General open quantum dynamics

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1 Introduction: closed systems vs. open systems

1.1 Closed systems

Consider a quantum system S and let $\mathcal{H}$ be the corresponding system's Hilbert space. The evolution of the *closed system* is fully governed by the system Hamiltonian H via the Schrödinger equation

$$i\dot{\psi}_t = H\psi_t \ , \quad (\hbar = 1), \tag{1}$$

and hence

$$\psi \ \longrightarrow \ \psi_t = U_t\psi, \tag{2}$$

where the unitary operator U_t is defined by

$$U_t = e^{-iHt} \ , \tag{3}$$

and $\psi \in \mathcal{H}$ is an initial ($t = 0$) state. Mixed states represented by density operators evolve according to von Neumann equation

$$\dot{\rho}_t = -i[H, \rho_t] \ . \tag{4}$$

1. pure state evolves into pure state
2. mixed state ρ evolves

$$\rho \ \longrightarrow \ \rho_t = \mathbb{U}_t(\rho) := U_t\rho U_t^\dagger, \tag{5}$$

3. entropy $S(\rho) = -\mathrm{Tr}(\rho \log \rho)$ satisfies

$$S(\rho_t) = S(\rho), \tag{6}$$

4. purity $\mathrm{Tr}\rho_t^2$ is constant,
5. the evolution $\mathbb{U}_t$ is **reversible**, that is, $\mathbb{U}_t^{-1} = \mathbb{U}_{-t}$.

1.2 Open systems

Consider now a quantum system S interacting with another system E – environment – and let

$$\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E$$

be the corresponding '$S+E$' Hilbert space. The Hamiltonian of the total closed '$S+E$' system reads

$$H = H_0 + H_{\rm int} = H_S \otimes \mathbb{1}_E + \mathbb{1}_S \otimes H_E + H_{\rm int}. \tag{7}$$

Note, that the splitting is not unique.

Let the initial state of '$E+S$' be as follows

$$\rho_{SE} = \rho \otimes \rho_E, \tag{8}$$

that is, initially (at $t=0$) S and E are not correlated. Since '$S+E$' is a closed system its evolution reads as follows

$$\rho_{SE} \ \longrightarrow \ \rho_{SE}(t) := U_t^{SE} \rho \otimes \rho_E U_t^{SE\dagger}, \tag{9}$$

where $U_t^{SE} = e^{-iHt}$.

Question: what is the evolution of the system S itself? The state of the system S evolves according to

$$\rho_t := {\rm Tr}_E \rho_{SE}(t) \tag{10}$$

and it is called **reduced evolution** of the system S.

The map

$$\boxed{\rho \to \Lambda_t(\rho) := {\rm Tr}_E \Big(U_t^{SE} \rho \otimes \rho_E U_t^{SE\dagger} \Big)} \tag{11}$$

enjoys the following properties:

- completely positive (CP)
- trace-preserving (TP)
- $\Lambda_{t=0} = {\rm id}$.

Λ_t is called a **dynamical map**.

1.3 Positive and completely positive maps

Let $L(\mathcal{H})$ be a space of linear operators in $\mathcal{H}$ (in this notes I assume that $\dim \mathcal{H} = d < \infty$).

Definition 1 *A linear map (super-operator)* $\Phi : L(\mathcal{H}) \to L(\mathcal{H})$ *is called*

- **positive** *iff*

$$X \geq 0 \implies \Phi(X) \geq 0.$$

- n**-positive** *if*

$$\mathrm{id}_n \otimes \Phi : M_n(\mathbb{C}) \otimes L(\mathcal{H}) \to M_n(\mathbb{C}) \otimes L(\mathcal{H})$$

  is positive

- **completely positive** *if it is n-positive for* $n = 1, 2, 3, \ldots$.

A linear map $\Phi : L(\mathcal{H}) \to L(\mathcal{H})$ is

- **trace-preserving** if $\mathrm{Tr}\Phi(X) = \mathrm{Tr}X$ for all $X \in L(\mathcal{H})$
- **unital** if $\Phi(\mathbb{1}) = \mathbb{1}$.

Note, that fixing an othonormal basis $|k\rangle$ in $\mathcal{H}$ one may define a matrix

$$\boxed{T_{ij} := \mathrm{Tr}(P_i \Phi(P_j))} \tag{12}$$

If Φ is positive and trace-preserving, then T_{ij} is stochastic.

Frobenius-Perron theorem — some remarks (classical vs. quantum).

Let E_{ij} be a matrix unit in $M_n(\mathbb{C})$. Any operator $X \in M_n(\mathbb{C}) \otimes L(\mathcal{H})$ has a following form

$$X = \sum_{i,j=1}^{n} E_{ij} \otimes X_{ij}, \quad X_{ij} \in L(\mathcal{H}).$$

One has

$$(\mathrm{id}_n \otimes \Phi)(X) := \sum_{i,j=1}^{n} E_{ij} \otimes \Phi(X_{ij}). \tag{13}$$

Proposition 1 Φ *is CP iff it is d-positive.*

Corollary 1 *One has*

$$\mathrm{CP} = \mathcal{P}_d \subset \mathcal{P}_{d-1} \subset \ldots \subset \mathcal{P}_1 = \mathrm{Positive}. \tag{14}$$

Theorem 1 (Stinespring, 1955) *$\Phi : \mathcal{A} \to L(\mathcal{H})$ is CP ($\mathcal{A}$ is a C^*-algebra) iff there exist*

- *a Hilbert space $\mathcal{K}$*
- *a $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$*
- *a linear operator $V : \mathcal{K} \to \mathcal{H}$*

such that

$$\Phi(a) = V\pi(a)V^{\dagger}. \tag{15}$$

for all $a \in \mathcal{A}$.

Theorem 2 *Φ is CP iff the* **Choi matrix**

$$C_{\Phi} := \sum_{i,j=1}^{d} E_{ij} \otimes \Phi(E_{ij}) \geq 0. \tag{16}$$

Theorem 3 (Stinespring,Sudarshan,Kraus) *A map Φ is CP if and only if*

$$\boxed{\Phi(X) = \sum_i K_i X K_i^{\dagger}} \tag{17}$$

where $K_i \in L(\mathcal{H})$ are called **Kraus operators**.

The map Φ represented in (17) is

- trace-preserving if
$$\sum_i K_i^{\dagger} K_i = \mathbb{1}. \tag{18}$$
- unital if
$$\sum_i K_i K_i^{\dagger} = \mathbb{1}. \tag{19}$$

Example 1 *Some examples of positive but not CP maps – they are important in entanglement theory!*

Basic properties of quantum channels: $\mathcal{E} : L(\mathcal{H}) \to L(\mathcal{H})$

- $\|\mathcal{E}(X)\|_1 \leq \|X\|_1$
- $S(\mathcal{E}(\rho)||\mathcal{E}(\sigma)) \leq S(\rho||\sigma)$
- $F(\mathcal{E}(\rho), \mathcal{E}(\sigma)) \geq F(\rho, \sigma)$

$$D(\rho||\sigma) = \begin{cases} \mathrm{Tr}[\rho(\log\rho - \log\sigma])\ , & \text{if } \mathrm{supp}\,\rho \subseteq \mathrm{supp}\,\sigma \\ +\infty\ , & \text{otherwise} \end{cases} . \tag{20}$$

and

$$F(\rho,\sigma) = \left(\mathrm{Tr}\sqrt{\sqrt{\rho}\,\sigma\,\sqrt{\rho}}\right)^2. \tag{21}$$

Example 2 (Pure decoherence) *Consider d-level system S coupled to the environment*

$$H = H_S \otimes \mathbb{1}_E + \mathbb{1}_S \otimes H_E + \sum_k P_k \otimes B_k \tag{22}$$

where

$$H_S = \sum_k E_k P_k. \tag{23}$$

One has

$$H = \sum_k P_k \otimes Z_k\ ; \quad Z_k = E_k \mathbb{1}_S + H_E + B_k. \tag{24}$$

One finds

$$U_t = e^{-iHt} = \sum_k P_k \otimes e^{-iZ_k t}, \tag{25}$$

and hence

$$\Lambda_t(\rho) = \sum_{k,l} C_{kl}(t)\, P_k \rho P_l \tag{26}$$

with

$$C_{kl}(t) = \mathrm{Tr}\Big(e^{-iZ_k t}\rho_E e^{iZ_l t}\Big). \tag{27}$$

The evolution of the density operator is very simple:

$$\rho_{kl} \ \longrightarrow\ C_{kl}(t)\,\rho_{kl},$$

that is, it is defined by the Hadamard product of $C(t)$ and ρ. Recall, that

$$(A \circ B)_{kl} := A_{kl} B_{kl}. \tag{28}$$

The map

$$\Phi_C(X) := C \circ X \tag{29}$$

is CP if and only if $C \geq 0$.

2 Markovian semigroup

The simplest evolution is provided by the following master equation

$$\dot{\rho}_t = \mathcal{L}(\rho_t), \tag{30}$$

which generalizes von Neumann equation

$$\dot{\rho}_t = -i[H, \rho_t] =: \mathcal{L}_H(\rho_t), \tag{31}$$

that is, the super-operator $\mathcal{L}_H : L(\mathcal{H}) \to L(\mathcal{H})$ is defined by

$$\mathcal{L}_H(\rho) := -i[H, \rho]. \tag{32}$$

The solution to (30) has the following form

$$\Lambda_t = e^{t\mathcal{L}}. \tag{33}$$

Theorem 4 (Gorini,Kossakowski,Sudarshan,Lindblad) *A linear map $\mathcal{L} : L(\mathcal{H}) \to L(\mathcal{H})$ generates legitimate dynamical map if and only if*

$$\mathcal{L}(\rho) = -i[H, \rho] + \sum_k \gamma_k \Big(V_k \rho V_k^\dagger - \frac{1}{2}\{V_k^\dagger V_k, \rho\} \Big) \tag{34}$$

where $\{A, B\} = AB + BA$, and $\gamma_k > 0$.

2.1 Examples of Markovian semigroups

Example 3 (Qubit decoherence)

$$\mathcal{L}(\rho) = \frac{\gamma}{2}(\sigma_z \rho \sigma_z - \rho); \quad \gamma > 0. \tag{35}$$

Note that

$$\begin{aligned} \mathcal{L}(E_{11}) &= 0 \\ \mathcal{L}(E_{22}) &= 0 \\ \mathcal{L}(E_{12}) &= -\gamma E_{12} \\ \mathcal{L}(E_{21}) &= -\gamma E_{21} \end{aligned}$$

and hence

$$\begin{aligned} \Lambda_t(E_{11}) &= E_{11} \\ \Lambda_t(E_{22}) &= E_{22} \\ \Lambda_t(E_{12}) &= e^{-\gamma t} E_{12} \\ \Lambda_t(E_{21}) &= e^{-\gamma t} E_{21} \end{aligned}$$

Now finds the following Kraus representation

$$\Lambda_t(\rho) = \frac{1+e^{-\gamma t}}{2}\,\rho + \frac{1-e^{-\gamma t}}{2}\,\sigma_z\rho\sigma_z. \tag{36}$$

Another way is a direct computation of $e^{\mathcal{L}t}$.

Example 4 (Qubit dissipation) *Let us consider*

$$\Phi(\rho) = \frac{1}{2}\Big(\gamma_+\mathcal{L}_+ + \gamma_-\mathcal{L}_-\Big) \tag{37}$$

where where

$$\begin{aligned}\mathcal{L}_1(\rho) &= [\sigma_+, \rho\sigma_-] + [\sigma_+\rho, \sigma_-]\ , \\ \mathcal{L}_2(\rho) &= [\sigma_-, \rho\sigma_+] + [\sigma_-\rho, \sigma_+]\ ,\end{aligned} \tag{38}$$

$\mathcal{L}_+$ *corresponds to pumping (heating) process,* $\mathcal{L}_-$ *corresponds to relaxation (cooling). To solve the master equation* $\dot{\rho}_t = L\rho_t$ *let us parameterize* ρ_t *as follows*

$$\rho_t = p_1(t)P_1 + p_2(t)P_2 + \alpha(t)\sigma_+ + \overline{\alpha(t)}\sigma_-\ , \tag{39}$$

with $P_k = |k\rangle\langle k|$. *Using the following relations*

$$\begin{aligned}\mathcal{L}(P_1) &= -\gamma_+\,\sigma_3\ , \\ \mathcal{L}(P_2) &= \gamma_-\,\sigma_3\ , \\ \mathcal{L}(\sigma_+) &= \gamma\,\sigma_+\ , \\ \mathcal{L}(\sigma_-) &= \gamma\,\sigma_-\ ,\end{aligned}$$

where

$$\gamma = \frac{\gamma_+ + \gamma_-}{2}.$$

one finds the following Pauli master equations for the probability distribution $(p_1(t), p_2(t))$

$$\dot{p}_1(t) = -\gamma_+\,p_1(t) + \gamma_-\,p_2(t)\ , \tag{40}$$

$$\dot{p}_2(t) = \gamma_+\,p_1(t) - \gamma_-\,p_2(t)\ , \tag{41}$$

together with $\alpha(t) = e^{-\gamma t}\alpha(0)$. *The corresponding solution reads*

$$p_1(t) = p_1(0)\,e^{-(\gamma_+ + \gamma_-)t} + p_1^*\Big[1 - e^{-(\gamma_+ + \gamma_-)t}\Big]\ , \tag{42}$$

$$p_2(t) = p_2(0)\,e^{-(\gamma_+ + \gamma_-)t} + p_2^*\Big[1 - e^{-(\gamma_+ + \gamma_-)t}\Big]\ , \tag{43}$$

where we introduced

$$p_1^* = \frac{\gamma_+}{\gamma_+ + \gamma_-}\ , \qquad p_2^* = \frac{\gamma_+}{\gamma_+ + \gamma_-}\ . \tag{44}$$

Hence, we have purely classical evolution of probability vector $(p_1(t), p_2(t))$ *on the diagonal of* ρ_t *and very simple evolution of the off-diagonal element* $\alpha(t)$. *Note, that asymptotically one obtains completely decohered density operator*

$$\rho_t \quad\longrightarrow\quad \left(\begin{array}{cc} p_1^* & 0 \\ 0 & p_2^* \end{array}\right)\ .$$

In particular if $\gamma_+ = \gamma_-$ *a state* ρ_t *relaxes to maximally mixed state (a state becomes completely depolarized).*

3 Beyond Markovian semigroup – non-Markovian evolution

Consider now

$$\dot{\Lambda}_t = \mathcal{L}_t \Lambda_t \ , \quad \Lambda_0 = \mathrm{id}, \tag{45}$$

with time dependent generator $\mathcal{L}_t$. The formal solution reads

$$\Lambda_t = \mathcal{T} \exp\left(\int_0^t \mathcal{L}_u du \right) = \mathrm{id} + \int_0^t \mathcal{L}_u du + \int_0^t dt_2 \int_0^{t_2} dt_1 \mathcal{L}_{t_2} \mathcal{L}_{t_1} + \dots . \tag{46}$$

If $[\mathcal{L}_t, \mathcal{L}_u] = 0$, then

$$\Lambda_t = \exp\left(\int_0^t \mathcal{L}_u du \right) = \mathrm{id} + \int_0^t \mathcal{L}_u du + \frac{1}{2} \left(\int_0^t \mathcal{L}_u du \right)^2 + \dots . \tag{47}$$

Evolution Λ_t is called **divisible** if

$$\Lambda_t = V_{t,s} \Lambda_s \ ; \quad t \geq s. \tag{48}$$

It is called

- P-divisible if $V_{t,s}$ is PTP
- CP-divisible if $V_{t,s}$ is CPTP

Theorem 5 *If Λ_t is P-divisible, then*

$$\frac{d}{dt} \| \Lambda_t(X) \|_1 \leq 0, \tag{49}$$

for all $X \in L(\mathcal{H})$. If Λ_t is CP-divisible, then

$$\frac{d}{dt} \| [\mathrm{id} \otimes \Lambda_t](X) \|_1 \leq 0, \tag{50}$$

for all $X \in L(\mathcal{H}) \otimes L(\mathcal{H})$.

For invertible the converse is also true.

The evolution Λ_t is **Markovian** iff it is CP-divisible.

We stress, that there are many other approaches. For example the one based on distinguishability of states:

$$D(\rho, \sigma) := \frac{1}{2} \| \rho - \sigma \|_1 \tag{51}$$

According to Breuer-Laine-Piilo (BLP) the evolution Λ_t is **Markovian** if

$$\frac{d}{dt}\|\Lambda_t(\rho) - \Lambda_t(\sigma)\|_1 \leq 0, \tag{52}$$

for all states ρ and σ.

Example 5 *Consider*

$$\mathcal{L}_t(\rho) = \frac{1}{2}\sum_{k=1}^{3}\gamma_k(t)(\sigma_k\rho\sigma_k - \rho), \tag{53}$$

with time dependent rates $\gamma_k(t)$. The corresponding map $\Lambda_t = \exp(\int_0^t \mathcal{L}_\tau d\tau)$ has the following form

$$\Lambda_t(\rho) = \sum_{\alpha=0}^{3} p_\alpha(t)\sigma_\alpha\rho\sigma_\alpha, \tag{54}$$

where $\sigma_0 = \mathbb{1}$, and time-dependent probability distribution $p_\alpha(t)$ read:

$$\begin{aligned}
p_0(t) &= \frac{1}{4}\Big(1 + \lambda_1(t) + \lambda_2(t) + \lambda_3(t)\Big),\\
p_1(t) &= \frac{1}{4}\Big(1 + \lambda_1(t) - \lambda_2(t) - \lambda_3(t)\Big),\\
p_2(t) &= \frac{1}{4}\Big(1 - \lambda_1(t) + \lambda_2(t) - \lambda_3(t)\Big),\\
p_3(t) &= \frac{1}{4}\Big(1 - \lambda_1(t) - \lambda_2(t) + \lambda_3(t)\Big),
\end{aligned}$$

with $\lambda_k(t)$ being eigenvalues of the map Λ_t: $\Lambda_t(\sigma_k) = \lambda_k(t)\sigma_k$ defined by

$$\lambda_i(t) = e^{-\Gamma_j(t) - \Gamma_k(t)}, \tag{55}$$

where $\{i, j, k\}$ is a permutation of $\{1, 2, 3\}$, and $\Gamma_k(t) = \int_0^t \gamma_k(\tau)d\tau$.

Proposition 2 *Time-local generator (53) gives rise to a legitimate dynamical map iff $p_\alpha(t) \geq 0$ for $t \geq 0$, that is,*

$$\lambda_i(t) + \lambda_j(t) \leq 1 + \lambda_k(t), \tag{56}$$

where $\{i, j, k\}$ is a permutation of $\{1, 2, 3\}$.

Note, that (56) provides highly nontrivial condition for the rates $\gamma_i(t)$.

Proposition 3 *Λ_t is P-divisible iff*

$$\gamma_1(t) + \gamma_2(t) \geq 0\ ,\ \ \gamma_2(t) + \gamma_3(t) \geq 0\ ,\ \ \gamma_3(t) + \gamma_1(t) \geq 0\ , \tag{57}$$

for all $t \geq 0$.

Proof: note that conditions (56) are necessary. Indeed, P-divisibility requires $\frac{d}{dt}\|\Lambda_t(\sigma_k)\|_1 \leq 0$. *One has*

$$\frac{d}{dt}\|\Lambda_t(\sigma_k)\|_1 = \frac{d}{dt}|\lambda_k(t)|\|\sigma_k\|_1 = -2[\gamma_i(t) + \gamma_j(t)],$$

where again $\{i,j,k\}$ *is a permutation of* $\{1,2,3\}$ *and we used the formula* $\lambda_k(t) = \exp(-\Gamma_i(t) - \Gamma_j(t))$. *Now, the corresponding propagator* $V_{t,s}$ *is given by* $V_{t,s} = e^{\int_s^t \mathcal{L}_\tau d\tau}$, *and hence* $V_{t,s}$ *is PTP iff* $\mathcal{L}_t$ *is a generator of a family of positive trace-preserving maps, that is, for any* ψ *and* ϕ *such that* $\langle\psi|\phi\rangle = 0$ *one has*

$$\langle\psi|\mathcal{L}_t(|\phi\rangle\langle\phi|)|\psi\rangle \geq 0,$$

for all $t \geq 0$. *Introducing the corresponding rank-1 projectors* $P_\psi = |\psi\rangle\langle\psi|$ *and* $P_\phi = |\phi\rangle\langle\phi|$ *let us observe that* $P_\phi = \mathbb{1} - P_\psi$ *(due to orthogonality of* ψ *and* ϕ*) and hence*

$$\begin{aligned}\langle\psi|\mathcal{L}_t(|\phi\rangle\langle\phi|)|\psi\rangle &= \mathrm{Tr}(P_\psi\mathcal{L}_t(\mathbb{1} - P_\psi)) = -\mathrm{Tr}(P_\psi\mathcal{L}_t(P_\psi)) \\ &= -\frac{1}{2}\sum_k \gamma_k(t)\mathrm{Tr}(P_\psi\sigma_k P_\psi\sigma_k) = \frac{1}{2}\sum_k \gamma_k(t)(1 - |\langle\psi|\sigma_k|\psi\rangle|^2),\end{aligned}$$

due to $\mathcal{L}_t(\mathbb{1}) = 0$. *Observe that at any* t *at most one* $\gamma_k(t)$ *may be negative. Indeed, suppose that* $\gamma_1(t) < 0$ *and* $\gamma_2(t) < 0$. *Taking* $|\psi\rangle = |0\rangle$ *one finds*

$$|\langle\psi|\sigma_1|\psi\rangle|^2 = |\langle\psi|\sigma_2|\psi\rangle|^2 = 0 \ , \quad |\langle\psi|\sigma_3|\psi\rangle|^2 = 1,$$

and hence

$$\langle\psi|\mathcal{L}_t(|\phi\rangle\langle\phi|)|\psi\rangle = \gamma_1(t) + \gamma_2(t) < 0.$$

Now, let $\gamma_1(t) < 0$. *One finds*

$$\langle\psi|\mathcal{L}_t(|\phi\rangle\langle\phi|)|\psi\rangle \geq \min\{\gamma_1(t) + \gamma_2(t), \gamma_1(t) + \gamma_3(t)\}$$

which implies (57). □

Proposition 4 *Let* ρ *be an arbitrary initial state. One has*

$$\frac{d}{dt}S(\Lambda_t(\rho)) \geq 0, \tag{58}$$

iff Λ_t *is P-divisible, that is, conditions (57) are satisfied.*

Proof: clearly P-divisibility implies (58). Now, suppose that (58) is satisfied for any ρ. *Taking the Bloch representation* $\rho = \frac{1}{2}(\mathbb{1} + \sum_k x_k\sigma_k)$, *one finds* $\rho_t = \frac{1}{2}(\mathbb{1} + \sum_k x_k(t)\sigma_k)$, *with*

$$x_1(t) = e^{-\Gamma_2(t)-\Gamma_3(t)}x_1 \ , \ x_2(t) = e^{-\Gamma_1(t)-\Gamma_3(t)}x_2 \ , \ x_3(t) = e^{-\Gamma_1(t)-\Gamma_2(t)}x_3,$$

that is, the Bloch vector evolves as follows $\mathbf{x}(t) = (\lambda_1(t)\,x_1, \lambda_2(t)\,x_2, \lambda_3(t)\,x_3)$. *The corresponding eigenvalues* $x_\pm(t)$ *of* ρ_t *read*

$$x_\pm(t) = \frac{1}{2}(1 \pm |\mathbf{x}(t)|).$$

Now, one has for the entropy

$$S(t) = -x_+(t)\log x_+(t) - x_-(t)\log x_-(t),$$

and hence

$$\frac{d}{dt}S(t) = -\dot{x}_+(t)\ \log\frac{x_+(t)}{x_-(t)}. \tag{59}$$

Note that $\log\frac{x_+(t)}{x_-(t)} \geq 0$. *Finally*

$$\dot{x}_+(t) = \frac{1}{|\mathbf{x}(t)|}\sum_{k=1}^{3}\dot{\lambda}_k(t)\lambda_k(t)x_k,$$

and hence since x_k *are arbitrary condition* $\dot{x}_+(t) \leq 0$ *reproduces (57).* □

4 Memory kernel master equation

4.1 Quantum jump representation of Markovian semigroup

Consider Markovian semigroup Λ_t governed by

$$\dot{\Lambda}_t = \mathcal{L}\Lambda_t. \tag{60}$$

Note taht

$$\mathcal{L} = B - Z, \tag{61}$$

where the operators $B, Z : L(\mathcal{H}) \to L(\mathcal{H})$ are defined as follows:

$$B(\rho) = \sum_k V_k\rho V_k^\dagger \tag{62}$$

and

$$Z(\rho) = i(C\rho - \rho C), \tag{63}$$

with $C \in L(\mathcal{H})$ given by

$$C = H + \frac{i}{2}\sum_k V_k^\dagger V_k. \tag{64}$$

Evidently, B is a CP map. Moreover, its dual $B^* : L(\mathcal{H}) \to L(\mathcal{H})$ reads $B^*(X) = \sum_k V_k^\dagger X V_k$ and hence $B^*(\mathbb{I}) = \sum_k V_k^\dagger V_k$. Now, let us denote by N_t a solution of the following equation

$$\dot{N}_t = -ZN_t\ , \quad N_{t=0} = \mathrm{id}. \tag{65}$$

One immediately finds

$$\boxed{N_t(\rho) = e^{-Zt}\rho = e^{-iCt}\rho e^{iC^\dagger t}}. \tag{66}$$

Proposition 5 *If $[B, Z] = 0$, then the solution to (60) reads*

$$\Lambda_t = N_t \sum_{k=0}^{\infty} \frac{t^k}{k\,!} B^k. \tag{67}$$

Proof: one has

$$\Lambda_t = e^{t\mathcal{L}} = e^{t(B-Z)} = e^{-tZ} e^{tB} = N_t \sum_{k=0}^{\infty} \frac{t^k}{k\,!} B^k, \tag{68}$$

where we used $e^{X+Y} = e^X e^Y$ for commuting X and Y. □

Now, since N_t and e^{tB} are CP, the map Λ_t is CP as well.

Proposition 6 *The map N_t is trace non-increasing.*

Proof: one has for arbitrary density operator ρ

$$\frac{d}{dt}\mathrm{Tr}[N_t(\rho)] = \mathrm{Tr}[(-iC + iC^\dagger)\rho] = -\mathrm{Tr}[B^*(\mathbb{I})\rho] \leq 0, \tag{69}$$

due to $B^*(\mathbb{I}) \geq 0$. □

Theorem 6 *The solution to (30) may be represented as follows*

$$\Lambda_t = N_t * \sum_{k=0}^{\infty} Q_t^{*n}, \tag{70}$$

where $X_t * Y_t := \int_0^t X_{t-\tau} Y_\tau d\tau$ denotes convolution, $Q_t := BN_t$, and $Q_t^{*n} := Q_t * \ldots * Q_t$ (n factors).

Proof: passing to the Laplace transform (LT) of (30) and (65) one finds

$$\widetilde{\Lambda}_s = \frac{1}{s - B + Z}\ , \qquad \widetilde{N}_s = \frac{1}{s + Z} \tag{71}$$

and hence

$$\widetilde{\Lambda}_s = \widetilde{N}_s \frac{1}{\mathrm{id} - B\widetilde{N}_s}, \tag{72}$$

where $\widetilde{f}_s := \int_0^\infty f_t e^{-ts} dt$. Now, introducing $\widetilde{Q}_s := B\widetilde{N}_s$ one obtains

$$\widetilde{\Lambda}_s = \widetilde{N}_s \sum_{k=0}^{\infty} \widetilde{Q}_s^n, \tag{73}$$

which implies (70) in the time domain. □

Representation (70) is often called a *quantum jump* representation of the dynamical map Λ_t and the CP map B is interpreted as quantum jump

$$\Lambda_t = e^{t\mathcal{L}} = \mathbb{1} + \mathcal{L}t + \frac{(\mathcal{L}t)^2}{2} + \ldots, \tag{74}$$

$$\Lambda_t = N_t + N_t * BN_t + N_t * BN_t * BN_t + \ldots. \tag{75}$$

4.2 Beyond Markovian semigroup

Consider now

$$\dot{\Lambda}_t = \int_0^t K_{t-\tau}\Lambda_\tau d\tau, \quad \Lambda_0 = \mathrm{id}. \tag{76}$$

Any memory kernel K_t has the following general structure

$$K_t = B_t - Z_t, \tag{77}$$

where maps $B_t, Z_t : L(\mathcal{H}) \to L(\mathcal{H})$ are Hermitian and satisfy $\mathrm{Tr}[B_t(\rho)] = \mathrm{Tr}[Z_t(\rho)]$. This condition guarantees that K_t annihilates the trace, that is, $\mathrm{Tr}[K_t(\rho)] = 0$ for any ρ, and hence Λ_t is trace-preserving. Now, let

$$\dot{N}_t = \int_0^t Z_{t-\tau}N_\tau d\tau, \quad N_0 = \mathrm{id}. \tag{78}$$

and

$$Q_t = B_t * N_t. \tag{79}$$

Theorem 7 *Let $\{N_t, Q_t\}$ be a pair of CP maps such that*

1. $N_{t=0} = \mathrm{id}$*,*
2. $\mathrm{Tr}[Q_t(\rho)] + \frac{d}{dt}\mathrm{Tr}[N_t(\rho)] = 0$ *for any* $\rho \in L(\mathcal{H})$*,*
3. $||\widetilde{Q}_s||_1 < 1$.

Then the following map

$$\Lambda_t = N_t + N_t * Q_t + N_t * Q * Q_t + \ldots \tag{80}$$

defines a legitimate dynamical map.

Proof: condition 3) guarantees that the series

$$\widetilde{\Lambda}_s = \widetilde{N}_s \sum_{k=0}^{\infty} \widetilde{Q}_s^n = \widetilde{N}_s \frac{1}{\mathrm{id} - \widetilde{Q}_s},$$

is convergent and hence (80) defines a CP map. Condition 1) implies that $\Lambda_{t=0} = N_{t=0} = \mathrm{id}$. Finally, condition 2) implies that the map Λ_t is trace-preserving. Indeed, passing the Laplace transform domain one finds

$$\mathrm{Tr}[\widetilde{Q}_s(\rho)] + \mathrm{Tr}[s\widetilde{N}_s(\rho) - \rho] = 0. \tag{81}$$

Now,

$$\widetilde{\Lambda}_s(\mathrm{id} - \widetilde{Q}_s) = \widetilde{N}_s, \tag{82}$$

and hence

$$\frac{1}{s}\mathrm{Tr}([\mathrm{id} - \widetilde{Q}_s](\rho)) = \mathrm{Tr}[\widetilde{N}_s(\rho)), \tag{83}$$

due to

$$\mathrm{Tr}[\widetilde{\Lambda}_s(X)] = \frac{1}{s}\mathrm{Tr}\, X. \tag{84}$$

This proves that (81) is equivalent to the trace-preservation condition (83). □

Semigroup

$$\Lambda_t = N_t + N_t * BN_t + N_t * BN_t * BN_t + \dots. \tag{85}$$

and beyond

$$\Lambda_t = N_t + N_t * B_t * N_t + N_t * B_t * N_t * B_t * N_t + \dots. \tag{86}$$

Example 6 *Let*

$$N_t = \left(1 - \int_0^t f(\tau)d\tau\right)\mathrm{id}, \tag{87}$$

where the function $f : \mathbb{R}_+ \to \mathbb{R}$ *satisfies:*

$$f(t) \geq 0\ , \qquad \int_0^\infty f(\tau)d\tau \leq 1.$$

Moreover, let $Q_t = f(t)\mathcal{E}$*, where* $\mathcal{E}$ *is an arbitrary quantum channel. Then one finds the following formula for the memory kernel*

$$K_t = \kappa(t)(\mathcal{E} - \mathrm{id}), \tag{88}$$

where the function $\kappa(t)$ *is defined in terms of* $f(t)$ *as follows*

$$\widetilde{\kappa}(s) = \frac{s\widetilde{f}(s)}{1 - \widetilde{f}(s)}. \tag{89}$$

In particular taking $f(t) = \gamma e^{-\gamma t}$ *one finds* $K_t = \delta(t)\mathcal{L}$*, with*

$$\mathcal{L} = \gamma(\mathcal{E} - \mathrm{id}), \tag{90}$$

being the GKSL generator.

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