Lecture 1 :

→ In everything I will be saying, I will only consider :

~~Ungauged Sugra~~ Susy BH's preserving 4 Supercharges in Ungauged Sugra

→ Moreover I will work always on the near-horizon geometry

↳ Susy BH's have an AdS_2 near-horizon geometry

↳ 4 Susy's imply in addition that they have a spherical horizon S^2

Near-horizon : $AdS_2 \times S^2$

(because it is ungauged Sugra $\Lambda = 0$, this implies that size of AdS_2 = size of S^2

~~→ The SD point of view is particularly useful for what I want to say~~

→ On top of this we have

constant Vec scalar ϕ^I
covariantly const $F^I_{\mu\nu} = e^I \omega_{AdS_2} + p^I \omega_{S^2}$

$$e^I = \cancel{\phi} e^I(\phi^I) \text{ and } \phi^I = \phi^I(q, p)$$

q : electric | $q_I \in \mathbb{Z}$ note up & down
p : magnetic | $p^I \in \mathbb{Z}$

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→ ~~M~~ (quick explanation):

$$q_I \phi^I \quad \text{(minimal coupling)}$$

$$\int_{S^2} F^I \propto p^I \in \mathbb{Z}$$

→ The 5D pt of view is particularly useful

Family of Bh solutions:

$$AdS_2 \times S^2 \times S^1$$

fibered

On one extremum:

$$(AdS_2 \times S^1) \underset{\text{product}}{\times} S^2$$

(Black string geometry)

On the other extremum:

$$AdS_2 \times \underbrace{(S^2 \times S^1)}_{= S^3}$$

(Non-rotating 5D Black hole)

→ In other words

$$ds^2 = ds^2_{AdS_2 \times S^2} + e^{-2\phi}(\underbrace{dy}_{\text{circle } S^1} + A)^2$$

$$A = \phi^0 \underbrace{e^0_1}_{\text{1-forms}} + p^0 \underbrace{\tilde{e}^0_1}_{\text{1-form}}$$

Such that $de^0_1 = F^0 = \phi^0 \omega_{AdS_2}$ → volume-form

$d\tilde{e}^0_1 = \tilde{F}^0 = p^0 \omega_{S^2}$ → volume-form

Focus will be on

$$(AdS_2 \times S^1) \times S^2$$

$\simeq$ locally AdS_3

means that A is purely electric

But in all cases, under dimensional reduction to 4D, one obtains (Bh geometry)

$$AdS_2 \times S^2 + \text{KK gauge field } A$$

+ any other matter in 5D

This (4D) setup is the one explained by Sameer in his lecture.

→ Key point:

Locally $AdS_3 = AdS_2 \times S^1$ = Extremal Bh in AdS_3

→ Use Holography

$$Z_{AdS_3} = Z_{CFT_2}$$

Bh states = # BPS states in 2D CFT

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→ On the 2D CFT we compute an index rather than the statistical partition function

$$\chi(\tau, z) = \mathrm{Tr}_{RR} (-1)^F \, q^{L_0 - \frac{c_L}{24}} \, \bar{q}^{\bar{L}_0 - \frac{c_R}{24}} \, y^{J_L}$$

$F = J_L - J_R$ or R-symmetry

(I'm using $\mathcal{N}=(2,2)$ notation)

(computation on the cylinder w/ Periodic Bnd conditions in both time and spatial directions)

→ The index or elliptic genus can be computed exactly in a regime where the CFT looks free! which is not the black hole regime! But since the index is invariant under continuous deformations

$$\chi(\tau, z)\Big|_{\text{BH regime}} = \chi(\tau, z)\Big|_{\text{Free regime}}$$

This would not be true for the statistical partition function

$$Z = \mathrm{Tr} \, q^{L_0 - \frac{c_L}{24}} \, \bar{q}^{\bar{L}_0 - \frac{c_R}{24}} \, y^{J_L}$$

w/ $\bar{L}_0 - \frac{c_R}{24} > 0$

→ Because of Susy, states come in pairs graded by their Fermion number and so they cancel in the Trace. Only the ground states survive:

$$\chi(\tau, z) = \mathrm{Tr} \, (-1)^F \, q^{L_0 - \frac{c_L}{24}} \, y^{J_L} \Big|_{\text{for } \bar{L}_0 - \frac{c_R}{24} = 0}$$

and so it is an holomorphic object!!

→ Comments :

· I will be mainly interested in the case of $N=(0,4)$ SCFT's which are associated to the low-energy physics of wrapped M5-branes on CY's

· In this case, the elliptic genus is not holomorphic because there can be BPS states due to non-linear supersymmetry. In any case, what I'm going to say applies exclusively to (4,4) or (2,2) SCFT's but can be easily generalized to (0,4)

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$\chi(\tau, \bar{\tau})$ and Number Theory

→ Superconformal sym ⊕ reparametrization invariance

$\chi(\tau, z)$ is a modular object

→ what does this mean?

Lightning review of modular forms and Jacobi forms

$$\phi(\tau) : \underset{\mathrm{Im}(\tau)>0}{\mathbb{H}} \longrightarrow \mathbb{C}$$

w/ the following property

$$\phi\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{w} \phi(\tau) \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

w: weight of modular form

→ In particular

$$\phi(\tau+1) = \phi(\tau)$$ So it has a Fourier expansion

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\phi(\tau) = \sum d(n) q^n \quad q = e^{2\pi i \tau}$$

and

$$\phi\left(-\frac{1}{\tau}\right) = \tau^{\omega} \phi(\tau)$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

→ In mathematics, $\phi(\tau)$ (w/ weight $\neq 0$) is required to ~~have~~ have special growth properties at infinity

- Their absolute value is majorized by a polynomial in $\max\{1, \mathrm{Im}(\tau)^{-1}\}$

(this cannot happen for $j(\tau)$ function because of Liouville's theorem) but it can have a majorized absolute value due to the weight

→ ~~The~~ The ring of modular forms is finitely generated by:

- Define Eisenstein series:

$$G_k(\tau) = \frac{(k-1)!}{2(2\pi i)^k} \sum_{\substack{m,n \\ \neq (0,0)}} \frac{1}{(m\tau+n)^k}$$

$k \geq 2$
weight

$$G_4(\tau) = \frac{1}{240} + q + 9q^2 + 28q^3 + \cdots$$

→ $G_6(\tau) = -\frac{1}{504} + q + 33q^2 + 244q^3 + \cdots$

→ Discriminant function Δ:

$$\Delta(\tau) = q \prod_{n=1}^{\infty} (1-q^n)^{24}$$

$$\Delta(\tau) = q - 24q^2 + 252q^3 + \cdots$$

$$\Delta(\tau) = \frac{1}{1728}\Big(\underbrace{240\, G_4}_{= E_4}\Big)^3 - \Big(\underbrace{504\, G_6}_{= E_6}\Big)^2$$

→ $j(\tau) = q^{-1} + 744 + 196884\, q + \cdots$

$$= \frac{(240\, G_4)^3}{\Delta(\tau)}$$

Note that $j(\tau)$ diverges at $\mathrm{Im}(\tau) \to \infty$ because of the pole at $q=0$ ($q^{-1} + \cdots$)

→ Jacobi forms

$$\phi(\tau, z) : \mathbb{H} \times \mathbb{C} \longrightarrow \mathbb{C}$$

they are characterized by a weight and an index:

$$\phi_{w,m}\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau+d)^w e^{\frac{2\pi i m c z^2}{c\tau+d}} \phi_{w,m}(\tau, z)$$

(modular transformations)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$$

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$$\phi_{\omega,m}(\tau, z + \lambda\tau + \mu) =$$
$$= e^{-2\pi i m(\lambda^2\tau + 2\lambda z + \mu)} \phi_{\omega,m}(\tau, z)$$
$$\lambda, \mu \in \mathbb{Z}$$

if $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\phi_{\omega,m}(\tau+1, z) = \phi_{\omega,m}(\tau, z)$

$\lambda = 0 \Rightarrow \phi_{\omega,m}(\tau, z+\mu) = \phi_{\omega,m}(\tau, z)$

So it has a Fourier expansion in both $q = e^{2\pi i\tau}$ and $y = e^{2\pi i z}$ (always considering $m \in \mathbb{Z}$

$$\phi_{\omega,m}(\tau, z) = \sum_{n \geq 0, \ell} c(n, \ell) q^n y^\ell$$

→ ~~a weak Jacobi form is~~

Define Polarity $\Delta = n - \frac{\ell^2}{4m}$

By definition a Jacobi form $(n > 0)$ always has $\Delta \geq 0$

A weak Jacobi form $(n \geq 0)$ can have $-\frac{\ell}{4} \leq \Delta < 0$

→ Because of the elliptic translations, a weak Jacobi form can be decomposed into a vector-valued modular form and theta functions:

$$\phi_{w,m}(\tau,z) = \sum h_\mu(\tau)\,\Theta_{m,\mu}(\tau,z)$$

μ is a coset representative of $\mathbb{Z}/2m\mathbb{Z}$

$\Theta_{m,\mu}(\tau,z)$ is a Jacobi-theta function defined as

$$\Theta_{m,\mu}(\tau,z) = \sum_{n\in\mathbb{Z}} q^{(\mu+2mn)^2}\, y^{(\mu+2mn)}$$

and

$$h_\mu(\tau) = \cancel{q^{\#}\sum_{n\geq 0} c_\mu(n)\, q^{n}}$$

$$= q^{A}\sum_{n\geq 0} c_\mu(n)\, q^n$$

→

$$c(n,\ell) = \iint d\tau\, dz\, \underbrace{\phi_{w,m}(\tau,z)}_{\sum h_\mu(\tau)\,\Theta_\mu(\tau,z)}\, e^{-2\pi i n\tau}\, e^{-2\pi i \ell z}$$

The integral on z projects on sector $\mu = \ell \bmod (2m)$ which means that:

$$A + n + \frac{\ell^2}{4m} = n \Rightarrow A + n = n - \frac{\ell^2}{4m} = \Delta$$

$\Rightarrow$ but $l = \mu \mod 2m)$

So $\frac{l^2}{4m} = \frac{\mu^2}{4m} + \mathbb{Z}$

and so we deduce that $A = \frac{-\mu^2}{4m}$

$\Rightarrow$ $C_\mu(n)$, therefore, only depends on

Δ, μ

$\rightarrow$ The generators $S = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

act on $\Theta_{m,\mu}(\tau, z)$ as

$$\Theta_{m,\mu}\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = \sqrt{\frac{\tau}{2mi}}\, e^{2\pi i \frac{m z^2}{\tau}} \sum_{\nu \in \mathbb{Z}/2m\mathbb{Z}} e^{-2\pi i \frac{\mu\nu}{2m}} \Theta_{m,\nu}(\tau, z)$$

$$\Theta_{m,\mu}(\tau+1, z) = e^{2\pi i \frac{\mu^2}{4m}} \Theta_{m,\mu}(\tau, z)$$

More generally we have:

$$\Theta_{m,\mu}\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau+d)^{\frac{1}{2}} e^{2\pi i \frac{m c z^2}{c\tau+d}} M\begin{pmatrix} a & b \\ c & d \end{pmatrix}_{\mu,\nu} \Theta_{m,\nu}(\tau, z)$$

$M\begin{pmatrix} a & b \\ c & d \end{pmatrix}_{\mu\nu}$ is a $(2m) \times (2m)$ Matrix !!

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→ In order to $\phi_{w,m} = \sum h_\mu(\tau)\,\Theta_{m,\mu}(\tau,z)$ to transform as a Jacobi form we need

$$h_\mu\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{w-\frac{1}{2}}\, M^{-1}_{\nu\mu}\begin{pmatrix} a & b \\ c & d \end{pmatrix} h_\nu(\tau)$$

$\left(h_\nu M^{-1}_{\nu\mu} M_{\mu\nu} \Theta_\nu \right.$

So a modular transformation "relates" $h_\mu(\tau)$ between themselves !!

→ The matrix $M\begin{pmatrix} a & b \\ c & d \end{pmatrix}_{\mu\nu}$ is called a Multiplier matrix

→ We are interested in computing the Fourier coefficient $C(n,l)$ of $\phi_{w,m}(\tau,z)$. But as we have seen, it is fully encoded in the Fourier coef. of $h_\mu(\tau)$

(Comment: for $N=(0,4)$ SCFT's

$$\chi(\tau,\bar{\tau},z) = \sum_\mu \underbrace{h_\mu(\tau)}_{\text{holomorphic}}\, \underbrace{\Theta(\tau,\bar{\tau},z)}_{\text{non-holomorphic}}$$

→ $h_\mu(\tau)$ are holomorphic in q except ~~at~~ for ~~$\Delta = n - \frac{l^2}{4m}$~~ the terms with $\Delta < 0$

→ We define the polar terms as the terms w/ negative exponent in the expansion of $h_\mu(\tau)$

$$h_\mu(\tau) = \underbrace{q^{-\frac{\mu^2}{4m} + \cancel{n}}}_{-\frac{\mu}{4m} + \frac{n}{m} < 0} \sum_{n>0} C_\mu(n)\, q^n$$

→ It happens that we can use the holomorphicity of $h_\mu(\tau)$ and its modular properties to write a close expression for the Fourier coefficients of $c(n,\ell)$ w/ $n - \frac{\ell^2}{4m} \geq 0$ in terms solely of the Fourier coefficients of the polar terms:

normalization

$$C_\mu(n)_{n-\frac{\mu^2}{4m}\geq 0} = \frac{1}{\#} \sum_{\bar{n} - \frac{\nu^2}{4m} < 0} C_\nu(\bar{n}) \sum_{c=1}^{\infty} \frac{1}{c} Kl\left(n - \frac{\mu^2}{4m}, \bar{n} - \frac{\nu^2}{4m}; \mu, \nu; c\right) \times$$

$$\times \boxed{\int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{dt}{t^{-w+2}} e^{2\pi \frac{1}{c}\left(n - \frac{\mu^2}{4m}\right)\frac{1}{t} - 2\pi\left(\bar{n} - \frac{\nu^2}{4m}\right)\frac{t}{c}}}$$

w = weight of vector-valued mod. form

= Modified Bessel function of 1st kind

$$I_\nu(z) = \frac{1}{2\pi i} \int_{\epsilon - i\infty}^{+\epsilon + i\infty} \frac{dt}{t^{\nu+1}} e^{\frac{z^2}{4t} + t}$$

This is called Hardy – Ramanujan – Rademacher expansion.

Key observation:

Integer = Analytic formula

(⇒ analytic Number theory)

→ $$Kl\left(m-\frac{\mu^2}{4m}, \bar{n}-\frac{\nu^2}{4m}; \mu, \nu; c\right) :=$$ Generalized Kloosterman Sums

$$= \sum_{\substack{(d,c)=1 \\ ad=1 \bmod (c)}} e^{2\pi i \left(n-\frac{\mu^2}{4m}\right)\frac{d}{c}} M\begin{pmatrix} a & b \\ c & d \end{pmatrix}_{\mu\nu} e^{2\pi i \left(\bar{n}-\frac{\nu^2}{4m}\right)\frac{a}{c}}$$

→ Kloosterman Sums (due to H. D. Kloosterman)

$$Kl(n, m, c) = \sum_{\substack{d \in (\mathbb{Z}/c\mathbb{Z})^* \\ ad=1 \bmod (c)}} e^{2\pi i \frac{d}{c} n + \pi i \frac{a}{c} m} \qquad n, m \in \mathbb{Z}$$

→ The generalized version is due to the presence of the Multiplier system.

→ We want to study the asymptotic growth of $C_\mu(n)$

- For fixed, therefore finite, level m $C_{\mu}(\bar{n})$ is of order 1. Also, one can show that Kloosterman Sums are bounded by some power of c.
- So we just need to analyse the Bessel function

$$\to \quad \frac{1}{2\pi i}\int_{\varepsilon - i\infty}^{\varepsilon + i\infty} \frac{dt}{t^{\nu+1}} \, e^{2\pi \frac{a}{t} + |b| t 2\pi}$$

first we rescale $t \to t\sqrt{\frac{a}{|b|}}$

$$\hookrightarrow \quad \frac{1}{2\pi i}\left(\sqrt{\frac{|b|}{a}}\right)^{\nu} \int \frac{dt}{t} \, e^{2\pi\sqrt{a|b|}\left(\frac{1}{t} + t\right)}$$

$$\sim \int e^{-\lambda S(x)} \, dx$$

→ The spectrum of $|b\rangle$, that is, the spectrum of negative polarities is given and so it is kept fixed. The only large parameter that we have is $a = n - \frac{\mu^2}{4m} > 0$

→ Therefore for large

$$n - \frac{\mu^2}{4m} \gg 1$$

$$\sim \frac{1}{2\pi i}\left(\sqrt{\frac{|b|}{a}}\right)^{\nu} e^{4\pi\sqrt{a|b|}}$$ + corrections in an expansion in $\frac{1}{\sqrt{a|b|}}$

→ This means that we can approximate each Bessel in the Rademacher expansion:

$$C_\mu(n) \simeq \sum_{\bar{n}-\frac{\nu^2}{4m}<0} C_\nu(\bar{n}) \times \sum_{c=1}^{\infty} Kl$$

$$\times\; e^{\frac{4\pi}{c}\sqrt{\left(n-\frac{\mu^2}{4m}\right)\left|\bar{n}-\frac{\nu^2}{4m}\right|}} + \ldots \text{corrections}$$

$$\mathrm{Min} \leq \left|\bar{n}-\frac{\nu^2}{4m}\right| \leq \mathrm{Max}$$

Therefore

$$C_\mu(n) \simeq C_\nu(\bar{n})\Big|_{\left|\bar{n}-\frac{\nu^2}{4m}\right|=\mathrm{Max}} e^{4\pi\sqrt{\left(n-\frac{\mu^2}{4m}\right)\mathrm{Max}}}$$

$$\underline{+ \cdots + e^{4\pi\sqrt{\left(n-\frac{\mu^2}{4m}\right)\mathrm{Min}}}}$$

$$\underline{+ e^{\frac{4\pi}{c=2}\sqrt{\left(n-\frac{\mu^2}{4m}\right)\mathrm{Max}}} + \cdots + e^{\frac{4\pi}{c=2}\sqrt{\left(n-\frac{\mu^2}{4m}\right)\mathrm{Min}}}}$$

$$\vdots$$

$$c>2$$

All the ___ (underlined) terms ~~are~~ grow exponentially slower than the leading term

$$e^{4\pi\sqrt{\left(n-\frac{\mu^2}{4m}\right)\mathrm{Max}}}$$

→ So when I take the $\log(|C_\mu(n)|)$

$$\log(|C_\mu(n)|) = 4\pi\sqrt{\left(n-\frac{\mu^2}{4m}\right)\mathrm{Max}} + \text{perturbative expansion in } \frac{1}{\sqrt{\left(n-\frac{\mu^2}{4m}\right)\mathrm{Max}}}$$

$$+ e^{-4\pi\left(\sqrt{n-\frac{\mu^2}{4m}}\right)\left(\sqrt{\mathrm{Max}}-\sqrt{(\mathrm{Max}-1)}\right)}$$

+ ∞ other Non-perturbative corrections

→ One can check that indeed

$$\frac{A}{4G} = 4\pi\sqrt{\left(n-\frac{\mu^2}{4m}\right)\mathrm{Max}} \quad \left(\begin{array}{c}\text{Wald's}\\ \text{entropy}\end{array}\right)$$

→ So for large $\left(n-\frac{\mu^2}{4m}\right)$ and fixed m, one has a closed form expression for quantum entropy which includes all the <u>perturbative</u> ~~quan~~ <u>quantum corrections</u>

$$C_\mu(n) \simeq \underset{\substack{(\mathcal{O}(1))\\ \text{factor}}}{C_\nu(\bar{n})\Big|_{\max}} \times \underset{\substack{(\mathcal{O}(1)\\ \text{factor})}}{\mathcal{K}\ell(\dots)} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{dt}{t^{-w+2}} \times$$

$$\times\, e^{2\pi\left(n-\frac{\mu^2}{4m}\right)\frac{1}{t} + 2\pi\,\mathrm{Max}\,t}$$

→ Note that the Bessel function is a sort of universal as I have not described any dynamics of the SCFT!! It only follows from holomorphicity (Susy) and modularity.

→ This would mean that there should also be an universal sector in Supergravity valid for any compactification!! This where Chern-Simons theory comes in.

→ There is yet another very interesting limit we can take, which corresponds to

$$\left(n - \frac{\mu^2}{4m}\right) \sim m \sim \lambda >> 1$$

that is, we rescale
$$n \to \lambda n$$
$$\mu \to \lambda \mu$$
$$m \to \lambda m$$

→ In this case, the Fourier coefficient of the polar term $C_r(\bar{n})$ will scale w/ λ, and so it is possible that $C_r(\lambda \bar{n})$ grows exponentially.

→ It is a non-trivial question to determine what are the asymptotics of $C_\mu(n)$, because in this case terms that were non-perturbative in the previous limit can become leading!!

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→ Schematically:

$$C_v(\bar{m})\Big|_{\substack{Max \\ \bar{m}\sim 0}} e^{4\pi\sqrt{\left(m-\frac{M^2}{4n}\right)Max}} + \dots + C_v(\bar{m})\Big|_{\substack{Min \\ \bar{m}\sim\lambda}} e^{4\pi\sqrt{\left(m-\frac{M^2}{4n}\right)Min}}$$

+ ... (c-dependent terms that can also become comparable)

→ ~~It is possible that~~ If $C_v(\bar{m})$ grows ~~sufficiently fast~~ sufficiently fast, it is possible that the leading saddle moves from the most polar term (Max) to a not so polar term.

↳ Typical for SU(3) CY w/ $\chi \neq 0$

→ One can show that if CY = $T^2 \times T^4$ / Quotients $(N=4)$
$T^2 \times K^3$ / Quotients

$C_v(\bar{m})$ still grows exponentially but not fast enough, so that the leading saddle does not change.

In this case, we say that Bekenstein-Hawking entropy holds in a more extensive regime!

→ This is the regime where 4D Black Hole physics holds: geometry becomes $AdS_2 \times S^2$ w/ $R(S^1) << 1$

→ Some Known examples:

$\mathcal{N}=8$ compactification $(T^2 \times T^4)$

$$\phi_{w,m}(\tau,z) = \underbrace{\phi_{-2,1}}_{\substack{\text{weight } -2 \\ \text{index (level)} = 1}}(\tau,z)$$

$$\vartheta(\tau,z) = q^{1/8}(y^{\frac{1}{2}}-y^{-1/2})\prod_{m=1}^{\infty}(1-q^m)(1-yq^m)(1-y^{-1}q^m)$$

$$\phi_{-2,1}(\tau,z) = \frac{\vartheta(\tau,z)^2}{\eta(\tau)^6} = \cancel{\ldots}$$

$$= (y^{\frac{1}{2}}-y^{-1/2})^2 \prod_{m=1}^{\infty} \frac{(1-yq^m)^2(1-y^{-1}q^m)^2}{(1-q^m)^4}$$

$$= y + y^{-1} - 2 + \mathcal{O}(q)$$

→ Level $m=1$ ⇒

$$\phi_{-2,1}(\tau,z) = \sum_{\mu=0,1} h_\mu(\tau)\,\Theta_\mu(\tau,z)$$

→ Polar terms $\Delta_{polar} = n - \frac{\mu^2}{4} < 0 \Rightarrow n=0$ and $\mu=1$

So only "one" polar term (there's actually an infinity of them due to spectral flow symm., but they all have the same degeneracy

→ $c_{\mu=1}(0) = (y + y^{-1} + \ldots) = 1$

Max = $\frac{1}{4}$

→ # states:

$$C_\mu(m) \simeq N \int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{dt}{t^{-\omega-2}}\, e^{2\pi\overbrace{\left(m - \frac{\mu^2}{4m}\right)}^{>0}\frac{1}{t} + 2\pi\overbrace{\left(\frac{1}{4}\right)}^{\|}t}$$

$\phi_{-2,1}$ has weight -2, therefore ω, the weight of the vector-valued modular form, equals

$$\omega = -2 - \frac{1}{2} = -\frac{5}{2}$$

$$C_\mu(m) \simeq N \int \frac{dt}{t^{\frac{5}{2}+2 = \frac{9}{2}}}\, e^{2\pi\left(m - \frac{\mu^2}{4m}\right)\frac{1}{t} + \frac{\pi}{2}t}$$

→ Also, since there is only one polar term

$$C_\mu(m) = \int \frac{dt}{t^{9/2}}\, e^{2\pi\left(m - \frac{\mu^2}{4m}\right)\frac{1}{t} + \frac{\pi}{2}t}$$

$$+ \sum_{c=2}^{\infty} \frac{1}{c} \sum_{\substack{(d,c)=1 \\ ad = 1\,\mathrm{mod}(c)}} e^{2\pi i\left(m - \frac{\mu^2}{4m}\right)\frac{d}{c}} M\begin{pmatrix} a & b \\ c & d\end{pmatrix}_{\mu\nu} e^{-2\pi i\left(\frac{1}{4}\right)\frac{a}{c}}$$

$$\times \int \frac{dt}{t^{9/2}}\, e^{2\pi\left(m - \frac{\mu^2}{4m}\right)\frac{1}{ct} + \frac{\pi t}{2c}}$$

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→ $N=4$ example ($T^2 \times T^4$ / CHL Quotients)
$T^2 \times K3$, $T^2 \times K3$/ CHL Quotients)

Take $T^2 \times K^3$ for ex.:

$$\phi_{w,m}(\tau,z) = \phi_{-10,m}(\tau,z)$$

There is a family of SCFT's w/ increasing level $m = 1 \ldots \infty$

(Comment: though this is a meromorphic Jacobi form we can do similar manipulations but it requires introducing the theory of mock-modular form)

$$C_\mu(n) = \frac{(2m+4)}{\sqrt{2m}} \times$$

$$\times \int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{dt}{t^{25/2}} \, e^{\frac{2\pi}{t}\left(n - \frac{\mu^2}{4m}\right) + 2\pi t\left(1 + \frac{m}{4}\right)}$$

when m is fixed

$$n - \frac{\mu^2}{4m} \gg 1$$

→ For $\mathcal{N}=2$ (Denef & Moore)

$$C_\mu(n) = \frac{(p^3 + c_2 \cdot p)}{\sqrt{\det(D_{ab})}\,2} \int \frac{dt}{t^{\frac{b_2}{2}+3}}\, e^{\frac{2\pi}{t}\left(n - \frac{\mu^2}{4m}\right) + \frac{\pi t}{24}\left(p^3 + c_2 p\right)}$$

M5 wrapping Divisor poincaré dual to $p^a \omega_a$ w/ $\omega_a \in H^{1,1}(CY)$; $b_2 = \dim H^{1,1}$

$c_2 \cdot p = c_{2a} \cdot p^a$ w/ c_{2a} = Second Chern-Class of Tangent bundle

$p^3 = C_{abc}\, p^a p^b p^c$ w/ C_{abc} = triple Intersection Matrix $\int \omega_a \wedge \omega_b \wedge \omega_c$

$D_{ab} = C_{abc}\, p^c$