Part II: Chern-Simons theory

$$\int e^{iS} \quad \text{w/} \quad S = \frac{K}{4\pi} \int_M \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)$$

A is Lie-algebra valued for gauge group G

→ Gauge transformation

$$A_\mu \to g A_\mu g^{-1} - dg\, g^{-1} \quad , \text{ for } g \in G$$

→ Field strength:

$$F = dA + A \wedge A$$

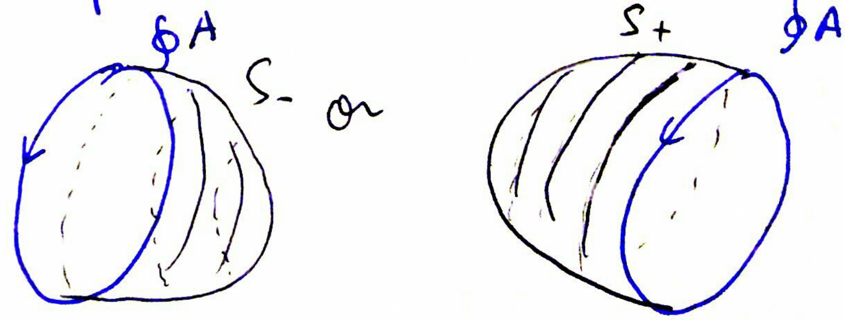
under gauge transformation

$$F \to g F g^{-1}$$

→ K is the level and it is quantized.

Sketch of the proof:

simple example



→ Both integrals must be the same because it was a choice. Therefore:

$$e^{2\pi i q \int_{S^+} F} = e^{2\pi i q \int_{S^-} F}$$

$$\Leftrightarrow \quad e^{2\pi i q \left(\int_{S^+} F - \int_{S^-} F\right)} = 1 \Rightarrow e^{2\pi i q \int_{S^+ \cup S^-} F}$$

Since $\int_{S^+ \cup S^-} F$ is by itself quantized we obtain a quantization for q.

→ We can repeat the same idea for the manifold M.

$$e^{i \frac{k}{4\pi} \int_M \mathrm{tr}\left(A \wedge dA + \frac{2}{3} A^3\right)} = e^{i \frac{k}{4\pi} \int_{S_+} d\mathcal{L}}$$

$$= e^{i \frac{k}{4\pi} \int_{S_-} d\mathcal{L}}$$

$$\partial S_+ = \partial S_- = M$$

$$d\mathcal{L} = \mathrm{tr}(F \wedge F)$$

we obtain the condition that:

$$e^{i \frac{k}{4\pi} \int_{S^+ \cup S^-} \mathrm{tr}(F \wedge F)} = 1$$

but $\frac{1}{2} \int \mathrm{tr}\left(\frac{F}{2\pi} \wedge \frac{F}{2\pi}\right)$ is quantized

$$SU(N) \to \mathbb{Z} \quad (\pi_3(G) \neq 0)$$

→ Observables:

Partition function: $Z = \int DA \, e^{iS}$

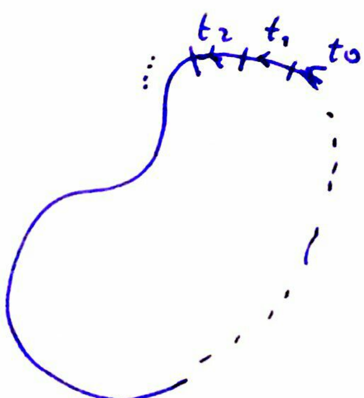
Wilson loops:

$$W_R(\mathcal{C}) = \mathrm{Tr}_R \, P \, e^{\oint_{\mathcal{C}} A}$$

Tr_R: rep R of G; P: Path ordered; $\mathcal{C}$: Path

$$\langle W_R \rangle = \frac{1}{Z} \int DA \; W_R(\mathcal{C}) \, e^{iS}$$

→ Path ordered exponential:



$$\mathrm{Hol} = P \, e^{\oint A} = \prod_{t=t_0}^{t_f} e^{A_\mu^{(t)} \delta X^\mu}$$

(Holonomy)

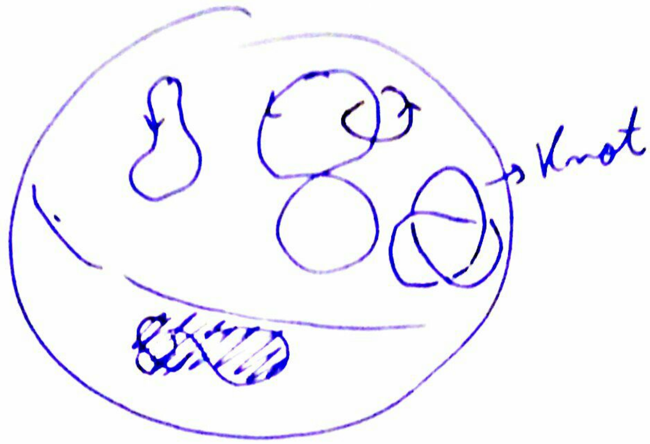
in this order (left → right)

→ Under gauge transformation $A \to g A g^{-1} - dg g^{-1}$

$$\mathrm{Hol} \to g(X(t_0)) \; \mathrm{Hol} \; g(X_0(t_f))^{-1}$$

so $W_R(\mathcal{C})$ is gauge invariant

→ One can compute more general Wilson loops



$$\left\langle \prod_{i=1}^{n} W_{R_i}(\gamma_i) \right\rangle = \frac{1}{Z} \underbrace{\int DA\, e^{iS} \prod_{i=1}^{n} W_{R_i}(\gamma_i)}_{\substack{\parallel \text{ unnormalized} \\ Z(M;\, \gamma_i,\, R_i)}}$$

→ Jones polynomial when all the R_i = N-dim rep of $SU(N)$

(Witten '88)

→ Counts knot invariants

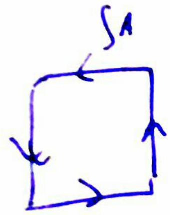
→ Perturbative analysis of partition function

$$Z = \int DA \, e^{i \frac{k}{4\pi} \int \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A^3\right)}$$

$k \to \infty$

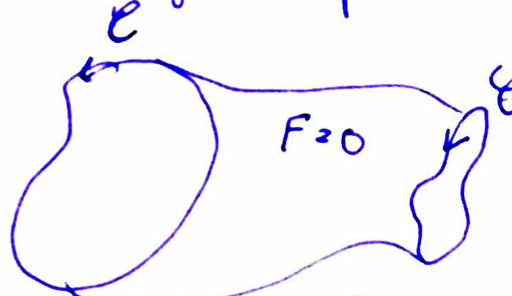
EOM: $dA + A \wedge A = F = 0$ (saddle pts are Flat-connections

→ One can show that



$$P e^{\int A} = e^{\int_{\square} F}$$

which means that the $P e^{\int A_{flat}}$ ~~is~~ only ~~independent of~~ depends on homotopy equivalent paths :



$$P e^{\int_C A} = P e^{\int_{C'} A}$$

→ Equivalence classes of flat connections correspond to homomorphisms :

$$\phi : \pi_1(M) \to G \quad \text{(up to conjugation)}$$

→ At large k we have:

$$Z = \sum_{A_{flat}} e^{ik\,CS(A_f)} \times \mathrm{Det}_{1\text{-loop}} + \underbrace{\cdots}_{\text{corrections}}$$

$$CS(A_f) = \frac{1}{4\pi}\int \mathrm{Tr}\left(A_f \wedge dA_f + \frac{2}{3}A_f^3\right) \quad \text{w/ } dA_f + A_f\wedge A_f = 0$$

→ To compute the Determinant we need to pick a metric in order to do proper gauge-fixing. It happens that the Determinant is a topological invariant and so independent of the metric chosen.

$$\mathrm{Det}_{1\text{-loop}} = \underset{\substack{\uparrow \\ \text{Reidemeister-Ray-Singer} \\ \text{Torsion (Top. invariant)}}}{\tau(M, A_f)^{1/2}} \times k^{(\dim H^1_A - \dim H^0_A)/2}$$

→ (Dependence on k due to zero modes!!)

→ In principle we could compute all the corrections order by order. Of course, we are not going to do this.

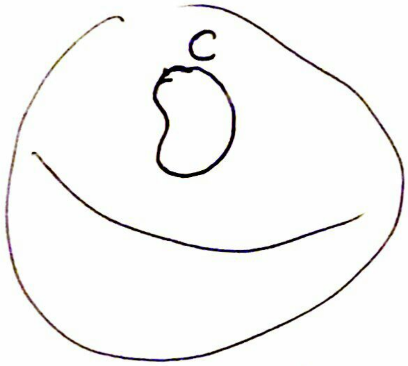
→ Non-perturbative approach (Witten '88)

• Theorem: any 3-manifold M can be obtained by surgery on links of manifold $\tilde{M}$. In particular, any 3-manifold can be obtained from S^3 (or $S^2 \times S^1$).

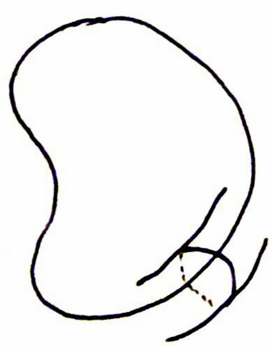
$$S^3 \xrightarrow{\text{surgery}} M$$

• What is surgery: Take manifold M and an arbitrary link C.

(there's no Wilson loop associated. This is just a geometric picture)



→ We thicken the link C to a "tubular neighborhood"



→ solid torus Tub(C)

ICTP

www.ictp.it

→ Removing this tube from M gives rise to a manifold M^* w/ boundary a torus:

$$M^* = M / Tub(C)$$

$$\partial M^* = T^2$$

→ we can further apply a diffeomorphism on the boundary of $Tub(C) \in SL(2,\mathbb{Z})$ and glue it back to M^* to construct new manifold $\tilde{M}$ (Cut and twist)

$$\tilde{M} = M^* \# \widetilde{Tub}(C)$$

→ How this works in more detail?

$$Tub(C) = \underbrace{D}_{\text{Disc}} \times \underbrace{S^1}_{\text{circle}}$$

So, $\partial Tub(C) = \bar{S}^1 \times S^1$

w/ $\bar{S}^1 = \partial D$

$\bar{S}^1$: parametrizes the 1-cycle that is contractible in the full geometry

S^1: parametrizes the non-contractible cycle

How to construct $\widetilde{\mathrm{Tub}}(C)$

→ Define cycles $C_1 \equiv S^1$; $C_2 \equiv \tilde{S}^1$

I can choose a different basis of cycles

$$\begin{pmatrix} \tilde{C}_1 \\ \tilde{C}_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

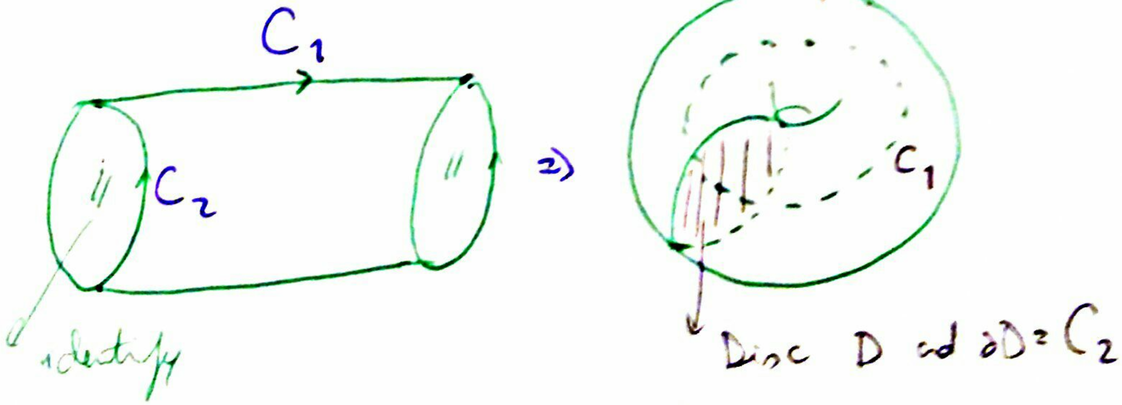
$\in SL(2,\mathbb{Z})$

(basically I want to keep the intersection of C_1 and C_2

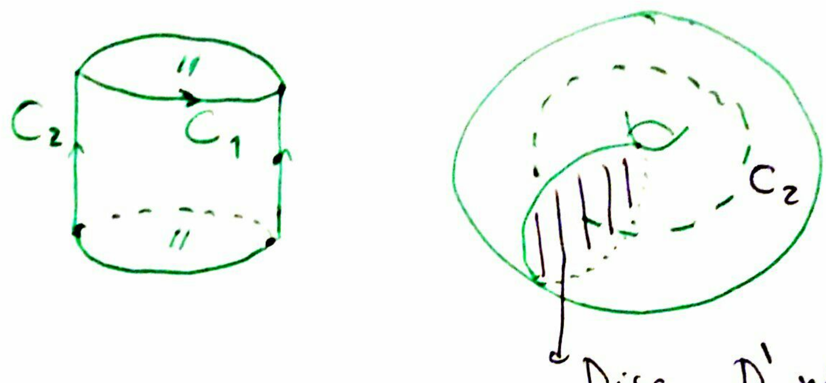
$C_1 \wedge C_2 = \tilde{C}_1 \wedge \tilde{C}_2$

→ Now I can declare that $\tilde{C}_1$ is the non-contractible cycle and $\tilde{C}_2$ is the contractible cycle.

→ How do I do this?

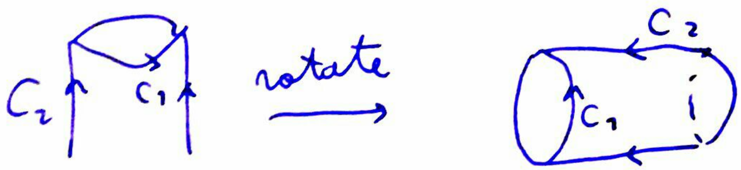


→ Or I could do



Disc D' w/ $\partial D' = C_1$

There's some detail about orientation:



Now we see that the cycle C_1 mapped to $-C_2$, and $C_2 \to C_1$, that is:

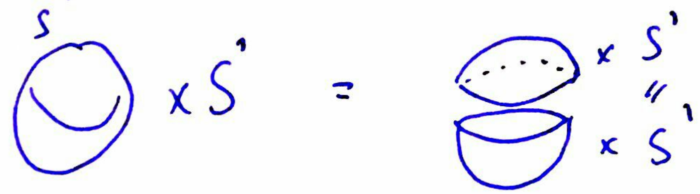
$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \to \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

→ This procedure is known as Filling. We could consider more complicated fillings but it is difficult to visualize.

→ For example:

$$S^1 \times S^2 = \text{Tub}_1 \# \text{Tub}_1$$

glued w/ identity

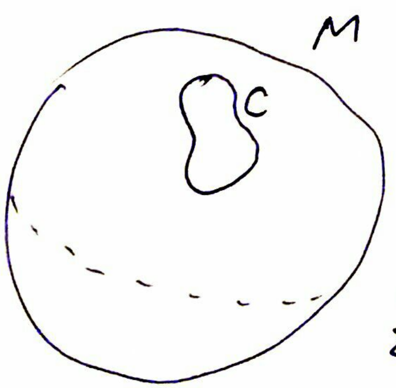


→ $S^3 = \text{Tub}_1 \# \text{Tub}_1$ glued w/ $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

→ lens space $= \text{Tub}_1 \# \text{Tub}_1$ glued w/ $\begin{pmatrix} a & b \\ \underset{\neq 0}{c} & d \end{pmatrix}$

→ The way Witten goes around w/ computing non-perturbatively is schematically of the form:

~~Take~~ $\cancel{S^1 \times S^2 = \text{Tub}_1 \# \text{Tub}_1}$



Suppose $\tilde{M}$ is obtained from M by surgery along C w/ element $g \in SL(2,\mathbb{Z})$, then:

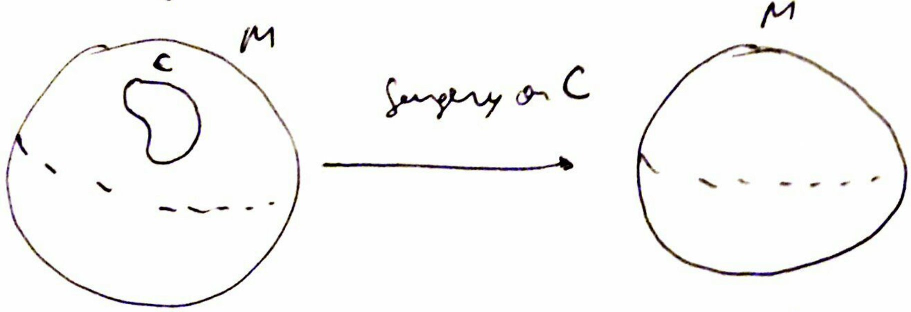
$$Z(\tilde{M}) = K_0^{\ i}\, Z(M; W_{R_i})$$

Put Wilson loop along C in rep R_i

$K_0^i(g)$ is a homomorphism from $SL(2,\mathbb{Z})$ to the Hilbert space defined by R_i

After surgery we obtain $\tilde{M}$ w/ no Wilson loop.

→ More generally, we have:



$$Z\left(\tilde{M}, W_{R_i}(C)\right) = K_i^{\ j}\, Z\left(M, W_{R_j}(C)\right)$$

$i = 0 \ldots \dim(\text{Hilbert space})$

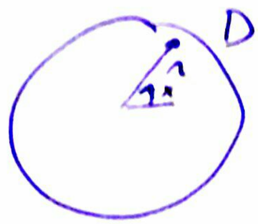
→ If we know $K(S)$ and $K(T)$ we can construct $K(S T^{m_1} S T^{m_2} \ldots)$

→ Since for many manifolds we can represent them as glued Tori, in order to compute the Chern-Simons action of Flat Connection we need to learn how to compute the CS for the Solid torus.

→ lets compute the Chern-Simons action of Flat Connection on Solid Torus $D \times S^1$

$$A = -dg\,g^{-1} \quad (\text{Flat } dA + A \wedge A = 0)$$

Parametrize Disc D by (x, r) and circle S^1 by y:



$\times\ S^1(y)$

$x \in [0, 2\pi]$

$r \in [0, 1]$

We can always bring $g(x,r,y)$ to what is called "normal" form, that is,

$$g(x,r,y) = f(x,r)\, e^{-i\hat{\beta} y}$$

$$\hat{\beta} = \frac{\beta}{2} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\sigma_3}$$

and $f(x,r) : D \to G$

w/ the condition that

$$\lim_{r \to 1} f(x,r) = e^{-i\hat{\alpha} x} \qquad \hat{\alpha} = \frac{\alpha}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Therefore at the boundary, the connection becomes "abelian"

$$\lim_{r \to 1} -dg\,g^{-1} = i\hat{\alpha}\,dx + i\hat{\beta}\,dy$$

→ We can compute the $\mathrm{Hol} = \mathcal{P} e^{\oint A}$. In this case the path ordering is easy because the conection "Commutes"

$$\mathrm{Hol} = e^{\oint_{S^1} A} = \begin{pmatrix} e^{i\pi\beta} & 0 \\ 0 & e^{-i\pi\beta} \end{pmatrix}$$

→ Compute flat connection:

$$-dg\,g^{-1} = -\frac{\partial f}{\partial r} f^{-1} dr - \frac{\partial f}{\partial x} f^{-1} dx + i f \hat{\beta} f^{-1} dy$$

→ $\int \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A^3\right)$

$$dA = -A \wedge A$$

$$= -\frac{1}{3} \int \mathrm{Tr}\left(A \wedge A \wedge A\right) \longleftarrow \text{Plug in}$$

$$-\frac{1}{3} \times 3 \int_{D \times S^1} \mathrm{Tr}\left(\left[\partial_r f f^{-1}, \partial_x f f^{-1}\right] i f \hat{\beta} f^{-1}\right) dr \wedge dx \wedge dy$$

(integrate y)

$$= 2\pi\beta \int_D \mathrm{Tr}\left(\left[\partial_x f f^{-1}, \partial_r f f^{-1}\right] f \frac{i}{2} \sigma_3 f^{-1}\right) dx \wedge dr$$

(choice of orientation $\int dx \wedge dy = +$

→ we can show that

$$\omega = \mathrm{Tr}\left(df \wedge i\frac{\sigma_3}{2} f^{-1} \right)$$

$$d\omega = + \mathrm{Tr}\left(df \wedge i\frac{\sigma_3}{2} f^{-1} df f^{-1} \right)$$

$$= - \mathrm{Tr}\left([\partial_x f f^{-1}, \partial_r f f^{-1}] f i\frac{\sigma_3}{2} f^{-1} \right) dx \wedge dr$$

$$= 2\pi\beta \int_D d\omega = 2\pi\beta \int_{\partial D} \omega$$

$$\omega \xrightarrow[r \to 1]{} -i\hat{\alpha}\, dx\, e^{-i\hat{\alpha}x}\, i\frac{\sigma_3}{2}\, e^{i\hat{\alpha}x}$$

$$= \hat{\alpha}\frac{\sigma_3}{2} dx = \frac{\alpha}{4}\sigma_3^2 dx$$

$$= 2\pi\beta \int_{\partial D} \omega = \boxed{2\pi^2 \beta\alpha}$$

Orientation $\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial y}, \frac{\partial}{\partial x} \right)$