trivial and it is known as JK contour (Jeffrey-Kirwan). It specifies in what order to take residues when many hyperplanes meet. oversimplifying here, for a  $U(1)$  theory, means: take two poles of fields with positive charge and poles at  $x=0$  ( $\text{if } k < 0$ ) or at  $x=\infty$  ( $\text{if } k > 0$ ). JK appeared in elliptic genus [Beinini-Eager-Hori-Tachikawa '13] and Witten index [Hori-Kim-Yin '14] Localization computations.

### TWO INTERPRETATIONS

- I) Compactifying an  $\Sigma_g$  we obtain a quantum mechanics. As usual supersymmetric path integral on  $S^1$  compute traces

$$Z_{\Sigma_g \times S^1} = \text{tr}(-1)^F e^{-\beta H_{\Sigma_g}} = \text{Witten index of the QM}$$

This is the QM of all modes on  $\Sigma_g$  in the presence of magnetic fluxes: Landau levels on  $\Sigma_g$ . Witten index is a number (of vacua) — pretty boring. But if we have flavor symmetries we can turn on supersymmetric backgrounds for them. Since  $U(1)$  flavor symmetry =  $U(1)$  background gauge symmetry we can turn on

$$\frac{1}{2\pi i} \int_{S^1} F^F = n \quad \text{magnetic flux}$$

$$U^F = \Delta^F + i \sigma^F \beta = \underset{\text{Wilson loop}}{=} (\text{chemical potential} + \text{red mass})$$

This is easy done in the 1-loop contribution

$$Z_{\text{dilute}}^{1\text{-loop}} \rightarrow \prod_{\rho \in R} \left( \frac{x^{\rho(F)} y}{1 - x^{\rho(F)}} \right)^{\rho(u) + n + (q-1)(v_F - 1)}$$

where  $x = e^{iu}$  is the gauge variable and  $y = e^{ivF}$  is a background. Now

$$Z_{\Sigma_g \times S^1}(y, u) = \frac{1}{|W|} \sum_{\mu} \int_{W} f dx Z_{\text{int}}(x, \mu; y, u) = \text{tr}(-1)^F e^{-\beta H} e^{i \Delta^F T}$$

where  $T$  is the  $U(1)$  flavor charge. This is an "equivariant index": it counts the number of ground-states of the QM weighted by powers of  $y^n$  that account for the  $U(1)$  charge of the vacuum

$$Z(y, u) = \sum g(n) y^n \quad g(n) = \# \text{ ground states (with sign: } \pm \text{ boson/ferm) with charge } n$$

This follows from the susy algebra:  $\{q, \bar{q}\} = H - \delta^F T$  so that indeed

$$\text{tr}(-1)^F e^{-\beta H} e^{i \Delta J} = \text{tr}(-1)^F e^{-\beta \frac{q}{2} \bar{q}} e^{i (C^F + i \beta \delta^F) J} = \bar{Z} g(u) y^n$$

only states with  
 $Q\bar{Q}=0$  contribute  
 in an index

Note that  $\delta^F$  is a real man and serves as a "regulator": we are left with a finite number of vacua with charges  $n$  even if the original  $\mathcal{L}$  has flat directions. We can also understand the 1-loop determinant for chiral fields as follows. Masses mode cancel in the index and we are left with zero modes. Susy in 1d admits two multiplets: chiral  $(\phi, \psi)$  and Fermi  $(\lambda, \text{Auxiliary})$ . A free chiral with charge 1 has index (Factors of  $\sqrt{y}$  are just a choice of overall charge)

$$\frac{\sqrt{y}}{1-y} = \sqrt{y}(1+y+y^2+\dots)$$

and a free Fermi  $\frac{1-y}{\sqrt{y}}$ . The number of such zero modes is then given by Riemann-Roch:  $p(n) + n + (g-1)(k_f - 1)$  and the sign decides chirality and chooses chiral multiplet versus Fermi.

**II)** Reduce on  $S^1$  to a two dimensional theory on  $\Sigma_g$ . It is convenient to look at an example. Take SQED,  $U(1)$  with  $q$  with charge +1 and  $\bar{q}$  with charge -1

	$U(1)$	$U(1)_R$	$U(1)_A$	$U(1)_T$
$q$	1	1	1	0
$\bar{q}$	-1	1	1	0

There is no superpotential. There are many  $R$ -symmetries, I just picked one,  $U(1)_R$ .  $U(1)_A$  is a flavor symmetry and  $U(1)_T$  the topological symmetry. Consider, for simplicity,  $g=0$  and ignore  $U(1)_T$

$$Z = \sum_m \int_{2\pi i x} \left( \frac{\sqrt{xy}}{1-xy} \right)^{m+n} \left( \frac{\sqrt{yx}}{1-yx} \right)^{-m+n} = \sum_m \int_{2\pi i x} \left( \frac{y-x}{1-xy} \right)^m \left( \frac{yx}{(1-x)(x-y)} \right)^n$$

JK instructs to take the poles for  $q$  (positive charge) that exist for  $m > -n$ . Resumming with some large cut-off  $m \geq -M$

$$Z = \int_{2\pi i x} \frac{P(x)^{-M}}{1-P(x)} Q(x) = \sum_{P(x^*)=1} \frac{Q(x^*)}{-x^* P'(x^*)} \quad (\text{residue theorem})$$

where  $x^*$  are the zeros of  $\frac{y-x}{1-xy} = 1 \Rightarrow x^* = -1$ . The explicit evaluation gives the partition function of 3 chiral with superpotential  $xyz$  which is indeed the mirror of SQED.

This is an example of the general situation:

The matrix model is always schematically of the form

$$x = e^{iu} \quad Z = \int dx_i \sum_{M_{JK}} e^{+im_i \frac{\partial W}{\partial u_i}} Q(u) = \int dx_i \prod_i \frac{e^{iM_i \frac{\partial W}{\partial u_i}}}{(1 - e^{+i \frac{\partial W}{\partial u_i}})} Q(u)$$

since JK typically selects "half" of the values for  $m_i$ ,  $m_i \geq -M_i$ . The function  $W$  is given by

$$W(u) = \frac{k}{2} u^2 + \frac{i}{2} x (\log x - 1) - L_2(x) \quad \text{for } y=1, p=1$$

This is precisely the twisted superpotential of the 2d theory on  $\mathbb{R}^2$  with all KK modes on  $S^1$ . Indeed

$$x = e^{iu} \quad L_2(x) = -i \sum_{n \in \mathbb{Z}} (u + 2\pi n) \left( \log \left( \frac{u + 2\pi n}{2\pi} \right) - 1 \right)$$

can be written as a sum of 1-loop contributions from KK modes.

The integrand has poles at

$$e^{i \frac{\partial W}{\partial u_i}} = 1$$

**Bethe vacua of  
the 2d theory**  
 [Witten; Beau-Thomas, ... '93]  
 [Nekrasov, Shatashvili '09]

Therefore  $Z$  can be written as a sum of Bethe vacua (st red  $P \circ g = 0$ )

$$Z = \sum_{u^*} \frac{Q(u^*)}{(\det \partial_{u_i} \partial_{u_j} W)}$$

[Nekrasov-Shatashvili '14  
 Closset-Kim-Wyllard '17]

This observation can be extended to  $g > 0$  (NS '14) and also to manifolds  $S^1 \rightarrow \Sigma_g$  with chern number  $p$  (Closset-Kim-Wyllard '17)

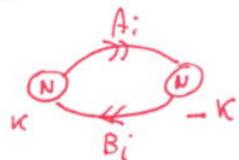
$$Z = \sum_{\substack{\text{Bethe} \\ \text{vacua } u^*}} F(u^*)^p H(u^*)^{g-1}$$

where  $F$  and  $H$  are constructed from the data of the 2d theory

## Lecture 3

### Black holes in $AdS_4 \times S^7$

The dual of  $AdS_4 \times S^7$  is known. For a review Marino 1104.0783. In  $N=2$  notations



$$W = A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1$$

$$U(N) \times U(N) \text{ with } CS(K, -K)$$

and bi-fundamentals

$A_i$ in $(N, \bar{N})$	$i=1, 2$
$B_i$ in $(\bar{N}, N)$	$i=1, 2$

Many interesting properties

- Free energy on  $S^3$  scales like  $N^{3/2}$   $F_{S^3} = \frac{\pi R^2}{3} J_K N^{3/2}$
- The theory has  $N=6$  supersymmetry, non-perturbatively enhanced to  $N=8$  for  $K=1, 2$
- For  $K=1, 2$ , R-symmetry is  $SO(8)$
- For  $N \gg K$  the dual is well described by a weakly coupled M-theory background

$$\begin{cases} ds_{11}^2 = L^2 \left( ds_{AdS_4}^2 + ds_{S^7/K}^2 \right) & \left(\frac{L}{e_P}\right)^6 \sim NK \\ F = L^3 \text{Vol}(AdS_4) \end{cases}$$

We will take  $K=1$  with  $SO(8)_R$  symmetry and dual  $AdS_4 \times S^7$ .

In  $N=2$  notations,  $SO(8)_R \rightarrow U(1)_R \times U(1)^3$ .  $U(1)^4 \subset SO(8)_R$  corresponds to a possible choice of different R-symmetries

	$R_1$	$R_2$	$R_3$	$R_4$
$A_1$	2	0	0	0
$B_1$	0	2	0	0
$A_2$	0	0	2	0
$B_2$	0	0	0	2

You can choose a reference R-symmetry, say the canonical one  $\frac{1}{2} \epsilon_{R\alpha}$ , and then  $J_\alpha = \frac{R_\alpha - R_4}{2}$ ,  $\alpha=1, 2, 3$  is a basis of flavor symmetries in  $N=2$  notations.

Recall that we have BHs with four charges  $\sum p^i = -\frac{K}{g}$  per  $U(1)^4$  and 6 charges  $q_\alpha$  with a constraint  $\rightarrow 3+3$  charges. The magnetic charges twist the theory - this you can do by choosing an R-symmetry (an integer one!) and introducing a background with a coefficient that cancel the spin connection. At this point there are 3 flavor symmetry and we can introduce 3  $\Delta$  and 3  $\eta$ .

$$\text{The index is } Z = \frac{1}{(N!)^2} \sum_{\mathbf{u}, \tilde{\mathbf{u}} \in \mathbb{Z}^N} \int Z_{\text{loop}} \left( \det \frac{\partial^2 \log Z_{\text{loop}}}{\partial u_a \partial \tilde{u}_b} \right)^g$$

where

$$Z_{\text{loop}} = \prod_{i \neq j} \left( 1 - \frac{x_i}{x_j} \right)^{1-g} \left( 1 - \frac{\tilde{x}_i}{\tilde{x}_j} \right)^{1-g} x_1^{k u_1} \tilde{x}_1^{-k u_1} \prod_{q=1,2} \prod_{i,j} \left( \frac{\sqrt{x_i y_a / x_j}}{1 - \frac{x_i y_a}{x_j}} \right)^{u_1 - \tilde{u}_j - h_a + 1-g}$$

$$x = e^{iu}, \tilde{x} = e^{i\tilde{u}}$$

$$\prod_{q=3,4} \prod_{i,j} \left( \frac{\sqrt{x_i y_b / x_j}}{1 - \frac{x_i y_b}{x_j}} \right)^{\tilde{u}_1 - u_j - h_a + 1-g} \frac{dx_i}{2\pi i x_i} \frac{d\tilde{x}_j}{2\pi i \tilde{x}_j}$$

where, in a democratic way, we assigned a fugacity  $y_a$  to each field and a flux  $h_a$  to each field

$$\begin{aligned} A_1 &\rightarrow y_1, u_1 \\ A_2 &\rightarrow y_2, u_2 \\ B_1 &\rightarrow y_3, u_3 \\ B_2 &\rightarrow y_4, u_4 \end{aligned}$$

$$y_i = e^{i\Delta_i}$$

but since  $W$  is invariant under global symmetries  $\prod_{a=1}^4 y_a = 1$  or  $\sum \Delta_a = 2\pi \mathbb{Z}$ . Moreover, also the  $h_a$  are not independent. Chiral contribute

$$\left( \frac{\sqrt{xy}}{1-x^p} \right)^{p(u) + (g-1)(v_\alpha - 1)}$$

for a choice of integer R-symmetry

There are many choices and each parameterize an inequivalent twist. But  $\sum v_\alpha = 2$  because  $R(W) = 2$ . Then we can write

$$(g-1)(v_\alpha - 1) = -h_a + 1-g$$

$$\sum_\alpha = -2(g-1) \quad g(1-g) - \sum h_a$$

in terms of four integer  $h_a \in \mathbb{Z}$  such that  $\boxed{\sum h_a = 2(1-g)}$ . This is the CFT dual of  $\sum p^\alpha = -k/g$ . The fact that we can introduce 3 independent  $\Delta$  allows to study 3 independent electric charges for the BH.

Working the integrand as  $e^{i u_i \frac{\partial W}{\partial u_i}}$  we read the twisted superpotential

$$W = \sum_{i=1}^N \frac{1}{2} (u_i^2 - \tilde{u}_i^2) + \sum_{j=3,4}^N \sum_{a=1,2} \text{Li}_2(e^{i(u_i - \tilde{u}_j + \Delta_a)}) - \sum_{b=1,2} \text{Li}_2(e^{i(\tilde{u}_1 - u_i - \Delta_a)})$$

and the BPS vacua are  $\frac{\partial W}{\partial u_i} = 2\pi n_i, \quad \frac{\partial W}{\partial \tilde{u}_i} = 2\pi \tilde{n}_i, \quad u_i, \tilde{u}_i \in \mathbb{Z}$

Explicitly

$$x_1^K \prod_{j=1}^N \frac{(1 - y_3 \frac{\tilde{x}_j}{x_1})(1 - y_4 \frac{\tilde{x}_j}{x_1})}{(1 - \frac{1}{y_1} \frac{\tilde{x}_j}{x_1})(1 - \frac{1}{y_2} \frac{\tilde{x}_j}{x_1})} = 1$$

$$\tilde{x}_j^K \left( \quad \right) = 1$$

$$\begin{cases} x \partial x \text{Li}_n(x) = \text{Li}_{n+1}(x) \\ \text{Li}_1(x) = -\log(1-x) \\ \text{Li}_n(x) = \sum_{k=1}^n \frac{x^k}{k^n} \end{cases}$$

Now  $Z(\Delta, u) = \sum_{\text{Bellvacua}} (\cdot)$ . In the large  $N$  limit we expect one vacuum to dominate the sum. You run numerics and discover that the imaginary part of  $U_i$  and  $\tilde{U}_i$  grows with  $N$

$$U_i = i N^\alpha t_i + v_i \quad \tilde{U}_i = i N^\alpha t_i + \tilde{v}_i$$

If you define  $t(\frac{i}{N}) = t_i$ ,  $v(\frac{i}{N}) = v_i$ ,  $\tilde{v}(\frac{i}{N}) = \tilde{v}_i$  and the density  $p(t) = \frac{1}{N} \frac{dv}{dt}$  with  $\int p(t) dt = 1$  you can define a function  $W[p(t), v(t), \tilde{v}(t)]$ . The interesting result is that  $W$  is local

$$W = i N^{\alpha+1} \int dt p(t) \delta v(t) + i N^{2-\alpha} \int dt p^2(t) \sum_{a=1}^4 g(\pm \delta v(t) + \Delta_a) \quad (\text{CS}) \quad (\text{Matter}) \quad \begin{matrix} \pm = 3 \\ \pm = 12 \end{matrix}$$

$$\begin{aligned} \delta v &= \tilde{v}(t) - v(t) \\ g(v) &= \frac{v^3}{6} - \frac{\pi v^2}{2} + \frac{\pi^2 v}{3} \end{aligned}$$

The reason for locality is that for  $i \neq j$ ,  $e^{i(v_i - v_j + \Delta_a)} \ll 1$  if  $i \neq j$  and  $L_2$  is suppressed. For  $i = j$  you can use  $L_2(e^{iv}) + L_1(e^{-iv}) = \frac{v^2}{2} - \pi v + \frac{\pi^2}{3}$  to obtain a suppressed term + polynomials and polynomials cancel with a clever choice of  $v_i, \tilde{v}_i$ .

Balancing the two contributions we find  $\alpha = 1/2$  and  $W \propto N^{3/2}$ .

The Bell eqs are now easy to solve

$$\frac{\delta W}{\delta p} = \frac{\delta W}{\delta v} = 0 \quad \Rightarrow$$



with a piecewise behavior with plateaux when  $\pm \delta v(t) + \Delta_a = 0, \pm \pi$ . The plateaux are artifact of the large  $N$  limit: at finite  $N$

$$\pm \delta v(t) + \Delta_a = e^{-i\sqrt{N} Y(a)} \ll 1 \text{ on the plateaux}$$

These exponentially suppressed contributions are important when one plugs the Bell vacua into  $Z_{\text{int}}(u=0)$ . Terms like  $\sum_{a=1}^N \log(1 - \gamma_a \frac{x_a}{x_1})$  that are usually suppressed as  $O(N)$  are now  $N \log(e^{-i\sqrt{N} Y}) \sim N^{3/2} \sqrt{Y}$  and contribute. One finds:

$$W = -i \frac{\sqrt{2}}{3} N^{3/2} \sqrt{\Delta_1 \Delta_2 \Delta_3 \Delta_4}$$

$$\log Z = (g-1) \left( \frac{2iW}{\pi} + i \sum_a \left( \frac{u_a}{1-g} - \frac{\Delta_a}{\pi} \right) \frac{\partial W}{\partial \Delta_a} \right) = i \sum_a u_a \frac{\partial W}{\partial \Delta_a}$$

since  $W$   
is homogeneous

I cheated a bit. Manipulations with  $\Delta_a$ , assumed  $\text{Re } \Delta_a \in [0, 2\pi]$  so the result is better written as  $\mathcal{W} = -i \frac{\sqrt{2}}{3} N^{3/2} \sqrt{[\Delta_1][\Delta_2][\Delta_3][\Delta_4]} \quad [\Delta] = \Delta \bmod 2\pi$  and since  $\sum \Delta_a \in 2\pi \mathbb{Z}$ ,  $\sum [\Delta_a] = 0, 2\pi, 4\pi, 6\pi, 8\pi$ . It turns out that the solution exists only for  $\sum [\Delta_a] = 2\pi$  (or  $6\pi$  but equivalent) related by a discrete symmetry.

The entropy is now obtained as the Legendre transform of  $\log Z$  (see lecture 1). Comparing with BH

### Localization

$$S_{BH} = \log Z - i \sum_{a=1}^4 q_a \Delta_a = \sum_{a=1}^4 N^{3/2} \sqrt{2 \Delta_1 \Delta_2 \Delta_3 \Delta_4} \frac{p_a}{\Delta_a} - i q_a \Delta_a$$

Valid for  $\sum \Delta_a = 2\pi \quad \text{Re } \Delta_a \in [0, 2\pi]$

### gravity

$$S_{BH} = -i \frac{\text{vol}(\Sigma)}{4\pi G_N} \frac{q_a x^1 - p_a \hat{x}^1}{q_a x^1} = \sum_{a=0}^3 N^{3/2} \sqrt{2 \hat{x}^0 \hat{x}^1 \hat{x}^2 \hat{x}^3} \frac{\hat{q}_a \hat{x}^1}{\hat{x}^1}$$

where  $\hat{x}^1 = \frac{2\pi x^1}{\Sigma x^1}$ ,  $F = -2i\sqrt{x^0 k^2 x^3}$

Dirac quantization:

$$\text{vol}(\Sigma) g_a p^1 = 2\pi \hat{p}^1, \quad \frac{\text{vol}(\Sigma)}{4\pi G_N} q_a = 2\pi \hat{q}_a$$

$$\hat{p}^1, \hat{q}_a \in \mathbb{Z}$$

$$\text{and } \frac{1}{2\pi^2 G_N} = \frac{2\sqrt{2}}{3} N^{3/2}$$

[Benini-Hristov-AZ 1511.04085  
1608.07294]

You see that the two sides are the same! Under the identification

$$(p_a, q_a) \rightarrow (\hat{p}^1, \hat{q}_a)$$

$$\Delta_a \rightarrow \hat{x}_a$$

Notice that in field theory  $\delta(\log Z - i \sum \Delta_a) = 0$  fixes the 3 independent  $\Delta_a$  in terms of 3 electric charges, for example  $q_1 - q_4$ . The index only sees the global symmetry. We cannot compute the electric charge for the R-symmetry (with this method - maybe there is another one). Gravity tells us that there is just a BH, for a given value of  $q_4$ . For this value  $S_{BH}$  is real. It would be interesting to determine  $q_4$  in another way!

## Generalizations

A) Large  $N$  can be done for other quivers dual to  $\text{AdS}_5 \times \text{SE}_7$ . It works only if the quiver is vector-like and baryonic symmetries are invisible in the large  $N$  limit. This is similar to computation of the F function (free energy on  $S_3$ ) in the large  $N$  limit. This is not a coincidence. You can prove that, the on-shell value of

$$W(\Delta) = F(\Delta)$$

as a function of a set of R-charges

For example, for ABJ4

$$W = \sqrt{\Delta_1 \Delta_2 \Delta_3 \Delta_4} N^{2/3} \quad F(\bar{\Delta}) = \frac{\sqrt{\Delta_1 \Delta_2 \Delta_3 \Delta_4}}{Z\Delta_1} N^{2/3} \quad [\text{Herzog-Jafferis-Pufu-Klebanov '11}]$$

$$\sum \Delta_i = 24$$

for other models {

[A2 with Hosseini-Mekareyai '16]  
[Azzurli-Bobev-Min-Cruchigno-A2 '17]  
[BOBEV-MIN-PILCH '17]

B) Full comparison done also for quivers dual to  $\text{AdS}_5 \times S^5$  in massive type IIA:  $U(N)_K + 3$  adjoint with scaling  $N^{5/3}$ . This time there are 2 global symmetries and

$$F = (X^0 X^1 X^2)^{2/3} N^{5/3} \quad W = (\Delta_1 \Delta_2 \Delta_3)^{2/3} N^{5/3} \quad F(\bar{\Delta}) = (\bar{\Delta}_1 \bar{\Delta}_2 \bar{\Delta}_3)^{2/3} N^{5/3}$$

BH found by [GURARDO '17]

HRISTOV-HOSSEINI-PASSIAS '17  
BENINI-KHACHATRYAN-MILAN '17

C)  $\log Z$  is a partition function. Can you get it holographically and does it match with the entropy? Yes

[HALMAGYI-LAL '17;  
CABO-BIZET-KOL-PANDOZAYAS-PAPADIMITRIOU-RATHEE '17]

D) Corrections in  $1/N$ ? There is an interesting  $\log N$  contribution that is universal. Computed and matched

[LIU-PANDOZAYAS-RATHEE-ZHAO '17]

E) Localization in supergravity performed by (at least the classical piece of the integrand)

[IGDATO-HRISTOV-REYS '17]

## Lecture 4

## BH from black strings

There is a necessary also in 5d. Here there are "black strings".  
Solution interpolating between

$$\text{AdS}_5 \longrightarrow \text{AdS}_3 \times \mathbb{Z}_g$$

[BENINI-BOBEV 112/13]

There are magnetic fluxes on  $\mathbb{Z}_g$

as before. So this is a RG flow from

$$\text{topologically twisted CFT}_6 \longrightarrow \text{CFT}_2^{(u_a)}$$

by a set of integers  $u_a$

For example, for  $N=4$  SYM,  $\text{AdS}_5 \times S_5$ , we have  $U(1)^3 \subset SO(6)$  and three integers  $u_a$  with  $\sum_{a=1}^3 u_a = 2(1-g)$ . Only two are independent and specify the twist. Every set of  $u_a$  is a different CFT. Benini and Bobev correctly matched the central charge  $C_{\text{FT}}$  with the Regge prediction! [12/13]

Now a standard mechanism to get a BH is to take a string and compactify it on a circle adding a momentum  $n$ . This is just Vafa-Sbranejger and the D1-D5 system! The entropy of the BH is obtained by Cardy formula for the high temperature behaviour of a CFT partition function

$$n \uparrow \begin{array}{c} \text{---} \\ \text{---} \end{array} \quad \tau = i \frac{\beta}{2\pi}$$

$$Z \sim e^{\frac{\pi i^2}{6\beta} C_{\text{CFT}}} \underset{\beta \gg 0}{\approx} \text{tr} (-1)^F q^{\frac{L_0 - 1}{24}} \quad q = e^{\frac{2\pi i \tau}{\beta}} = e^{-\beta}$$

$$\approx \sum c(n) q^{\frac{n-1}{24}} \quad c(n) = e^{S(n)}$$

The number of states with energy  $n$ , for large  $n$  is then the Legendre transform

$$S(n) = \log c(n) \approx \frac{\pi^2}{6\beta} C_{\text{CFT}} + \bar{\beta} n \quad \Big|_{\bar{\beta} \text{ extremum}}$$

$$\equiv 2\pi \sqrt{\frac{C_{\text{CFT}} \cdot n}{6}}$$

which is just another form of the Cardy formula.

This has a nice counterpart in localization [HOSSEINI-NEDELIN-AZ'16]. This time you compute the topologically twisted index of  $N=4$  SYM on  $\mathbb{E}^3$

$$Z(\Delta, h) = \frac{1}{|W|} \sum_{\substack{w \in Z^N \\ \Xi^{w \circ 0}}} \int_{i=1}^{N-1} \frac{dx_i}{2\pi i x_i} \left( \prod_{i \neq j} \frac{\Theta_1\left(\frac{x_i y_j}{x_j}\right)}{\ln(q)} \right)^3 \prod_{a=1}^3 \left( \frac{\ln(q)}{\Theta_1\left(\frac{x_i y_a}{x_j} q\right)} \right)^{u_i - u_j - p_a + 1} q = e^{2\pi i \theta}$$

where I write the expression for  $S^3$  for simplicity and  $y_1 y_2 y_3 = 1$ ,  $\Xi^{p_a} = 2$  and this time you try to solve it in the high temperature limit  $T = \frac{iB}{2\pi}$  for  $B \rightarrow 0$ . This can be done by using modular transformation for the Bethe (for  $q \rightarrow 1$ ). This time the Bethe eqs are

$$e^{i \frac{\partial W}{\partial u_i}} = \prod_{j=1}^N \prod_{a=1}^3 \frac{\Theta_1\left(\frac{x_j y_a}{x_i} q\right)}{\Theta_1\left(\frac{x_i y_a}{x_j} q\right)} = 1 + \frac{1}{\beta} \left( \frac{\beta^3 - \pi \Delta_a^2 + \frac{2}{3} \Delta_a^3}{\beta} \right)$$

$$\text{For } B \rightarrow 0 \text{ it is easy to compute } W \sim \frac{i}{\beta} \sum_{i,j} (u_i - u_j)^2 \frac{1}{2} \Xi(\Delta_a - \pi) = -\frac{i}{2B} \sum_i (u_i - u_j)^2$$

where again we choose  $\Re(\Delta_a \in [0, 2\pi])$  so that  $\Xi(\Delta_a) = 0, 2\pi, 4\pi, 6\pi$ . 0, 6\pi are inequivalent and  $2\pi \sim 4\pi$  so we used  $\Xi(\Delta_a) = 2\pi$ . Bethe eqs are  $\frac{\partial W}{\partial u_i} = 2\pi u_i$ ,  $u_i \in \mathbb{Z}$

$$-\frac{2\pi i}{B} \sum_{j=1}^N (u_i - u_j) = 2\pi u_i \quad \Rightarrow \quad u_i = \frac{iB}{N} h_i$$

$\sum u_i = 0$   
Parity ( $N$ )

Since  $u_i$  lies on a lattice  $u_i \equiv u_i + 2\pi n_i + 2\pi m_i \pi$  the only solution is (up to permutations)

$$u_k = \frac{iB}{N} \left( k - \frac{N+1}{2} \right)$$



The solution is actually exact all order in  $B$  [Hong-Liu '18 - they conjecture the full set of Bethe vacua and expression for  $\log Z$ ]

$W(\delta) = \frac{iN^2}{2B} \delta_1 \delta_2 \delta_3$	$\sum \delta_a = 2\pi$
--	------------------------

and

$\log Z(\delta) = -i h_a \frac{\partial W}{\partial \delta_a} = \frac{N^2}{2B} (u_1 \delta_2 \delta_3 + u_2 \delta_3 \delta_1 + u_3 \delta_1 \delta_2)$
---

Interestingly enough, defining  $\hat{\delta}_a = \frac{\delta_a}{\pi}$ , there are a set of R-charges for  $N=4$  SYM,  $\sum \hat{\delta}_a = 2$  and

$$W(\delta) \sim a(\tilde{\delta}) \quad \text{trial R-charge } N=4 \text{ SYM in 4d}$$

$$\log Z(\delta) = \frac{4\pi^2}{6B} C(\hat{\delta}) \quad \text{trial c-charge in 2d CFT}$$

Following Cardy argument we now perform a Legendre transform with respect to  $\Delta$  to introduce electric charges  $q_a$  and with respect to  $\beta$  to introduce momentum  $n$ . For zero electric charges  $q_a$

$$0 = \partial_\Delta \log Z(\Delta) = \frac{\pi^2}{6\beta} \partial_\Delta C(\Delta)$$

Just extremizing  $C(\Delta)$  and this gives the exact central charge of the CFT (Bevin-Baber). The Legendre transform with respect to beta then gives Cardy's formula

$$S_{BH}^{(p_a)} = 2\pi \sqrt{\frac{n C_{CFT}(p_a)}{6}} = \frac{\pi^2 C_{CFT}(p_a)}{6\beta} + n\beta$$

You can also introduce  $q_a \neq 0$  and you get

$$S(p_a, q_a) = \frac{\pi^2 C_{CFT}(p_a, \Delta)}{6\beta} + n\beta - i q_a \Delta_a = \sqrt{g(p_a)(n + q_a A_{ab}^{-1} q_b)}$$

where  $A_{ab}$  is a t'Hooft anomaly matrix

This has a nice counterpart in gravity [CHRISTOU '14]. the RG flow can be dimensionally reduced along a circle in AdS<sub>3</sub>

$$\begin{array}{c} \text{AdS}_5 \longrightarrow \text{AdS}_3 \times \Sigma_g \\ \Downarrow \\ \text{Ugelyk ring} \longrightarrow \text{AdS}_2 \times \Sigma_g \\ (\text{not AdS}_4!) \end{array}$$

You can follow what happens by reducing the gauged sugra in 5d to an N=2 gauged sugra in 4d. For AdS<sub>5</sub> × S<sub>5</sub>, in 5d we have 3 vectors. the compactification circle provides a new vector. The model is STU with

$$F = \frac{x^1 x^2 x^3}{x^0} \quad \text{where } x^0 \text{ is the new vector} \\ \text{and } g_0 = 0, g_i = g; g^a \equiv 0$$

Now the BH has magnetic charges with respect to 1,2,3 and electric charged with respect to 0

and again you can identify

$$\begin{aligned} \hat{x}^0 &\Rightarrow \beta \\ \hat{x}^a &\Rightarrow i\Delta^a \\ q_0 &\rightarrow n \quad p^a \rightarrow p^a \\ W(\Delta) &\rightarrow F(x) \end{aligned}$$

and the attractor mechanism with

$$S \sim p^a F_a(\hat{x}) - q_0 \hat{x}^0 = \sum_{a=1}^3 p^a \delta_a \left( \frac{\Delta_1 \Delta_2 \Delta_3}{\beta} \right) + n\beta - i q_a \Delta_a$$

|||

$W(\Delta)$

$$S \sim \text{leg}(\Delta) + n\beta - i q_a \Delta_a$$