Quantum Entanglement and Holography

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Ref.

QE → { Nielsen Chuang "QC and QI" Cambridge
　　　 Wilde "QIT" "

HEE → Rangamani-TT 1609.0465
　　　　　　　　springer lec. note

① Quantum Entanglement (QE)

(1-1) What is QE?

QE → correlations special to quantum mechanics.

$\hookrightarrow$ between H_A and H_B

(a) For pure states. $\rho_{tot} = |\psi\rangle\langle\psi|$, $|\psi\rangle \in H_{tot} = H_A \otimes H_B$

$|\psi\rangle$ has QE iff $|\psi\rangle \neq |\psi_A\rangle \otimes |\psi_B\rangle$!

(b) For mixed states ρ

ρ has QE iff ρ is Not separable !

$$\left[\rho_{sep} = \underbrace{\sum_k p_k \rho_A^{(k)} \otimes \rho_B^{(k)}}_{\text{classical states}} \quad \left(p_k \geq 0, \ \sum_k p_k = 1 \right) \right]$$

(1-2) Bell States (EPR states)

Consider two spin systems AB. A B

$$|\psi_c\rangle = \left(\frac{|\uparrow\rangle_A + |\downarrow\rangle_A}{\sqrt{2}}\right) \otimes \left(\frac{|\uparrow\rangle_B + |\downarrow\rangle_B}{\sqrt{2}}\right) \to \text{No QE}$$

$$|\psi_B\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_A \otimes |\uparrow\rangle_B + |\downarrow\rangle_A \otimes |\downarrow\rangle_B\right) \to \exists\, \text{QE}$$

(Maximally entangled)

4 Bell states $|\phi^\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$

$|\psi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$

a, a'　　b, b'

A　　B

Bell's inequality (CHSH)

$$\underbrace{|\langle ab\rangle - \langle ab'\rangle + \langle a'b\rangle + \langle a'b'\rangle|}_{S} \leq 2$$

$\uparrow$ classically true.

$$S = \langle a(\underbrace{b-b'}_{0 \text{ or } 2})\rangle + \langle a'(\underbrace{b+b'}_{2 \text{ or } 0})\rangle$$

a, a' → spin measurements for the A spin.

For Bell state $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

$a = \sigma_x$, $a' = \sigma_z$, $b = \frac{1}{\sqrt{2}}(\sigma_z + \sigma_x)$, $b' = \frac{1}{\sqrt{2}}(\sigma_z - \sigma_x)$

$$S = \frac{1}{2}\left(\langle 0|\underbrace{(b-b')}_{\sqrt{2}\sigma_x}|1\rangle + \langle 1|(b-b')|0\rangle\right)$$

$$+ \frac{1}{2}\left(\langle 0|\underbrace{(b+b')}_{\sqrt{2}\sigma_z}|0\rangle - \langle 1|(b+b')|1\rangle\right) = \underbrace{2\sqrt{2}}_{\text{violate ineq.}}$$

(1-3) Entanglement Entropy

For pure states, the amount of QE between A and B is measured by

Entanglement Entropy (EE).

$H_{tot} = H_A \otimes H_B$ we have in mind

$\rho_{tot} = |\Psi\rangle\langle\Psi|$

$\rho_A = \mathrm{tr}_B\, \rho_{tot}$ but in principle ρ_{tot} can be

Entanglement Entropy

$$\boxed{S_A = -\mathrm{tr}\, \rho_A \log \rho_A}$$

Ex. $|\Psi\rangle_C = |\Psi_A\rangle \otimes |\Psi_B\rangle \rightarrow S_A = 0$

$|\Psi\rangle_B = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \rightarrow \rho_A = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\downarrow$

$S_A = \log 2$

Properties of EE

For mixed states, $S_A \neq S_B$

(a) When $\rho_{tot} = |\Psi\rangle\langle\Psi|$, we have $S_A = S_B$.

(∵ $|\Psi\rangle = \sum_i \sqrt{\lambda_i}\, |i\rangle_A |i\rangle_B \to S_A = S_B = \sum_i -\lambda_i \log \lambda_i$)

(b) Strong subadditivity (SSA)

$$S_{AB} + S_{BC} \geq S_{ABC} + S_B$$

Renyi EE

$$S_A^{(n)} = \frac{1}{1-n} \log \mathrm{tr} (\rho_A)^n$$

$$\lim_{n \to 1} S_A^{(n)} = S_A$$

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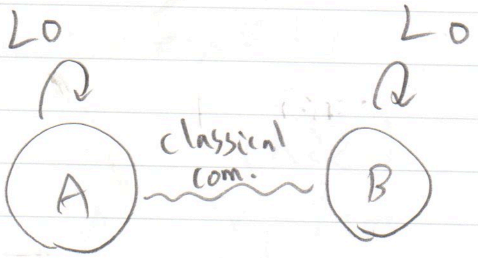
(1-4) Operational Meaning of EE

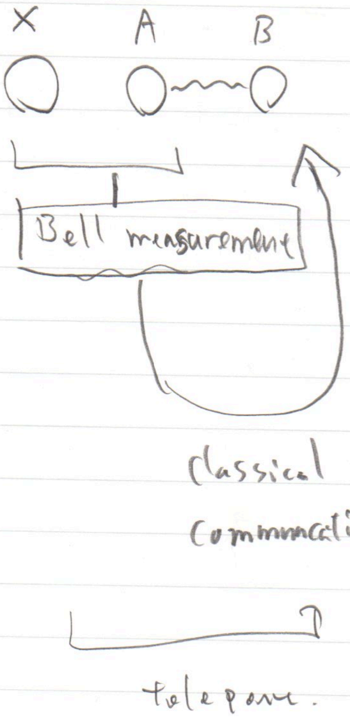
LOCC

LOCC = Local operation and Classical Communication

measurements inside A or B



Ex. Quantum teleportation



$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = |\phi^+\rangle_{AB}$$

$$(a|0\rangle_X + b|1\rangle_X) \otimes |\phi^+\rangle_{AB}$$

$$= \frac{1}{2}\Big[\, |\phi^+\rangle_{XA} \otimes (a|0\rangle_B + b|1\rangle_B)$$

$$+ |\phi^-\rangle_{XA} \otimes (a|0\rangle_B - b|1\rangle_B)$$

$$+ |\psi^+\rangle_{XA} \otimes (a|1\rangle_B + b|0\rangle_B)$$

$$+ |\psi^-\rangle_{XA} \otimes (a|1\rangle_B - b|0\rangle_B) \,\Big]$$

$$\downarrow U_4$$

$$a|0\rangle_B + b|1\rangle_B$$

1 Qubit entanglement

→ 1 Qubit teleport !

Definition of EE

$|\psi\rangle_{AB}$ → a given entangled state.

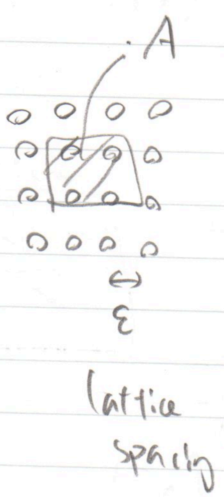
$$\lim_{n\to\infty}\left[\left(|\psi\rangle_{AB}\right)^{\otimes n} \underset{\text{LOCC}}{\overset{\text{LOCC}}{\rightleftarrows}} \left(|\phi^+\rangle\right)^{\otimes n S_A}\right]$$

$|\phi^+\rangle$: Bell state

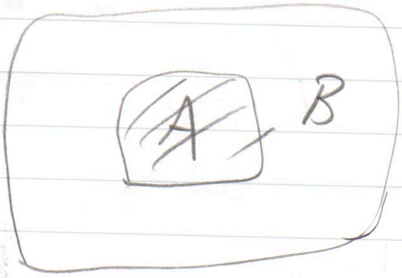
reversible! ⇓ This is not so for Mixed states!

S_A: Definition of EE!

(1-5) EE in QFT



$\xrightarrow{\varepsilon\to 0}$



$$H_{tot} = H_A \otimes H_B$$
$$\downarrow$$
$$S_A$$

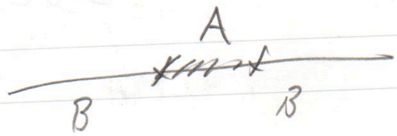
② Entanglement Entropy in CFTs

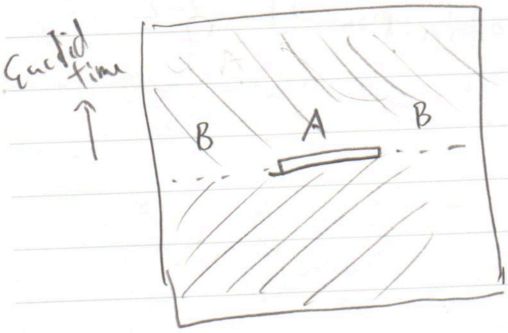
(2-1) Replica Method

$$S_A^{(n)} = \frac{1}{1-n}\log \boxed{\mathrm{tr}(\rho_A)^n}$$

compute

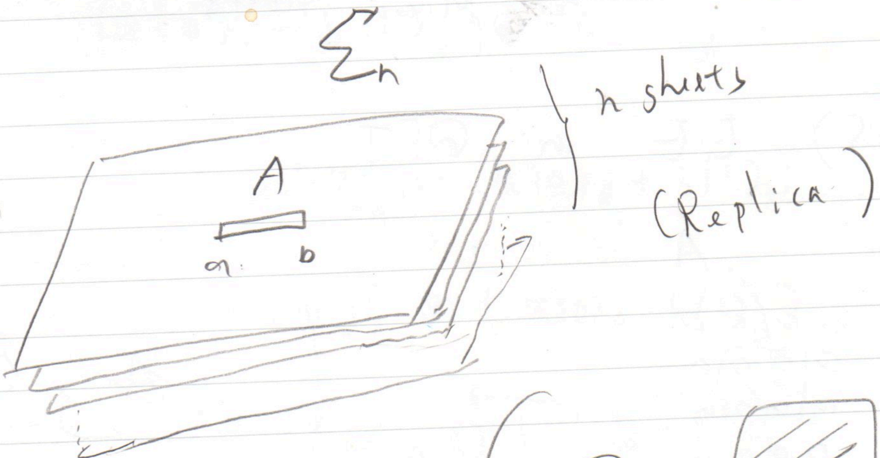
Consider 2D QFT as an example.





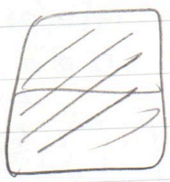
$$= \mathrm{tr}_B |\Psi\rangle\langle\Psi| = \rho_A$$

$\mathrm{tr}(\rho_A^n) =$



$$= \frac{Z_n}{(Z_1)^n}$$

$Z_1 =$



$\mathrm{tr}\rho_A = 1$

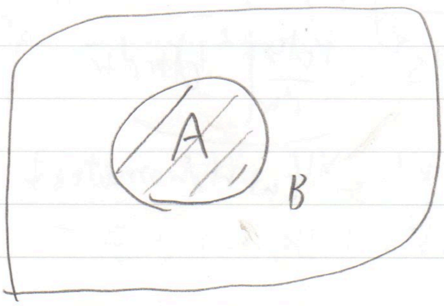
Twist operator

$$= \langle \sigma_n(a)\bar{\sigma}_n(b)\rangle_{CFT^{\otimes n}/\mathbb{Z}_n}$$

(2-2) Area law

Consider $(d+1)$ dim QFT with a UV fixed pt.

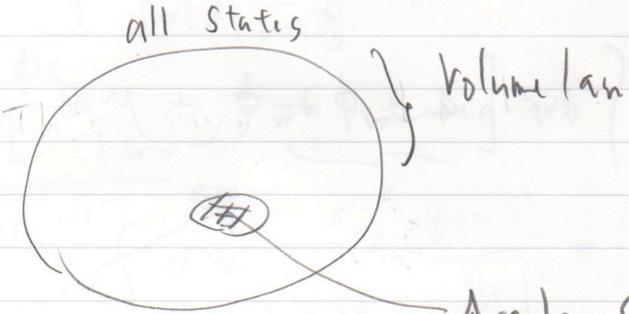
(S_A is UV div.)



$H_{tot} = H_A \otimes H_B$

$$\boxed{S_A = \underbrace{(\text{const.})}_{\substack{\text{depends on QFT} \\ \propto \ \#\text{ of fields}}} \times \frac{\text{Area}(\partial A)}{\varepsilon^{d-1}} + \underbrace{\cdots}_{\substack{O(\varepsilon^{d-3}) \\ \text{if } A \text{ is smooth}}}}$$

Note1



Arealaw $\leftrightarrow$ ground state of Local Hamiltonian

Note2

For 2D CFT, we have $S_A \sim \log|\partial A|$

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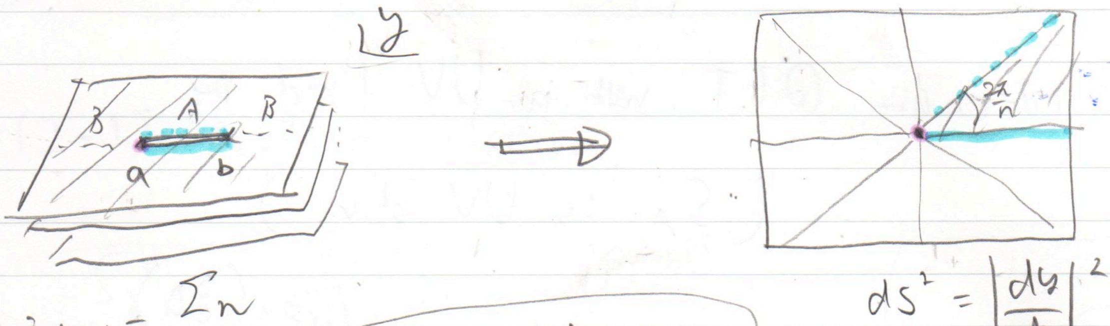
$$\frac{dy}{dw} = -n w^{n-1} \times \frac{b-a}{(w^n-1)^2}$$

(2-3) EE in 2D CFT



$ds^2 = dy d\bar{y}$

$$\boxed{w^n = \frac{y-a}{y-b}}$$

$$ds^2 = \underbrace{\left|\frac{dy}{dw}\right|^2 dw d\bar{w}}_{\text{curved metric}}$$

$$Z_{2D\,CFT}(\Sigma) = e^{I_L(\phi)}$$

$$ds^2_\Sigma = e^{2\phi} dw d\bar{w}$$

Weyl anomaly

UV divergence

$$I_L(\phi) = \frac{c}{24\pi}\int dw^2 \left[\underbrace{4\partial_w\phi\,\partial_{\bar{w}}\phi}_{(\partial_\mu\phi)^2} + \mu e^{2\phi}\right]$$

Weyl anomaly : $T^\mu{}_\mu = \frac{c}{24\pi}R = -\frac{c}{12\pi}\partial^\mu\partial_\mu\phi$

$$\overset{\shortparallel}{\frac{\delta I_L}{\delta\phi}}$$

In our setup

$$e^{2\phi} = \left|\frac{dy}{dw}\right|^2 = n^2(b-a)^2 \times \frac{|w|^{2(n-1)}}{|w^n-1|^4}$$

$$\Rightarrow \partial_w \phi = \frac{(1-n)-(1+n)w^n}{2(w^n-1)w}$$

$$I_L^{(n)}(\phi) = \frac{c}{24\pi}\int dw d\bar{w}\left[\frac{|(1-n)-(1+n)w^n|^2}{|w^n-1|^2|w|^2} + \underbrace{\mu\frac{n^2(b-a)^2|w|^{2(n-1)}}{|w^n-1|^4}}_{\text{Not important}}\right]$$

$$S_A^{(n)} = \frac{1}{1-n}\left[I_L^{(n)}(\phi) - n\cdot I_L^{(1)}(\phi)\right]$$

We focus on the UV divergences $\rightarrow$ $\begin{cases} \text{(i) } y\to a & |w| > \left(\frac{\varepsilon}{b-a}\right)^{\frac{1}{n}} \\ \text{(ii) } y\to b & |w| < \left(\frac{b-a}{\varepsilon}\right)^{\frac{1}{n}} \\ \text{(iii) } y\to\infty & |w^n-1| > \left(\frac{b-a}{y_\infty}\right) \end{cases}$

ε : cut off in $(y,\bar{y})$

(i) $y\to a$ $(w\to 0)$

$$I_L^{(n)} \simeq \int_{|w|>\left(\frac{\varepsilon}{b-a}\right)^{\frac{1}{n}}} dw^2 \frac{(1-n)^2}{|w|^2}\times\frac{c}{24\pi} \simeq \frac{c}{12}\times\frac{(1-n)^2}{n}\times\log\left(\frac{b-a}{\varepsilon}\right)$$

(ii) $y\to b$ $(w\to\infty)$

$$I_L^{(n)} \simeq \int_{|w|<\left(\frac{b-a}{\varepsilon}\right)^{\frac{1}{n}}} dw^2 \frac{(1+n)^2}{|w|^2}\times\frac{c}{24\pi} \simeq \frac{c}{12}\times\frac{(1+n)^2}{n}\log\left(\frac{b-a}{\varepsilon}\right)$$

(iii) $y\to\infty$ $(w\to 1)$

$$I_L^{(n)} \simeq \int_{|w^n-1|>\left(\frac{b-a}{y_\infty}\right)} dw^2 \left[\frac{4n^2}{|w^n-1|^2} + \mu\frac{n^2(b-a)^2}{|w^n-1|^4}\right]$$

$z\equiv w^n$

$$= \int_{|z|>\frac{b-a}{y_\infty}} dz d\bar{z}\left[\frac{4}{|z-1|^2} + \mu\frac{(b-a)^2}{|z-1|^4}\right]\times n$$

n sheets

Does Not contribute to EE

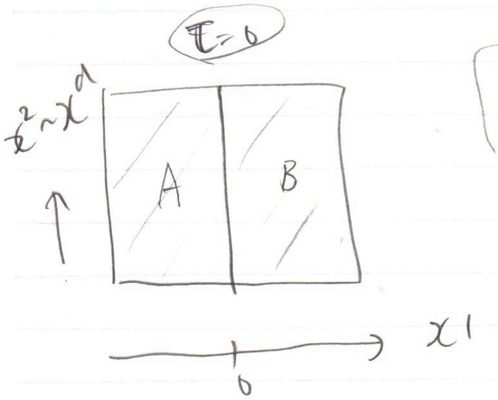
$$\Rightarrow I_L^{(n)}(\phi) - n I_L^{(1)}(\phi) \simeq \frac{c}{6}\times\frac{(1-n^2)}{n}\times\log\left(\frac{b-a}{\varepsilon}\right)$$

$$\boxed{S_A^{(n)} = \frac{c}{6}\left(1+\frac{1}{n}\right)\log\left(\frac{b-a}{\varepsilon}\right)}$$

(2-4) EE in Free QFTs.

Euclidean time

Consider Massive Free Scalar QFT on $\mathbb{R}^{d+1} \ni (\tau, x^1, \cdots, x^d)$



$$\boxed{A : x_1 \leq 0 , \qquad B : x_1 \geq 0}$$

$$\begin{cases} \int d\tau \, dx^1 \cdots dx^d = V_{d+1} \\ \int dx^2 \cdots dx^d = V_{d-1} \end{cases}$$

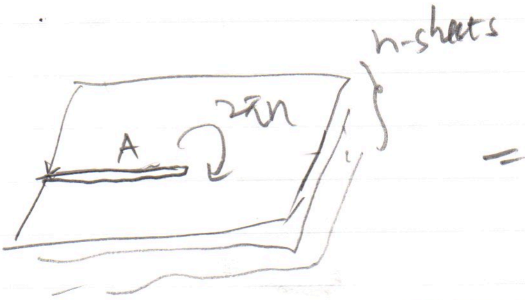
$$\log Z_{\mathbb{R}^{d+1}} = -\frac{1}{2} \times \frac{V_{d+1}}{(2\pi)^{d+1}} \int_{|k| < 1/\varepsilon} (dk)^{d+1} \log (k^2 + m^2)$$

$$= \frac{V_{d+1}}{(2\pi)^{d+1}} \times \int_{\varepsilon^2}^{\infty} \frac{ds}{2s} \int (dk)^{d+1} e^{-s(k^2+m^2)}$$

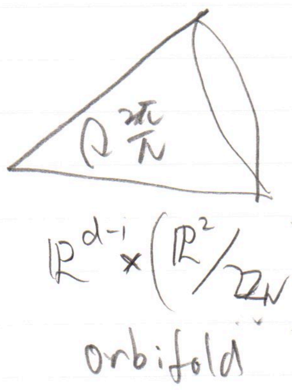
$$= V_{d+1} \int_{\varepsilon^2}^{\infty} \frac{ds}{2s} (4\pi s)^{-\frac{d+1}{2}} e^{-sm^2}$$

Now,

$\mathrm{Tr}(\rho_A)^n =$



$=$



$\mathbb{R}^{d-1} \times (\mathbb{R}^2 / \mathbb{Z}_N)$ orbifold

Assume this analytical continuation $\boxed{n = \frac{1}{N}}$

$z \equiv \tau + i x_1 \quad g \in \mathbb{Z}_N : z \to e^{\frac{2\pi i}{N}} z$

$\boxed{gx \sim x} \Rightarrow \mathbb{Z}_N$ orbifold $\mathbb{R}^2/\mathbb{Z}_N$

Projection to $g = 1$

$$\log Z_{\mathbb{R}^2/\mathbb{Z}_N \times \mathbb{R}^{d-1}} = V_{d-1} \times \int_{\varepsilon^2}^{\infty} \frac{ds}{2s} (4\pi s)^{-\frac{d-1}{2}} e^{-sm^2} \sum_{j=0}^{N} \mathrm{Tr}\left[\frac{g^j}{N}\right]$$

$j \neq 0$

$$\mathrm{Tr}[g^j] = \int dk_0 dk_1 \langle k | g^j | k \rangle$$
$$= \int dk_0 dk_1 \, \delta(k - e^{\frac{2\pi i}{N}} k)\, \delta(\bar{k} - e^{-\frac{2\pi i}{N}} \bar{k})$$
$$= \frac{1}{4\sin^2\left(\frac{\pi j}{N}\right)}$$

$$\Rightarrow \sum_{j=1}^{N-1} \mathrm{Tr}\left[\frac{g^j}{N}\right] = \frac{1}{4N} \sum_{j=1}^{N-1} \frac{1}{\sin^2\left(\frac{\pi j}{N}\right)} = \frac{1}{4N} \cdot \left(\frac{N^2-1}{3}\right)$$

correspond to $j=0$

$$S_A^{(n)} = \frac{1}{1-\frac{1}{N}} \left(\log Z_{\mathbb{R}^2/\mathbb{Z}_N \times \mathbb{R}^{d-1}} - \frac{1}{N} \log Z_{\mathbb{R}^{d+1}} \right)$$

$n = \frac{1}{N}$

$$= \frac{1}{12}\left(1 + \frac{1}{n}\right) V_{d-1} \times \int_{\varepsilon^2}^{\infty} \frac{ds}{2s} (4\pi s)^{-\frac{d-1}{2}} e^{-sm^2}$$

$n=1$

$$S_A = \begin{cases} d \geq 2 & \underbrace{\frac{V_{d-1}}{6(d-1)(4\pi)^{\frac{d-1}{2}}} \times \frac{1}{\varepsilon^{d-1}}} + \mathcal{O}(\varepsilon^{-(d-3)}) \\ d = 1 & \frac{1}{6} \log\left(\frac{1}{m\varepsilon}\right) \end{cases}$$

Area law

$\frac{c}{6}$

③ Holographic Entanglement Entropy

(3-1) AdS/CFT (Maldacena 1997) Large N, Strongly coupled

$$\text{Gravity on } \underbrace{(d+2)\text{ dim AdS}}_{M} = \underbrace{(d+1)\text{ dim. CFT}}_{\partial M}$$

↑ String theory

$\boxed{AdS_{d+2}}$: A hypersurface: $(X_0)^2 + (X_{d+2})^2 = (X_1)^2 + (X_2)^2 + \cdots + (X_{d+1})^2 + 1$

in $\mathbb{R}^{2,d}$ $ds^2 = -(dX_0)^2 - (dX_{d+2})^2 + (dX_1)^2 + \cdots + (dX_{d+1})^2$

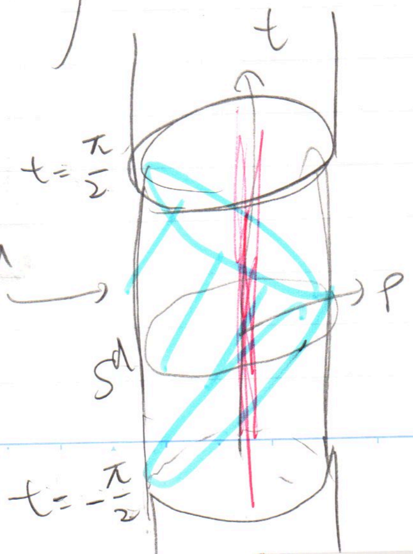
SO(2, d) sym.

$$\left.\begin{aligned} X_0 &= R\cosh\rho\cos t \\ X_{d+2} &= R\cosh\rho\sin t \\ X_a &= R\sinh\rho\,\Omega_a \\ a &= 1 \sim d+1 \end{aligned}\right\}$$

Global AdS

$$ds^2 = R^2\left(-\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, \underbrace{d\Omega_d^2}_{S^d}\right)$$

$\partial(\text{g}AdS_{d+2}) = \underbrace{\mathbb{R}}_{\text{time}} \times S^d$

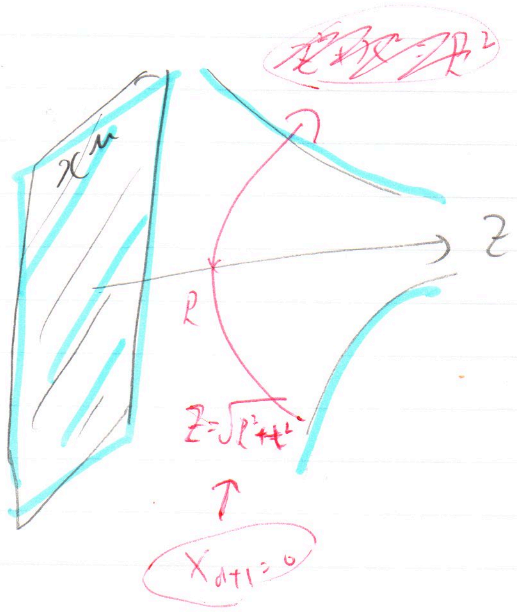


$$X_0 + X_{d+1} = z + \frac{x^\mu x_\mu}{z}$$
$$X_0 - X_{d+1} = \frac{R^2}{z}$$
$$X_{d+2} = \frac{R x_0}{z}$$
$$(i = 1 \sim d) \quad X_i = \frac{R x_i}{z}$$

→ **Poincare AdS**

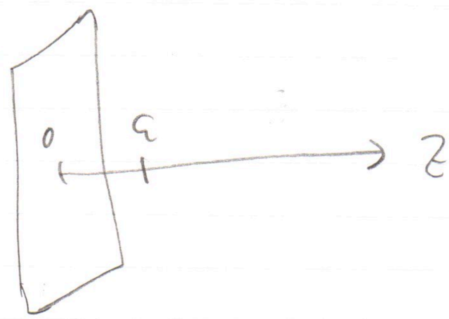
$$ds^2 = \frac{R^2}{z^2}\left(dx^\mu dx_\mu + dz^2\right)$$

$$\partial(\text{P}\,AdS_{d+2}) = \mathbb{R}^{1,d}$$



UV cut off

The metric gets divergent as $z \to 0$!



$z \to \varepsilon \leftrightarrow$ CFT UV cut off

$\Lambda = \frac{1}{\varepsilon}$.

Bulk to Boundary Relation (GKPW)

$$Z_{\text{gravity}} = Z_{\text{CFT}}$$

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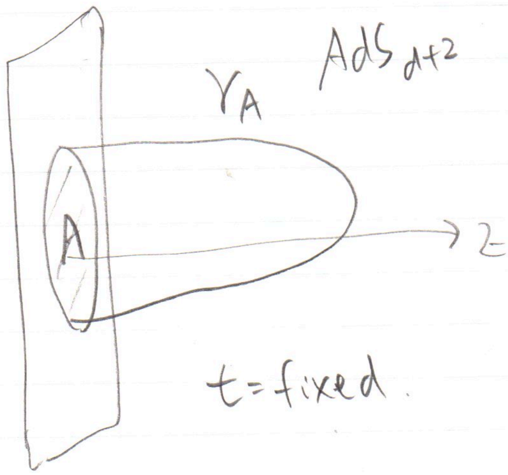
(3-2) Holographic Entanglement Entropy (HEE)

RT 2006

(a) HEE in Static b.g.



$$S_A^{RT} = \underset{\gamma_A}{\mathrm{Min}} \left[\frac{\mathrm{Area}(\gamma_A)}{4G_N} \right]$$

$\partial\gamma_A = \partial A$

$\gamma_A \sim A$

homologous.

Note 1 Area law

$$S_A \sim \frac{R^d}{4G_N} \times A(\partial A) \times \int_\varepsilon \frac{dz}{z^d} \sim \frac{1}{\varepsilon^{d-1}} \Rightarrow \mathrm{Area}$$

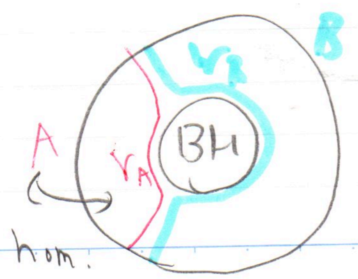
$\sim$ # fields (N^2)

central charge.

Note 2 Mixed vs Pure

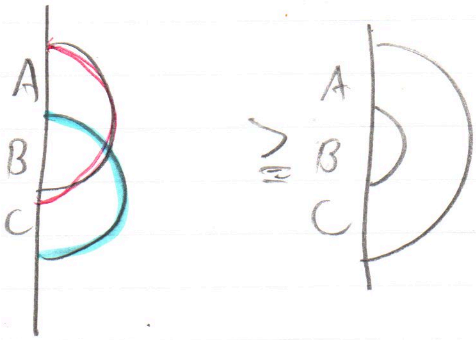
At finite temparature,

$$\lim_{A\to 0} |S_B - S_A| = S_{BH}$$

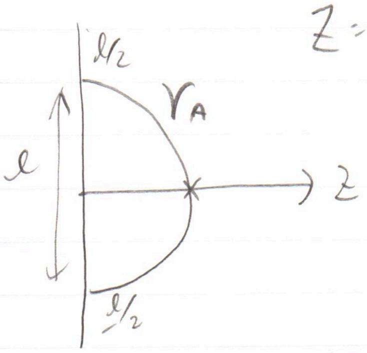


$\rightarrow$ $S_A \neq S_B$

Note 3　SSA



Note 4　2D CFT



$$z = \sqrt{\frac{l^2}{4} - x^2}$$

$$ds^2 = \frac{R^2}{z^2}\left(dz^2 + dx^2 + dt^2\right)$$

$$= R^2 \times \frac{l^2}{z^2\left(l^2 - 4z^2\right)} dz^2$$

$$L(\gamma_A) = 2R \int_{\varepsilon}^{l/2} \frac{dz}{z} \times \frac{l}{\sqrt{l^2 - 4z^2}} = 2R \log\left(\frac{l}{\varepsilon}\right)$$

$$\rightarrow \quad S_A = \frac{L}{4G_N} = \frac{c}{3} \log\left(\frac{l}{\varepsilon}\right) \qquad \left[\frac{3R}{2G_N} = c\right]$$

(b) HEE in Time-dep big　HRT 2007

$$S_A^{HRT} = \underset{\gamma_A^{sol}}{\mathrm{Min}}\left[\underset{\substack{\gamma_A \\ \partial\gamma_A = \partial A \\ \gamma_A \sim A}}{\mathrm{Ext}}\left[\frac{\mathrm{Area}(\gamma_A)}{4G_N}\right]\right]$$

extremal surface

(c) Derivation from Bulk to Bdy relation

[Lewkowycz - Maldacena 2013]

$$Z_{\text{gravity}} = e^{-I_G}$$

$$I_G = -\frac{1}{16\pi G_N}\int \sqrt{g}\,(R - 2\Lambda) + \cdots$$

$Z_n = \mathrm{tr}(\rho_A)^n \rightarrow$  $\rightarrow R = 4\pi(1-n)\,\delta(\gamma_A) + \cdots$

$$\log Z^{(n)}_{\text{gravity}} \sim \frac{4\pi(1-n)}{16\pi G_N}\int_{\gamma_A}\sqrt{g}$$

$$S_A = -\frac{\partial}{\partial n}\log\frac{Z^{(n)}}{(Z^{(1)})^n} = \underbrace{\frac{1}{4G_N}\int_{\gamma_A}\sqrt{g}}$$ //

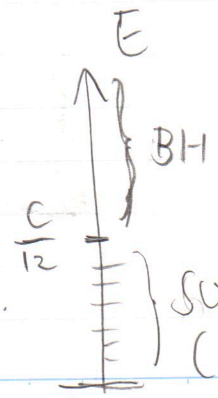
Einstein eq → extremalization of Area

(3-3) Holographic CFTs and HEE [Heemskerk 2010, Hartman... 2013]

Classical Gravity in AdS ⟺ Strongly coupled Large N CFTs

⇓

Called Holographic C

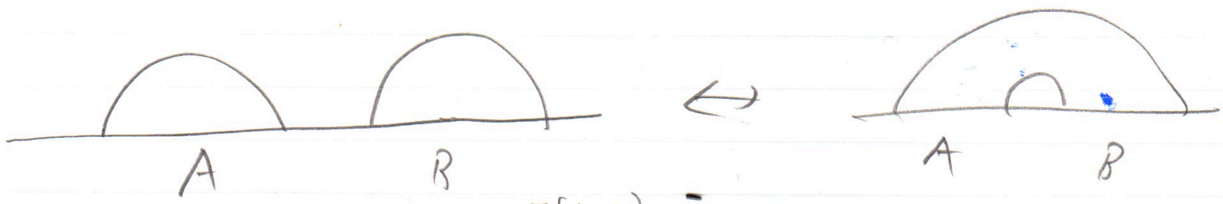


$L_0 - \frac{c}{24}$

e.g. 2D CFT

→ large c and sparse spectrum

Ex. Phase transition in HEE $\quad I(A:B) = S_A + S_B - S_{AB} > 0$



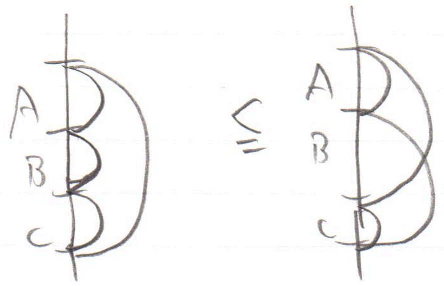
$I(A:B) = 0$

$\frac{d_{AB}}{d_A} > r_c \qquad\qquad \frac{d_{AB}}{d_A} < r_c$

Ex. Monogamy of MI [Hayden-Headrick-Maloney]

$$I(A,BC) \geq I(A,B) + I(A,C)$$

$$\Leftrightarrow \quad S_A + S_B + S_C - S_{AB} - S_{BC} - S_{CA} + S_{ABC} \leq 0$$



This is not true in general!

e.g. $|\psi\rangle_{ABCD} = (|0000\rangle + |1111\rangle)/\sqrt{2}$

$\Downarrow$

$\log 2 = S_A = S_B = S_C = S_{AB} = S_{BC} = S_{CA} = S_{ABC}$

$\to \quad (\qquad) = \log 2 > 0$!

(3-4) Corrections to HEE

We assumed Einstein Gravity Approximation.

(a) Stringy correction $\quad \frac{R_{AdS}^2}{\alpha'} \sim \underbrace{g^{\#}}_{\text{coupling constant in G}} \gg 1$

Fursner 2006, ...

$$I_G = -\frac{1}{16\pi G_N}\int\sqrt{g}\left[R-2\Lambda+\lambda\cdot\left(R_{abcd}R^{abcd}-4R_{ab}R^{ab}+R^2\right)\right]$$

$$\to \quad S_A = \frac{1}{4G_N}\,\underset{\gamma_A}{\mathrm{Min}}\left[\int_{\gamma_A}(dx)^d\sqrt{g}\left(1+2\lambda R_{\gamma_A}\right)\right]$$

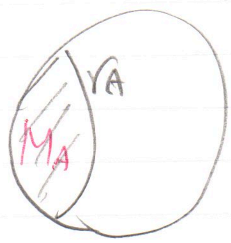
(b) Quantum Gravity Effects. $\quad \frac{R_{AdS}}{l_{pl}} \sim c^{\#} \gg 1$

FLM 2013

loop integrals in SUGRA

$$S_A = \underbrace{\frac{A(\gamma_A)}{4G_N}}_{O\left(\frac{1}{G_N}\right)} + \underbrace{O(1)}_{\text{We focus on this}} + O(G_N) + \cdots$$

$$S_A^{FLM} = \frac{A(\gamma_A)}{4G_N} + \underbrace{S_{M_A}}_{\text{bulk EE}} + \cdots$$



Jafferis - Lewkowycz - Maldacena - Suh.

(3-5) Entanglement wedge

JLMS 2015, ...

$(\mathrm{tr}\,\rho_A = 1)$

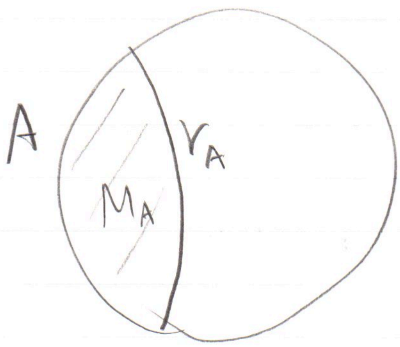
$$S_A^{CFT} = -\mathrm{tr}\left[\rho_A^{CFT} \log \rho_A^{CFT}\right] \qquad \rho_A^{CFT} = e^{-\underbrace{H_A^{CFT}}_{\text{Modular Hamiltonian}}}$$

$$= \mathrm{tr}\left[\rho_A^{CFT} H_A^{CFT}\right]$$

$$S_{M_A}^{bulk} = \mathrm{tr}\left[\rho_{M_A}^{bulk} H_{M_A}^{bulk}\right]$$

$$\mathrm{tr}\left(e^{-\frac{\hat{A}}{4G_N}} \frac{\hat{A}}{4G_N}\right) = \frac{A}{4G_N}$$

$$\Rightarrow \quad H_A^{CFT} = \underbrace{\frac{\hat{A}}{4G_N}}_{\text{edge mode}} + H_{M_A}^{bulk}$$

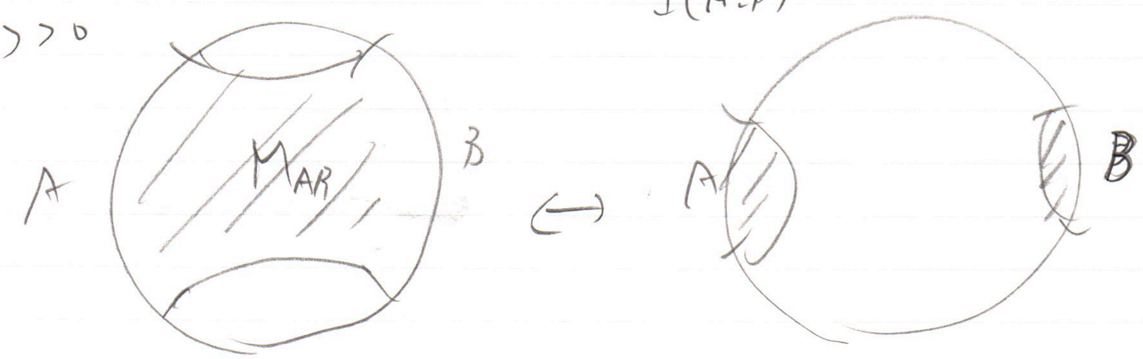


Dual

$$\rho_A^{CFT} \leftrightarrow \underbrace{\rho_{M_A}^{bulk}}_{\text{entanglement wedge}}$$

e.g. two disconnected system AB

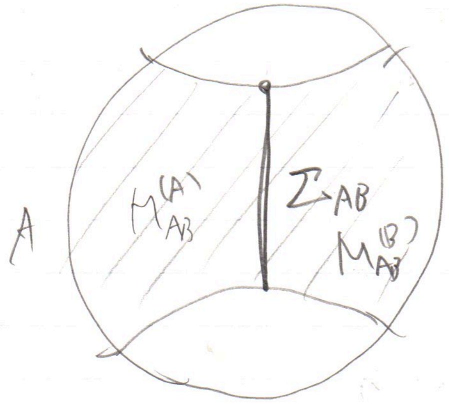
$I(A:B) > 0$ $\qquad$ $I(A:B) = 0$



$d_{AB} \to$ small $\qquad$ $d_{AB} \to$ large

(3-6) Holographic Entanglement of Purification

Umemoto - T 2017.



Σ_{AB} : Minimal surface which separates EW M_{AB} into A side and B side

$M_{AB}^{(A)} \cup M_{AB}^{(B)} = M_{AB}$

called

Entanglement Wedge Cross Section

Conjecture

$$\underbrace{E_p(\rho_{AB})}_{} = \frac{\mathrm{Area}(\Sigma_{AB})}{4G_N}$$

Entanglement of purification (EoP)

Def of EoP

$\mathcal{H}_A \otimes \mathcal{H}_B \to \mathcal{H}_A \otimes \mathcal{H}_B \otimes \underbrace{\mathcal{H}_{\tilde{A}} \otimes \mathcal{H}_{\tilde{B}}}_{\text{extra space}}$

$$E_p(\rho_{AB}) = \mathop{\mathrm{Min}}_{|\psi\rangle_{A\tilde{A}B\tilde{B}}} \left[S(\rho_{A\tilde{A}}) \right]$$

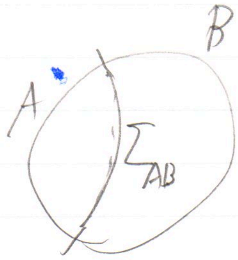
Purification

$$\rho_{AB} = \mathrm{Tr}_{\tilde{A}\tilde{B}} |\psi\rangle\langle\psi|$$

Properties

(i) $\rho_{AB} = |\tilde{\Psi}\rangle\langle\tilde{\Psi}|$: pure

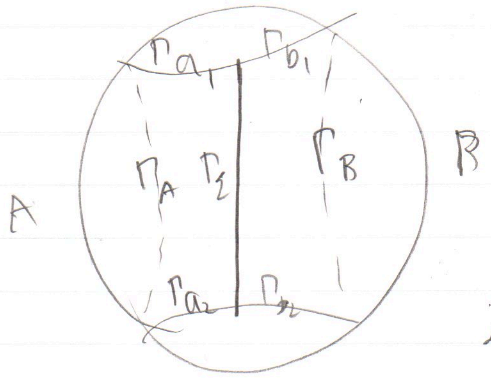
$\rightarrow \quad E_p(\rho_{AB}) = S_A = S_B$



(ii) $E_p(\rho_{AB}) \geqq \frac{1}{2} I(A:B)$

$\qquad\qquad\qquad\qquad \shortparallel$

$\qquad\qquad\qquad S_A + S_B - S_{AB}$



$$\begin{cases} |\Gamma_A| \leqq |\Gamma_{a_1}| + |\Gamma_\Sigma| + |\Gamma_{a_2}| \\ |\Gamma_B| \leqq |\Gamma_{b_1}| + |\Gamma_\Sigma| + |\Gamma_{b_2}| \end{cases}$$

$\Downarrow \quad \Gamma_{AB} = \Gamma_{a_1} + \Gamma_{a_2} + \Gamma_{b_1} + \Gamma_{b_2}$

$$|\Gamma_\Sigma| \geqq \frac{1}{2}\left(|\Gamma_A| + |\Gamma_B| - |\Gamma_{AB}|\right)$$

$$\shortparallel$$

$$\frac{1}{2}(S_A + S_B - S_{AB})$$

$$\shortparallel$$

$$I(A,B)$$

"Derivation"

(i) Tensor Network

(ii) CFT calculation

using path-int. optimization

Caputa - Miyaji - Umemoto - T.

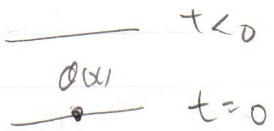
2018

④ EE under Local Quenches

(4-1) 3 versions of Local Quenches

Local Quenches ↑ Local excitations

(i) Local Opearator Quench



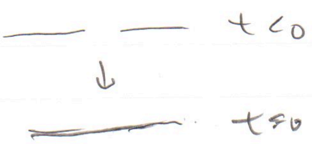
$$|\Psi\rangle = \underbrace{e^{-\alpha H}}_{\text{UV cut off}} \underbrace{O(x)}_{\text{local op}} |0\rangle$$

CFT Nozaki-Numasawa 1401.0539

Hol NNT 1302.5703

Asplund et al. → 1506.03772

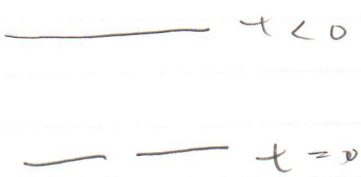
(ii) Local Joinning Quench



CFT Calabrese-Cardy 0708.3750

Hol { Ugajin 1311.2562
Shimaji-Wei-T 1812.01176

(iii) Local Splitting Quench

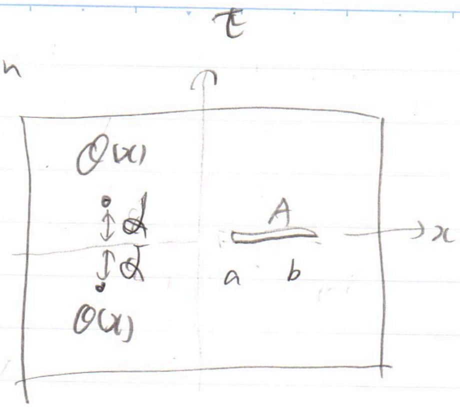


CFT, Hol } → SWT

(4-2) CFT Description

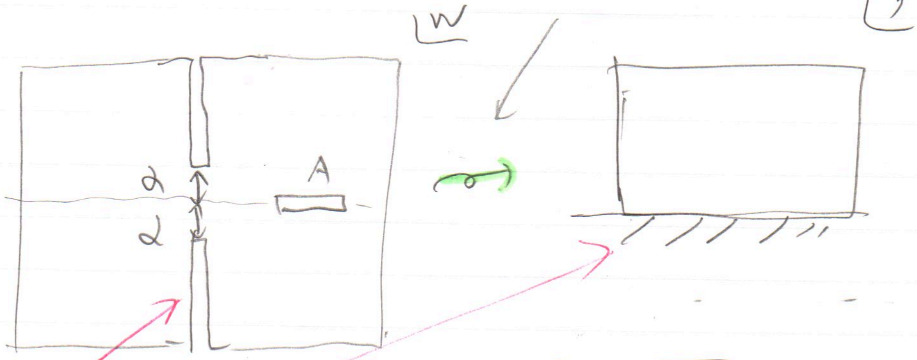
$w = i\tau + x$

(i) → $\rho_A =$



$\zeta = i\sqrt{\frac{i\alpha - w}{i\alpha + w}}$

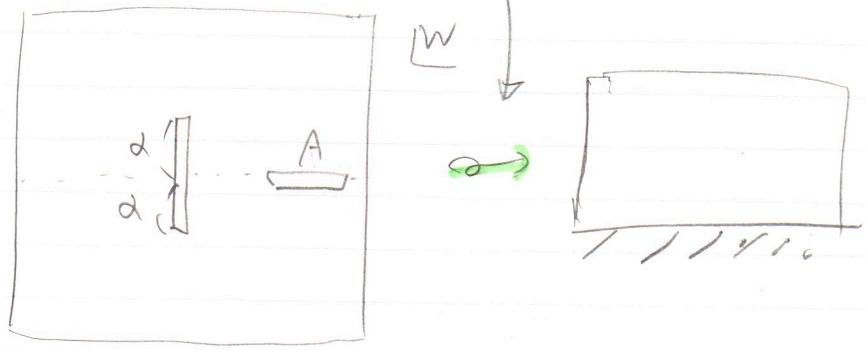
(ii) → $\rho_A =$



Conformal Boundary
(BCFT)

$\zeta = i\sqrt{\frac{w+i\alpha}{w-i\alpha}}$

(iii) → $\rho_A =$



$$\mathrm{tr}\,\rho_A^n = \langle \sigma_n(a)\,\sigma_n(b)\rangle_{CFT^{\otimes n}/\mathbb{Z}_n}$$

↑

$$h_n = \bar{h}_n = \frac{c}{24}\left(n - \frac{1}{n}\right)$$

Wick rotation

$\tau \to it$

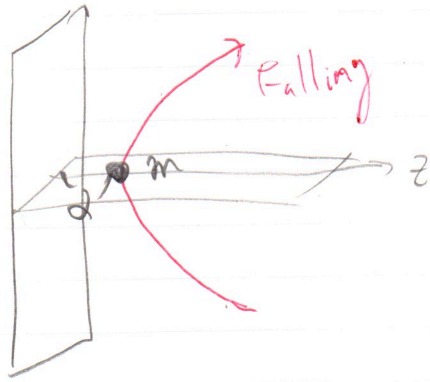
real time evolution

(i)~(iii) → the same EM tensor

(4-3) Holographic Description

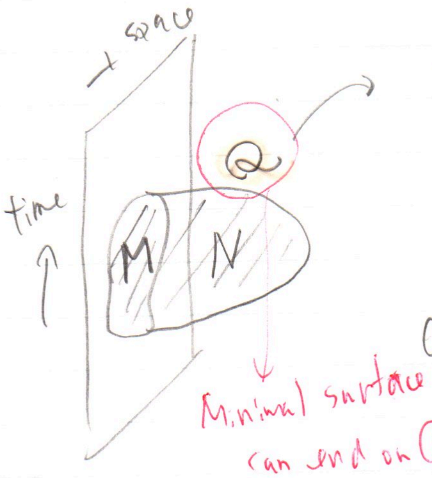
(ii) (iii) → the same metric (Shock wave geometry in AdS)

(i) →



$m\ell \simeq \Delta \gg 1$

(ii), (iii) use AdS / BCFT (T.T. 1105.5165)

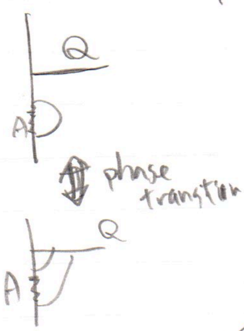


$K_{\mu\nu} - h_{\mu\nu} K = -T \cdot h_{\mu\nu}$

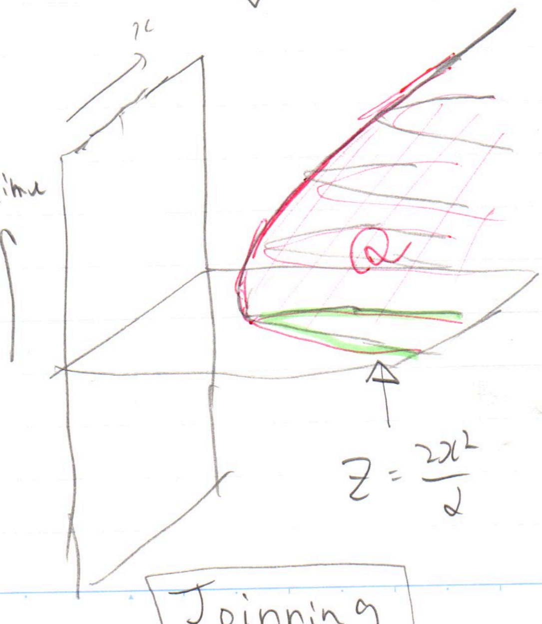
tension

CFT on M $\overset{\text{Dual}}{\Longleftrightarrow}$ Gravity on N

$(\partial M = \partial Q)$ $(\partial N = Q)$

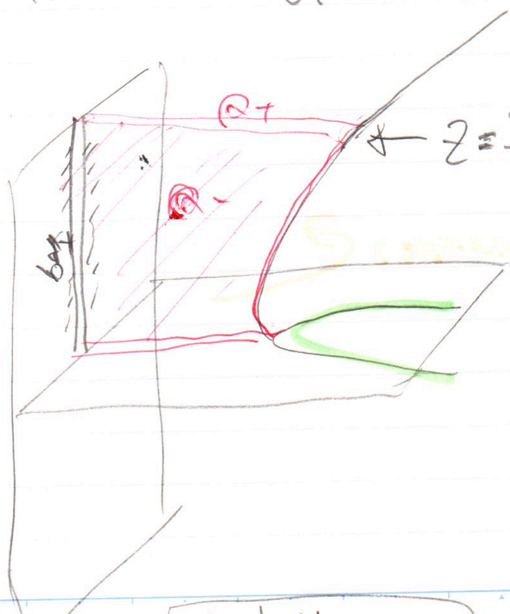


⇓ Apply to Hol local Quench



$z = \frac{2x^2}{\alpha}$

Joinning

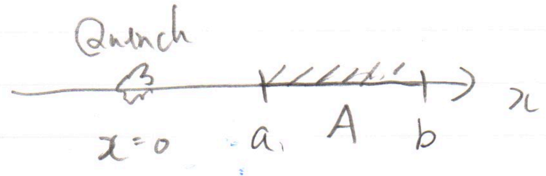


$z = \frac{2(\alpha^2+t^2)}{\sqrt{\alpha^2+4t^2}}$

s
t

Splitting

(4-4) Holographic EE



$\boxed{\text{Result}}$ $0 < a \ll t \ll b$ middle time zone

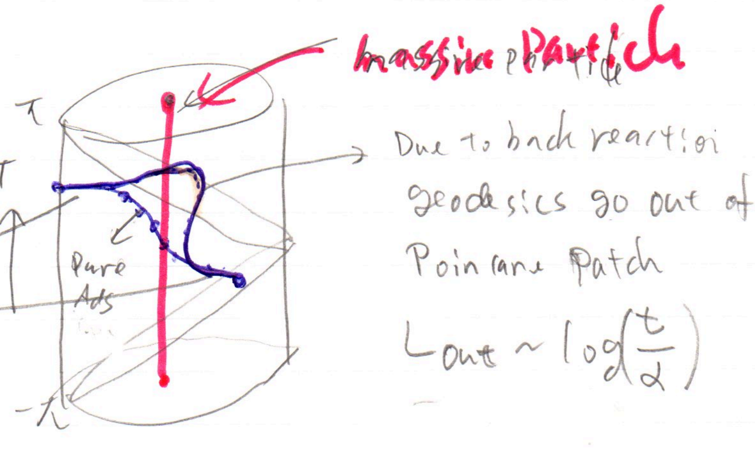
Op (i) $S_A = S_A^{con} \simeq \frac{c}{6}\log\frac{t}{\alpha} + \frac{c}{3}\log\frac{l}{\epsilon}$

Join (ii) $S_A = S_A^{dis} \simeq \frac{c}{6}\log\frac{t}{\epsilon} + \frac{c}{6}\log\frac{t}{\alpha} + \frac{c}{6}\log\frac{b}{\epsilon}$

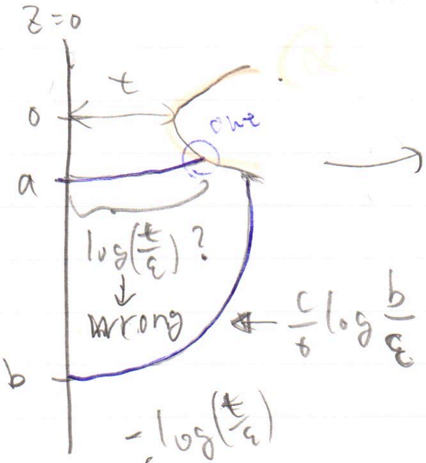
Sp (iii) $S_A = S_A^{dis} \simeq \frac{c}{6}\log\frac{a}{\epsilon} + \frac{c}{6}\log\frac{b^2}{\alpha\epsilon}$

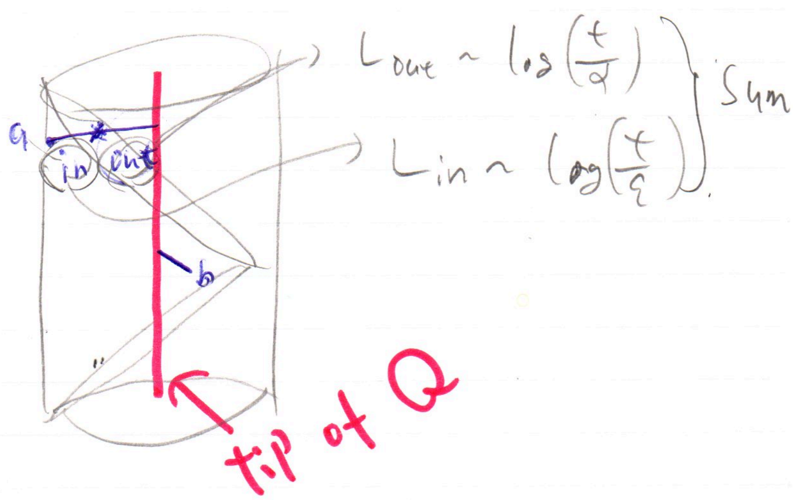
$\boxed{\text{Hol Explanations}}$

(i) Map into "global AdS_3"

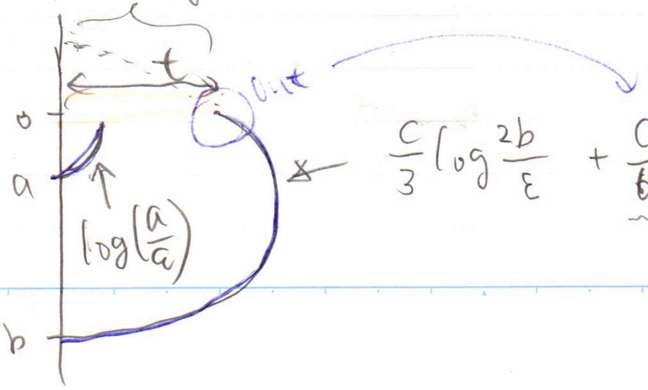


(ii)





(iii)



$\frac{c}{3}\log\frac{2b}{\epsilon} + \frac{c}{6}\log\frac{t}{\alpha} - \frac{c}{6}\log\frac{t}{\epsilon}$