Two Dimensional Critical Curves: CFT, self-similarity, fractals II

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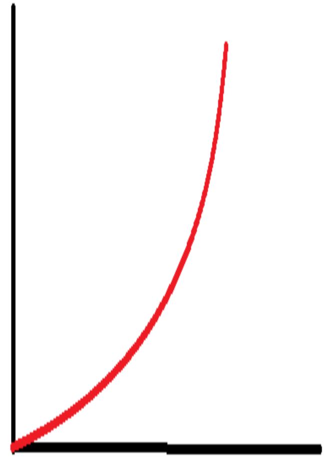
Scaling Graphs

$y(ax) = a\, y(x)$ self-similar

$y(ax) = b\, y(x), \;\; b \neq a$ self-affine



$eg\; Y(x) = c\, x^3 , \qquad b = a^{-3}$

Not Interesting !

Graph of Weierstrass function

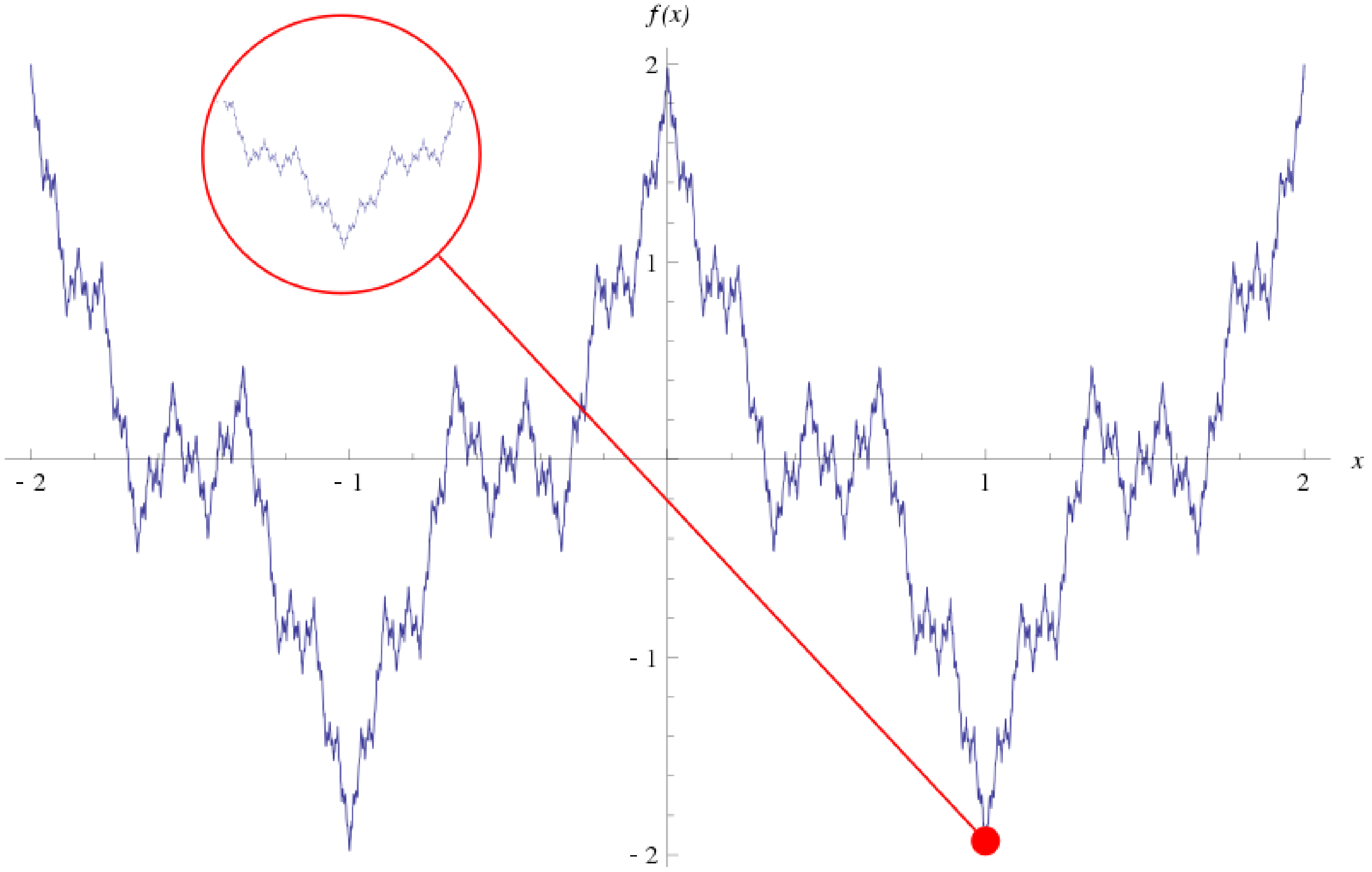


Image wikipedia

Weierstrass function

$$y(x) = \sum_{n=0}^{n=\infty} b^n \,\text{Cos}(a^n x)$$

b is positive odd integer such that

$$ab > 1 + \frac{3}{2}\pi$$

Continuous every where nowhere differentiable

Weierstrass function is self-affine

$$b\, y(a\, x) = y(x)$$

Hausdorff dimension:

$$d_f = 2 + \log(b) / \log(a)$$

Clearly $1 < d_f < 2$

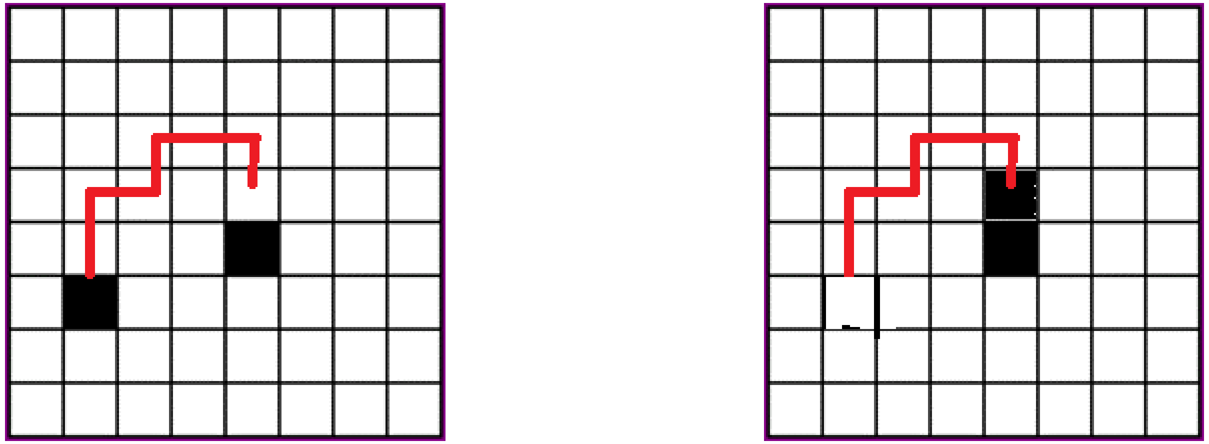
Hunt, Brian. "The Hausdorff dimension of graphs of Weierstrass functions." *Proceedings of the American mathematical society* 126.3 (1998): 791-800.

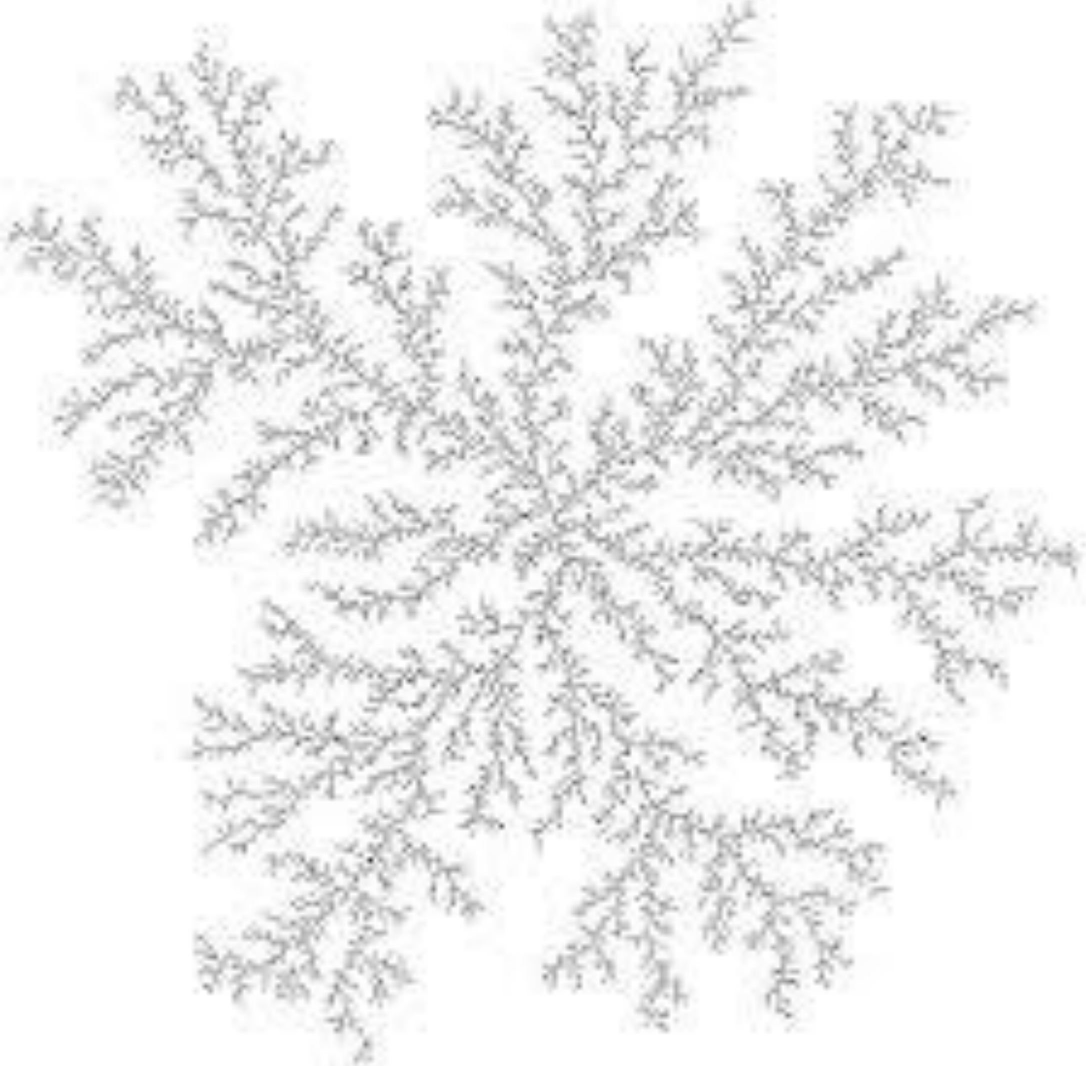
Calculate the fractal dimension using correlation methods

d_f=1.539..

Diffusion Limited Aggregate

Aggregation of particles by random walk





Calculate the fractal dimension using correlation methods

Box counting method ,

number of boxes in radius R is :

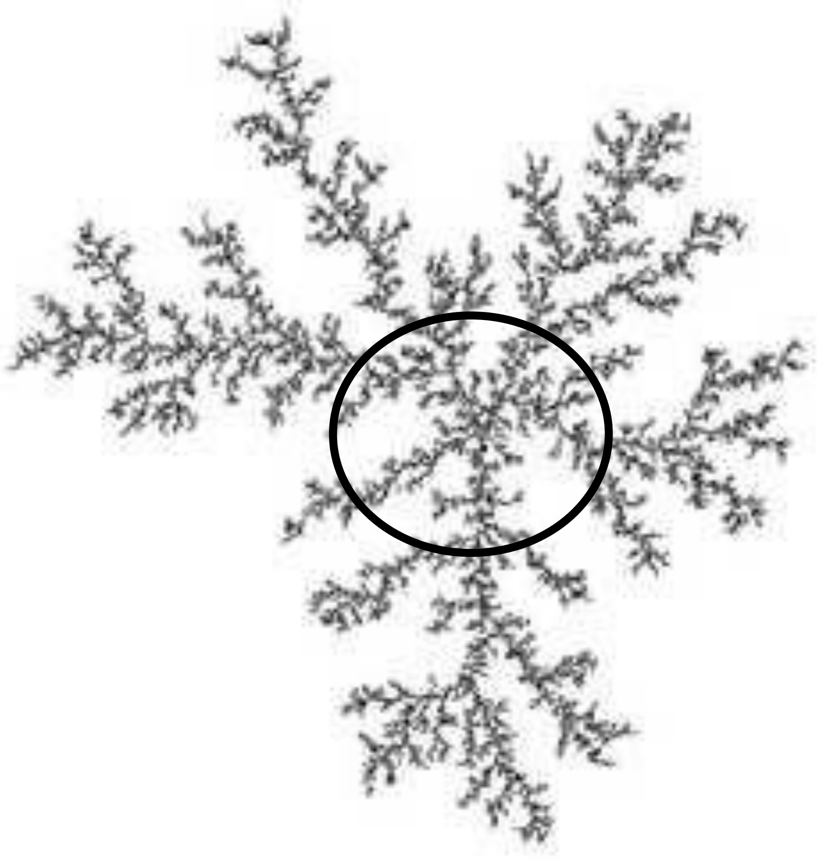
$$N(R) \sim R^{d_f}$$

number of boxes of size *a* is :

$$N(R) \sim a^{-d_f}$$

Or summing yup

$$N(a, R) \sim \ (R/a)^{d_f}$$



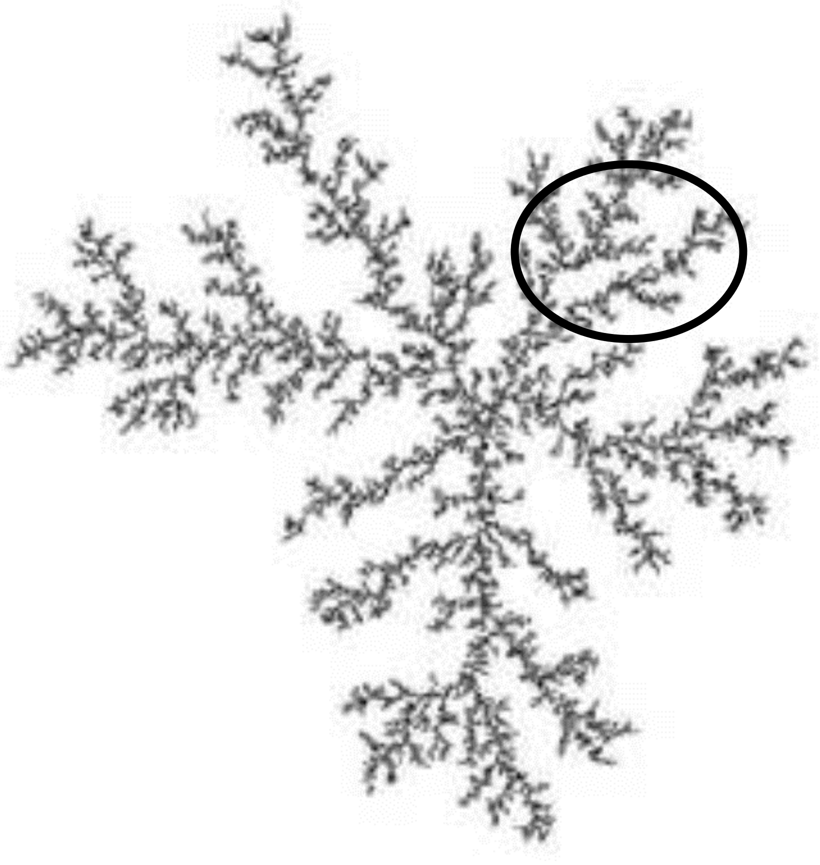
Calculate the fractal dimension using correlation methods

What if the center changes ?

$$C(r,x) = \langle \rho(x+r)\rho(x) \rangle$$

average density of particles around the particle at x

$$C(r) = \int C(r,x)\, dx \;\sim\; r^{d_f - 2}$$



Calculate the fractal dimension of images

Density autocorrelation method allows us to calculate the fractal dimension of any image !

Estimate fractal dimension of originally “objects” in 3d
Using gray scales on 2d images

d_f=1.334

d_f=2.867

- M Ghafari, M Ranjbar, S Rouhani; “Observation of a crossover in kinetic aggregation of Palladium colloids” Applied Surface Science, 2015 , 1143-1149 ; arXiv preprint arXiv:1412.8052 353,
- Shanmugavadivu, P., and V. Sivakumar. "Fractal dimension based texture analysis of digital images." Procedia Engineering 38 (2012): 2981-2986.

Fractal Dimensions of Critical Curves

- Curves we observe in critical theories of 2d Statisitcal physics models are scale invariant hence fractals.

- How can we calculate the fractal dimension of the critical curves we observe in our models ?

1. Traditional methods
2. Conformal Field Theory

Conformal Field Theory in 2d
a quantum field theory with conformal invariance

1. Operators

$$\varphi_{h,\overline{h}}\,(z,\overline{z})$$

2. Hilbert Space

$$\varphi_{h,\overline{h}}\,(0)\ |\,0\rangle = \ |\,h,\overline{h}\rangle$$

Field operators

Under conformal transformations $z \to w(z)$, all field operators must be representations of the Virasoro algebra.

Under a conformal transformation field operators transform as

$$\varphi(w,\overline{w}) \;\to (\frac{\partial w}{\partial z})^{-h} (\frac{\partial \overline{w}}{\partial \overline{z}})^{-\overline{h}} \varphi(z,\overline{z})$$

Conformal Field Theory - CFT

These are called quasi-primary fields. The conformal weights h and $\overline{h}$ are related to the scaling dimension Δ and spin s:

$$h = \frac{1}{2}(\Delta + s) \qquad \overline{h} = \frac{1}{2}(\Delta - s)$$

Generators of conformal Symmetry

1. Recall generators of symmetries = conserved currents
2. Here our main current is T(z)
3. Laurent expand T around origin:

$$T(z) = \sum_n z^{-n-2} L_n$$

$$L_n = \frac{1}{2\pi i} \oint T(z) z^{n+1}$$

Virasoro Algebra

These generators form an infinite extension of the sl(2,c) algebra, but with a central charge:

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12} m(m^2 - 1)\delta_{m+n,0}$$

Conformal Field Theory (CFT)

Conformal Field Theory is a quantum field theory with conformal invariance

In 2d this means that the operator content of the CFT must be a representation of the Virasoro Algebra

Highest weight Representations

Let there be a state $|h>$ such that:

$$L_n|h> = 0, \qquad n > 0$$
$$L_0|h> = h\,|h>$$

Then the Verma module V_h ,formed by the states:
$L_{-n1}L_{-n2}L_{-n3}$ ... $|h>$ is a representation of the Virasoro Algebra. If these modules were irreducible then we have for the Hilbert Space:

$$H = \oplus_{h,\bar{h}} V_h \otimes \overline{V}_{\bar{h}}$$

Operator-state correspondence

In 2d CFT we have a strict

operator –state correspondence.

If state $|\varphi >$ belongs to the Hilbert space then there must exist an operator φ such that :

$$\varphi\ |0> = |\varphi >$$

Where $|0>$ is the vacuum.

2,3 point functions

$$\langle \varphi_{h_1}(z_1)\varphi_{h_2}(z_2)\rangle = \frac{\delta_{h_1,h_2}}{|z_1-z_2|^{h_1+h_2}}$$

$$\langle \varphi_{h_1}(z_1)\varphi_{h_2}(z_2)\varphi_{h_3}(z_3)\rangle = \frac{C_{123}}{{x_{12}}^{a}{x_{23}}^{b}{x_{31}}^{c}}$$

$$a = h_1 + h_2 - h_3,\ldots$$

$$x_{12} = \ z_1 - z_{2,\ldots}$$

4 point function

Four point function is

$$\langle \varphi_{h_1}(z_1)\varphi_{h_2}(z_2)\varphi_{h_3}(z_3)\varphi_{h_4}(z_4)\rangle = f(\eta)\prod_{i<j} {x_{ij}}^{\frac{h}{3}-h_i+h_j}$$

$$h = h_1 + h_2 + h_3 + h_4$$

The only independent cross ratio is;

$$\eta = \frac{x_{12}x_{34}}{x_{13}x_{24}}$$

4 point function

- Symmetry does not determine the function $f(\eta)$
- To do so we need to specify the exact CFT we are dealing with then f satisfies a hyper-geometric function which is actually the expression of a null state

Null states

Using the highest weight representation, we end up with descendent states obtained by the action of the ladder operators:

$$L_{-n}|h >= |h + n >$$

At the weight h+n there will be degeneracy, there are P(n) states with equal weight

Null states

The minimal series are characterized by Null states equation which is a linear combination of the Virasoro generators which when acting on the state annihilate it. For example for the Ising model we have:

$$\left(L_{-2} - \frac{3}{4}{L_{-1}}^2\right) |\tfrac{1}{2} > = 0$$

Minimal series

$$c = 1 - \frac{6}{m(m+1)}\ , m = 2,3, ..$$

$\varphi_{p,q}$ has conformal dimensions:

$$h_{p,q} = \frac{((m+1)p - mq)^2 - 1}{4m(m+1)},\ 1 \leq p \leq m-1\ , 1 \leq q \leq p$$

Representations of the Virasoro Algebra

the minimal series representations of Virasoro algebra are finite.

1. For example for m=2, c=0,p=1,q=1, has only one state the vacuum

2. For m=3, c=1/2 ,

p=1,q=1 the vacuum state,

p=2,q=1 and p=2,q=2 , implying that his CFT has two primary fields. Since we know this model to correspond to the Ising model, these two fields have to be the energy density and spin, with conformal weights:

$$h_{1,1} = 0, h_{2,1} = \frac{1}{2}, h_{2,2} = 1/16$$

Minimal Series

Other low lying CFT’s in the minimal series are:

Table 1: Low lying CFT's and corresponding critical models in 2d.

m	c	Statistical model
3	1/2	Ising model
4	7/10	Tricritical Ising model
5	4/5	3-state Potts model
6	6/7	Tricritical 3-state Potts model