Chimera generalisations: planar oscillators, local coupling, and random networks

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- Chimeras have been known since 2002.

- Early papers considered a ring of identical phase oscillators with nonlocal coupling, or a pair of all-to-all coupled subnetworks.

- Natural questions:

1. Do we need identical oscillators?

2. Do we need all-to-all coupling?

3. Do we need nonlocal coupling?

4. Do chimeras occur in networks of more general oscillators?

- I will show how some of these questions were answered.

Nonlocal coupling

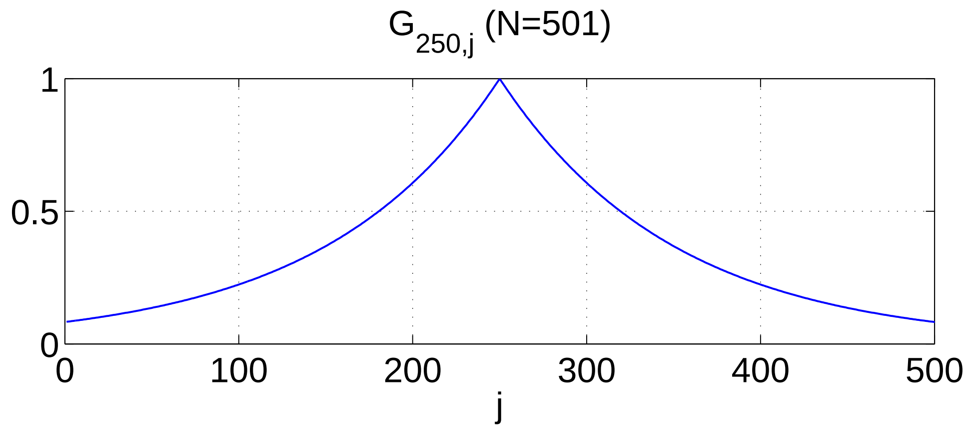
- Consider a network of phase oscillators on a ring with nonlocal coupling:

$$\frac{d\theta_i}{dt} = \omega - \frac{1}{N}\sum_{j=0}^{N} G_{ij}\sin\left(\theta_i - \theta_j + \alpha\right)$$

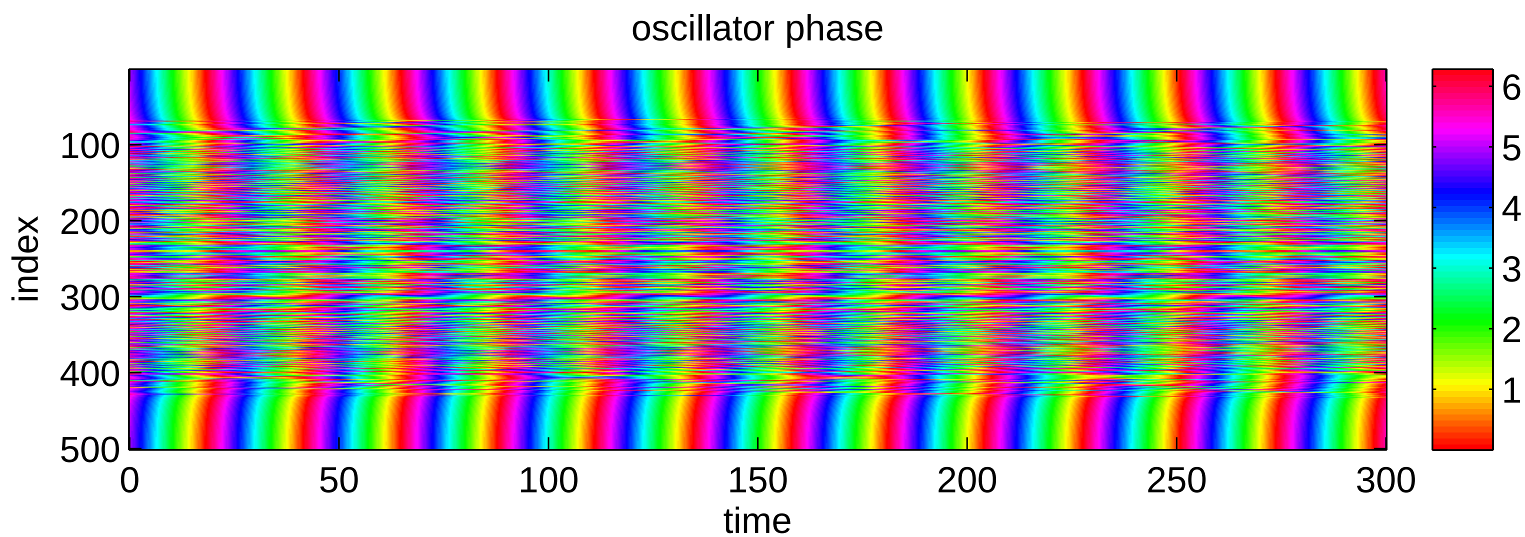
where

$$G_{ij} = e^{-5|i-j|/N}$$

and $|i-j|$ is shortest distance between oscillators i and j on ring.



For some initial conditions, and values of α, we get **chimeras**:



Even though oscillators are identical and coupled in highly-symmetric way, they split into synchronous and asynchronous groups.

Movie. Phases are shown in a coordinate frame rotating with synchronous oscillators.

- Thought to require nonlocal coupling.

- Is that true? Where does nonlocal coupling come from?

Consider general reaction-diffusion equation on 1D domain Ω with only local interactions via diffusion in one variable:

$$\frac{\partial u}{\partial t} = f(u) + v \quad (1)$$

$$\varepsilon \frac{\partial v}{\partial t} = g(u) - v + \frac{\partial^2 v}{\partial x^2} \quad (2)$$

ε small $\Rightarrow$ separation of timescales: u is “slow”, v is “fast.”

Taking limit of infinitely fast dynamics for v, i.e. setting $\varepsilon = 0$, gives

$$\left(1 - \frac{\partial^2}{\partial x^2}\right) v = g(u)$$

- If $h(x)$ is the Green's function associated with $\left(1 - \frac{\partial^2}{\partial x^2}\right)$ on Ω

$$v(x) = \int_\Omega h(x-y)g(u(y))\,dy$$

and substituting this into (1) we obtain nonlocal equation for u:

$$\frac{\partial u}{\partial t} = f(u) + \int_\Omega h(x-y)g(u(y))\,dy$$

[For $\Omega = \mathbb{R}$ and $\lim_{|x|\to\infty} h(x) = 0$, $h(x) = e^{-|x|}/2$.]

- We will do the network analogue of this, but not set $\varepsilon = 0$.

Model:

$$\frac{d\theta_j}{dt} = \omega_j - \text{Re}\left(z_j e^{-i\theta_j}\right)$$
$$\varepsilon \frac{dz_j}{dt} = A e^{i(\theta_j+\beta)} - z_j + \frac{z_{j+1} - 2z_j + z_{j-1}}{(\Delta x)^2}$$

for $j = 1, 2 \ldots N$, where $\Delta x = L/N$; A, β and ε are all constants.

• State of oscillator j described by two variables: $\theta_j \in [0, 2\pi)$ and $z_j \in \mathbb{C}$.

• ω_j randomly chosen from a Lorentzian distribution with half-width-at-half-maximum σ centred at ω_0, namely

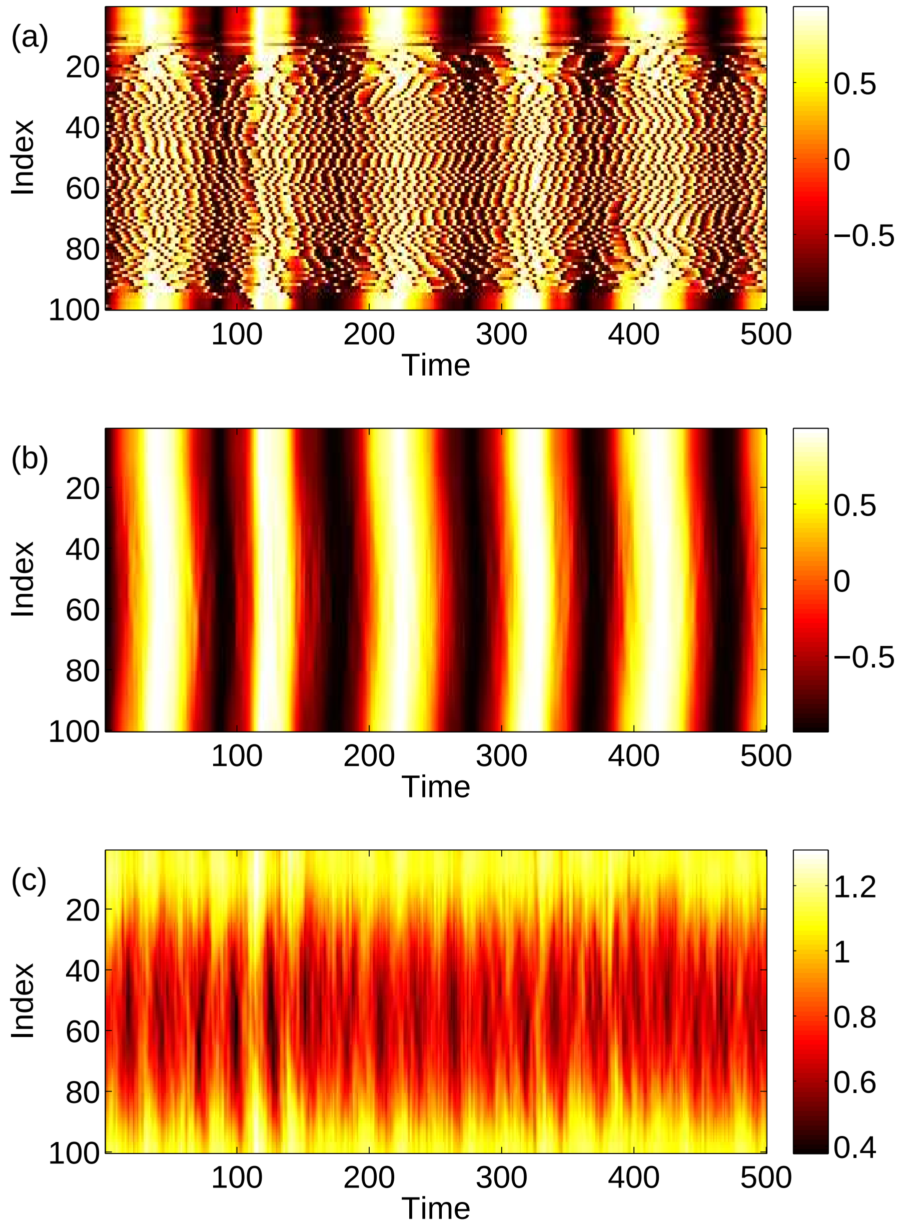
$$g(\omega) = \frac{\sigma/\pi}{(\omega - \omega_0)^2 + \sigma^2}$$

Does show chimera at $\varepsilon = \mathbf{0.2}$

(a): $\sin \theta_j$;

(b): $\sin(\arg(z_j))$;

(c): $|z_j|$.



Relationship to previous models:

• Set $\varepsilon = 0$. If z_j is the jth entry of the vector $\mathbf{z} \in \mathbb{C}^N$ and similarly for θ_j,

$$(I - D)\mathbf{z} = Ae^{i(\theta+\beta)}$$

where I is $N \times N$ identity matrix and D is matrix representation of classical second difference operator with periodic boundary conditions.

• Defining $G = (I - D)^{-1}$ we have

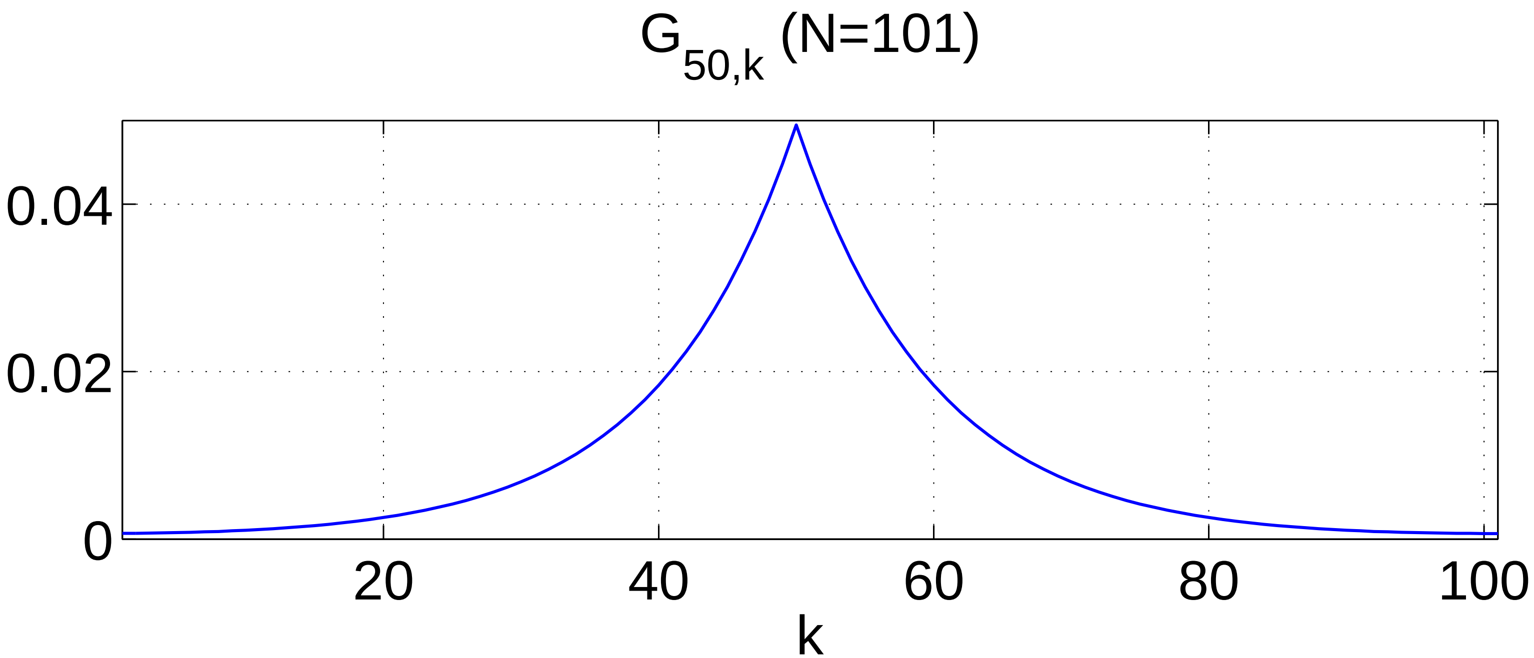
$$z_j = A \sum_{k=1}^{N} G_{jk} e^{i(\theta_k+\beta)}$$

and thus

$$\frac{d\theta_j}{dt} = \omega_j - A \sum_{k=1}^{N} G_{jk} \cos(\theta_j - \theta_k - \beta)$$

where

$$G_{jk} = \frac{1}{N} \sum_{r=0}^{N-1} \frac{\exp\left(-2\pi i r |j-k| / N\right)}{1 + \frac{2[1-\cos(2\pi r/N)]}{(\Delta x)^2}}.$$



Analysis for $\varepsilon \neq 0$ using self-consistency

• System is invariant under the simultaneous shift: $\theta_j \mapsto \theta_j + \gamma$ and $z_j \mapsto z_j e^{i\gamma}$ for all j.

• Move to a rotating coordinate frame: $\phi_j \equiv \theta_j - \Omega t$ and $y_j \equiv z_j e^{-i\Omega t}$, where Ω is to be determined.

• Take the limit $N \to \infty$:

$$\frac{\partial \phi}{\partial t} = \omega - \text{Re}\left(y e^{-i\phi}\right) - \Omega \tag{3}$$

$$\varepsilon \frac{\partial y}{\partial t} = A e^{i(\phi+\beta)} - y + \frac{\partial^2 y}{\partial x^2} - i\varepsilon\Omega y \tag{4}$$

• Search for solutions for which y is stationary, i.e. just a function of space. Let such a solution be $y(x) = R(x) e^{i\Theta(x)}$.

• To obtain a stationary solution of (4) replace $e^{i\phi}$ by its expected value, calculated using density of ϕ, inversely proportional to its velocity ($\partial\phi/\partial t$).

• Need to solve

$$0 = Ae^{i\beta} \int_{-\infty}^{\infty} \int_{0}^{2\pi} e^{i\phi} p(\phi|\omega) g(\omega) d\phi \, d\omega - y + \frac{\partial^2 y}{\partial x^2} - i\varepsilon\Omega y$$

where the density of ϕ given ω is

$$p(\phi|\omega) = \frac{\sqrt{(\omega - \Omega)^2 - R^2}}{2\pi|\omega - \Omega - R\cos(\Theta - \phi)|}$$

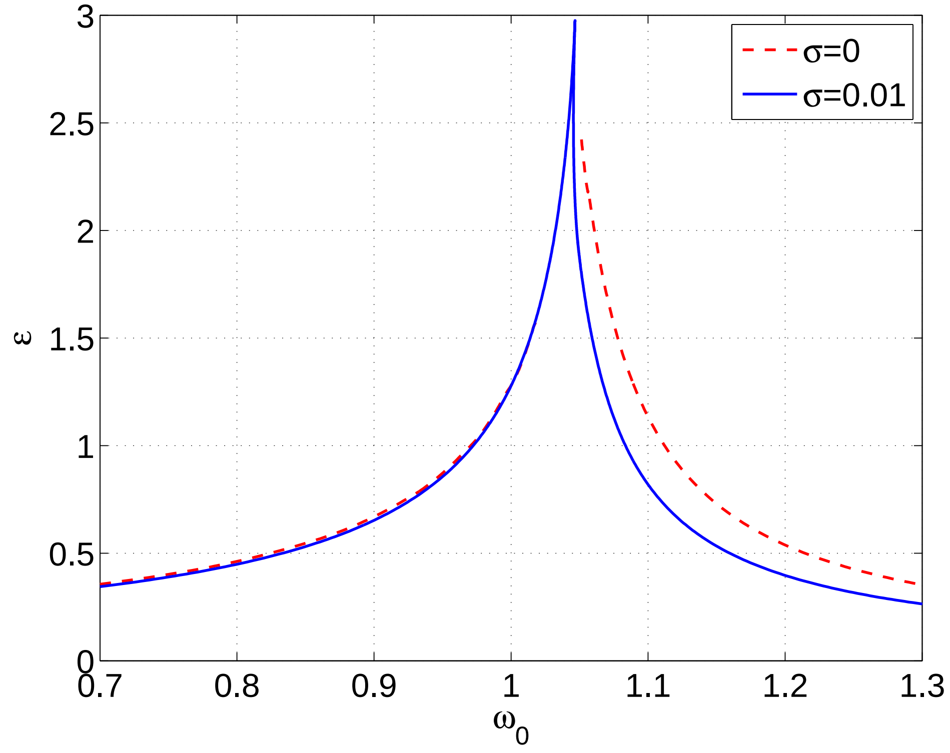
and $g(\omega)$ is the Lorentzian.

- Evaluating the integrals we obtain

$$\frac{Ae^{i(\Theta+\beta)}}{R}\left[\omega_0 + i\sigma - \Omega - \sqrt{(\omega_0 + i\sigma - \Omega)^2 - R^2}\right] - \left(1 + i\varepsilon\Omega - \frac{\partial^2}{\partial x^2}\right) Re^{i\Theta} = 0$$

- A nonlinear eigenproblem: unknowns $R(x), \Theta(x)$ and eigenvalue Ω.

Once a solution is found, follow it as ω_0 and ε are varied. Find they are destroyed in saddle-node bifurcations:



(σ is width of frequency distribution, ω_0 is centre.)

• Local coupling through diffusion (as here) more natural.

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Synchronization of Chemical Micro-oscillators

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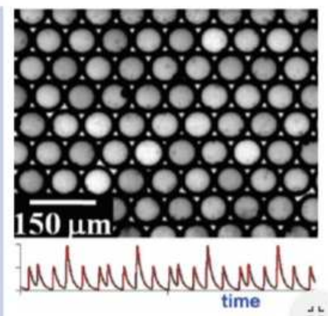
ABSTRACT Many phenomena of biological, physical, and chemical importance involve synchronization of oscillatory elements. We explore here, in several geometries, the behavior of diffusively coupled, nanoliter volume, aqueous drops separated by oil gaps and containing the reactants of the oscillatory Belousov–Zhabotinsky (BZ) reaction. A variety of synchronous regimes are found, including in- and antiphase oscillations, stationary Turing patterns, and more complex combinations of stationary and oscillatory BZ drops, including three-phase patterns. A model consisting of ordinary differential equations based on a simplified description of the BZ chemistry and diffusion of messenger (primarily inhibitory) species qualitatively reproduces most of the experimental results.

SECTION Kinetics, Spectroscopy



• What is missing: stability.

• “Chimeras in networks with purely local coupling”. Carlo R. Laing. Phys. Rev. E. (Vol. 92, 050904(R), 2015).

Planar oscillators

- Many systems studied use phase oscillators.

- Chimeras first observed in systems of planar (2D) oscillators, so they certainly exist, but what about analysis?

- Here we consider two coupled subnetworks of planar oscillators with a parameter, ϵ, such that as $\epsilon \to 0$ we recover previously studied system of phase oscillators.

The model we consider is

$$\begin{aligned}\frac{dX_j}{dt} &= i\omega X_j + \epsilon^{-1}\left\{1-(1+\delta\epsilon i)|X_j|^2\right\}X_j \\ &+ e^{-i\alpha}\left[\frac{\mu}{N}\sum_{k=1}^{N} X_k + \frac{\nu}{N}\sum_{k=1}^{N} X_{N+k}\right]\end{aligned}$$

for $j = 1, \ldots N$ and

$$\begin{aligned}\frac{dX_j}{dt} &= i\omega X_j + \epsilon^{-1}\left\{1-(1+\delta\epsilon i)|X_j|^2\right\}X_j \\ &+ e^{-i\alpha}\left[\frac{\mu}{N}\sum_{k=1}^{N} X_{N+k} + \frac{\nu}{N}\sum_{k=1}^{N} X_k\right]\end{aligned}$$

for $j = N+1, \ldots 2N$, where $X_j \in \mathbb{C}$, and $\omega, \epsilon, \alpha, \mu$ and $\nu \in \mathbb{R}$.

• Pair of populations of N Stuart-Landau oscillators; all-to-all coupling within each population of strength μ; all-to-all coupling between the two populations of strength ν.

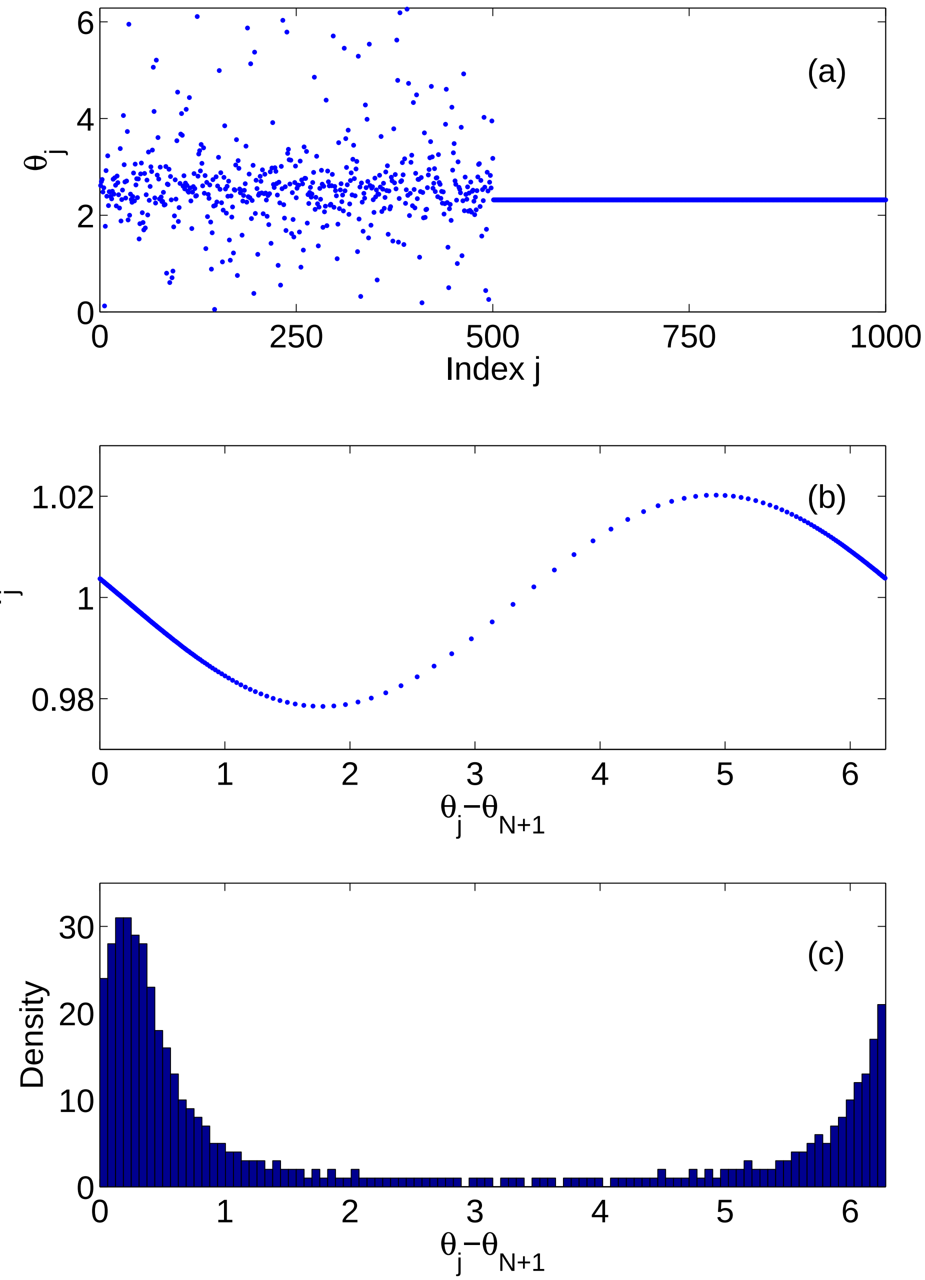
Defining $X_j = r_j e^{i\theta_j}$, have

$$\begin{aligned}\frac{dr_j}{dt} &= \epsilon^{-1}(1 - r_j^2)r_j + \frac{\mu}{N}\sum_{k=1}^{N} r_k \cos(\theta_k - \theta_j - \alpha) \\ &+ \frac{\nu}{N}\sum_{k=1}^{N} r_{N+k}\cos(\theta_{N+k} - \theta_j - \alpha) \\ \frac{d\theta_j}{dt} &= \omega - \delta r_j^2 + \frac{1}{r_j}\left[\frac{\mu}{N}\sum_{k=1}^{N} r_k \sin(\theta_k - \theta_j - \alpha)\right. \\ &+ \left.\frac{\nu}{N}\sum_{k=1}^{N} r_{N+k}\sin(\theta_{N+k} - \theta_j - \alpha)\right]\end{aligned}$$

for first population. Similarly for second.

As $\epsilon \to 0, r_j \to 1$, and recover system of Abrams et al., 2008.

Snapshot of chimera for $N = 500$. Population 2 perfectly sychronised.



• To analyse, set $X_j = Y$ for all j in population 2.

• Since dynamics depend on phase differences, go to rotating coordinate frame (speed Ω) such that Y is constant.

• Rotate frame such that Y is real.

Then

$$0 = i(\omega - \Omega)Y + \epsilon^{-1}\left\{1 - (1 + \delta\epsilon i)Y^2\right\}Y + e^{-i\alpha}\left(\mu Y + \nu\widehat{X}\right)$$

where

$$\widehat{X} \equiv \frac{1}{N}\sum_{k=1}^{N} X_k$$

Given $\widehat{X}$, can solve this for Y.

• Each oscillator in population 1 satisfies

$$\begin{aligned}\frac{dX}{dt} &= i\left[\delta Y^2 + \mu\sin\alpha - (\nu/Y)\mathrm{Im}\left\{e^{-i\alpha}\widehat{X}\right\}\right]X \\ &+ \epsilon^{-1}\left\{1-(1+\delta\epsilon i)|X|^2\right\}X + e^{-i\alpha}[\mu\widehat{X}+\nu Y] \qquad (5)\end{aligned}$$

• Waveforms are equally staggered in time (splay state) so that as $N \to \infty$, $\widehat{X}$ and Y become constant.

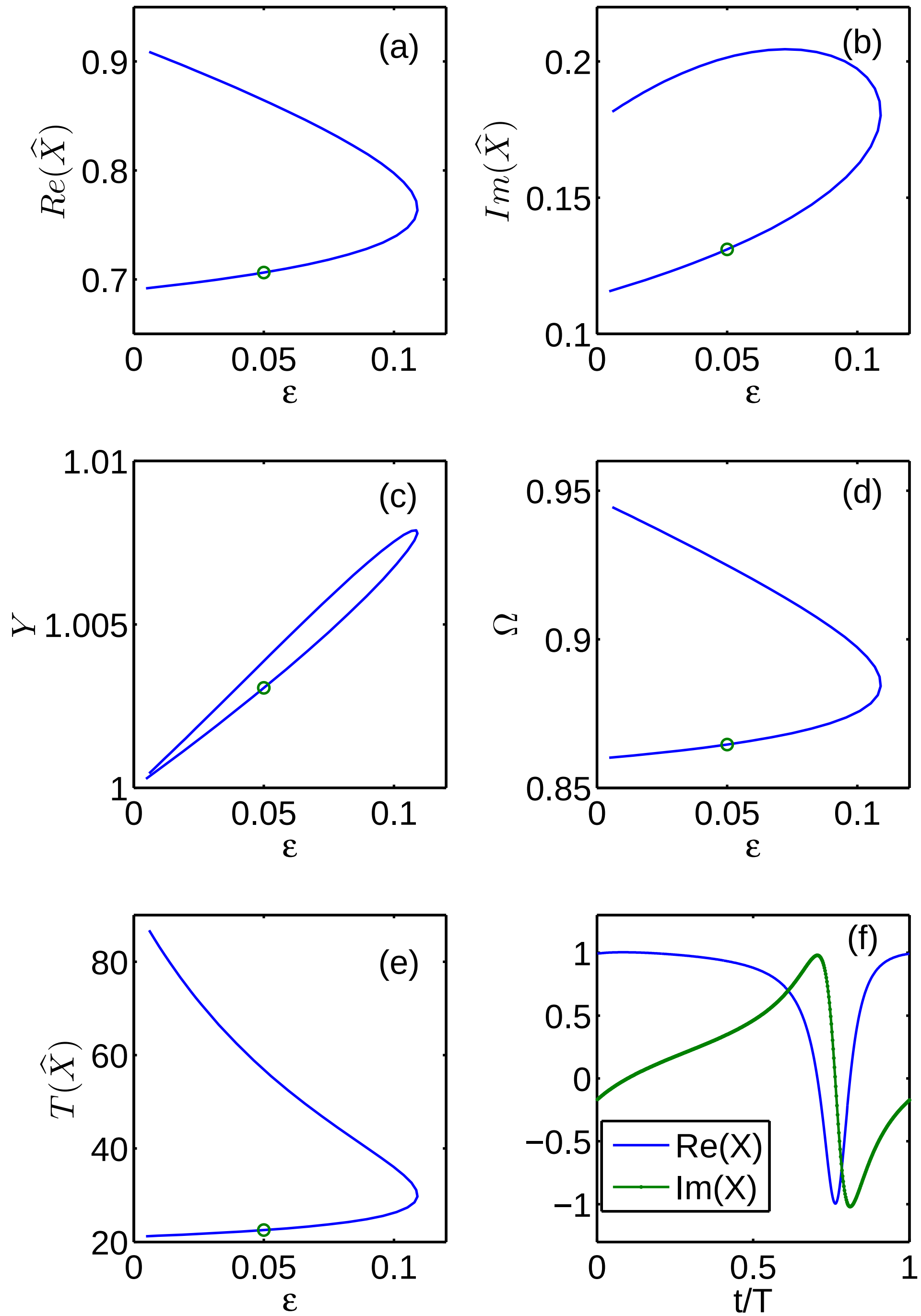
• Can replace mean of population 1 by integral over one period, so self-consistency equation defining chimera is

$$\widehat{X} = \frac{1}{T(\widehat{X})}\int_0^{T(\widehat{X})} X(t;\widehat{X})\,dt$$

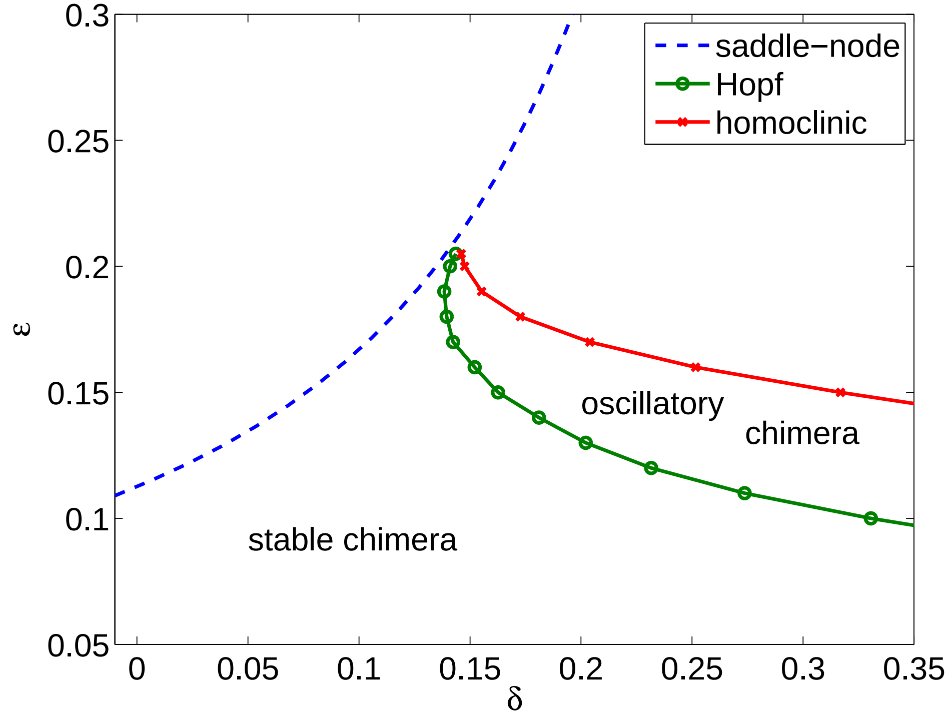
where $X(t;\widehat{X})$ is a periodic solution of Eq. (5) with period $T(\widehat{X})$.

Follow solutions of this complex equation as ϵ is varied.

Solutions destroyed in saddle-node bifurcation.



- Varying δ and ϵ:



- Stability and other bifurcation curves found numerically.

- What is missing: proper stability calculations.

- "Chimeras in networks of planar oscillators" C. R. Laing. Phys. Rev. E (Vol. 81, 066221, 2010)

Random networks

- Consider a pair of populations of N non-identical phase oscillators, with coupling both within and between populations.

$$\begin{aligned}
\frac{d\theta_j^1}{dt} &= \omega_j^1 + \frac{\mu}{N}\sum_{k=1}^{N} A_{jk}\sin\left(\theta_k^1 - \theta_j^1 - \alpha\right) + \frac{\nu}{N}\sum_{k=1}^{N} B_{jk}\sin\left(\theta_k^2 - \theta_j^1 - \alpha\right) \\
\frac{d\theta_j^2}{dt} &= \omega_j^2 + \frac{\mu}{N}\sum_{k=1}^{N} C_{jk}\sin\left(\theta_k^2 - \theta_j^2 - \alpha\right) + \frac{\nu}{N}\sum_{k=1}^{N} D_{jk}\sin\left(\theta_k^1 - \theta_j^2 - \alpha\right)
\end{aligned}$$

- Superscript labels population.

- All ω taken from Lorentzian distribution.

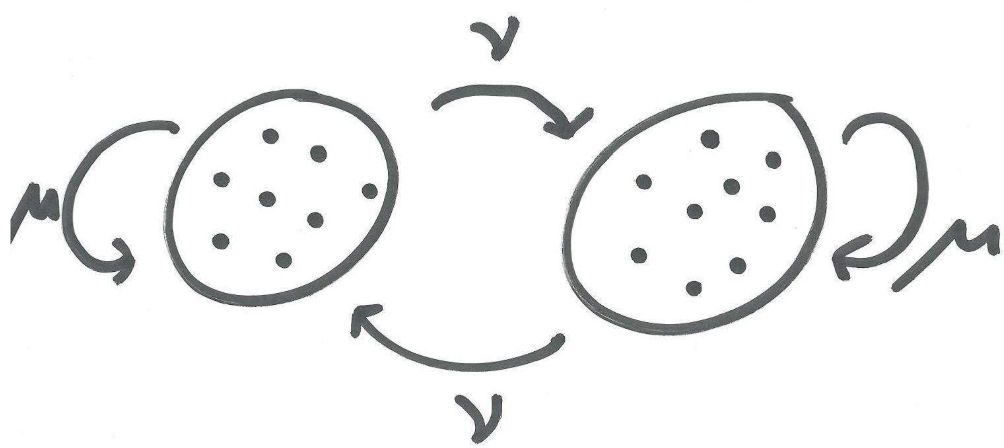
- μ and ν are overall coupling strengths within and between populations.

- A_{jk}, B_{jk}, C_{jk} and D_{jk} are constants, as is α.

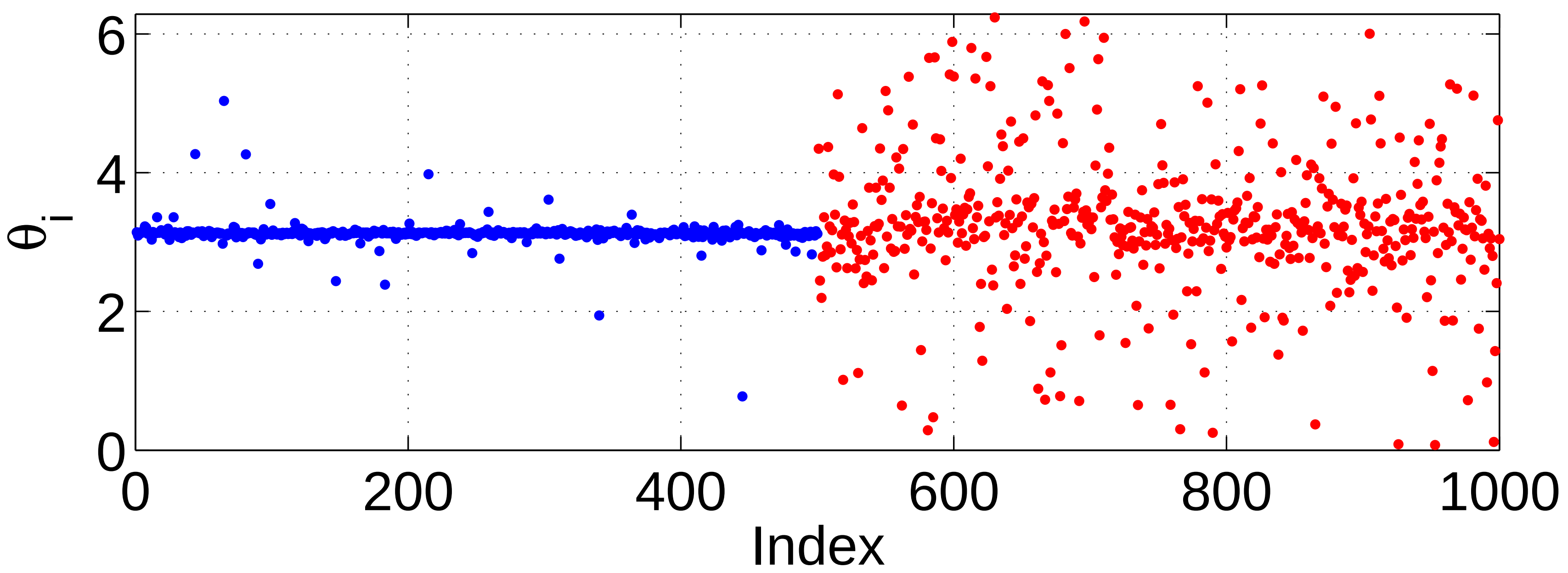
• When $A_{jk} = B_{jk} = C_{jk} = D_{jk} = 1$, have all-to-all coupling within and between populations:

$$\frac{d\theta_j^1}{dt} = \omega_j^1 + \frac{\mu}{N}\sum_{k=1}^{N}\sin\left(\theta_k^1 - \theta_j^1 - \alpha\right) + \frac{\nu}{N}\sum_{k=1}^{N}\sin\left(\theta_k^2 - \theta_j^1 - \alpha\right)$$

$$\frac{d\theta_j^2}{dt} = \omega_j^2 + \frac{\mu}{N}\sum_{k=1}^{N}\sin\left(\theta_k^2 - \theta_j^2 - \alpha\right) + \frac{\nu}{N}\sum_{k=1}^{N}\sin\left(\theta_k^1 - \theta_j^2 - \alpha\right)$$



- Known to support stationary chimera states. Movie



- And oscillating chimera states. Movie

• What happens when connectivity within and between populations is not all-to-all?

• We **fix** the connectivity by randomly setting $A_{jk} =$ constant or zero (details later) and similarly for B_{jk}, C_{jk}, D_{jk}.

• Then consider an infinite ensemble of networks with this fixed connectivity, but each having different realisations of the ω's [intrinsic frequencies, distribution $g(\omega)$].

• By averaging over this ensemble we obtain ODEs for the “order parameter” at each node.

- Still have $2N$ ODEs, but they give information about **typical** dynamics of node.

- Chimera states are **fixed points** of these ODEs (they are not fixed points of the oscillator dynamics) so can be followed as parameters are varied.

- Do this multiple times for statistically-similar networks to gain insight into effects of changing connectivities.

- Idea first used by Barlev et al. (2011).

- Consider an ensemble of networks with fixed connectivity.

- Let number of members of the ensemble go to infinity.

- We describe the state of population 1 by the probability density function

$$f^1(\theta_1^1, \theta_2^1, \ldots, \theta_N^1; \omega_1^1, \omega_2^1, \ldots \omega_N^1; t)$$

and population 2 by the function

$$f^2(\theta_1^2, \theta_2^2, \ldots, \theta_N^2; \omega_1^2, \omega_2^2, \ldots \omega_N^2; t)$$

which, by conservation of oscillators, satisfy

$$\frac{\partial f^\sigma}{\partial t} + \sum_{j=1}^{N} \frac{\partial}{\partial \theta_j^\sigma} \left[f^\sigma \left(\frac{d\theta_j^\sigma}{dt} \right) \right] = 0$$

for $\sigma = 1, 2$.

• Using Ott/Antonsen ansatz and properties of Lorentzian (width Δ) we obtain

$$\begin{aligned}\frac{da_k}{dt} &= -\Delta a_k + (\overline{R}_k - R_k a_k^2)/2 \\ \frac{db_k}{dt} &= -\Delta b_k + (\overline{S}_k - S_k b_k^2)/2\end{aligned}$$

for $k = 1, \dots N$ where

$$R_k = e^{-i\alpha}\,(\mu A_k + \nu B_k) \qquad S_k = e^{-i\alpha}\,(\mu C_k + \nu D_k)$$

and

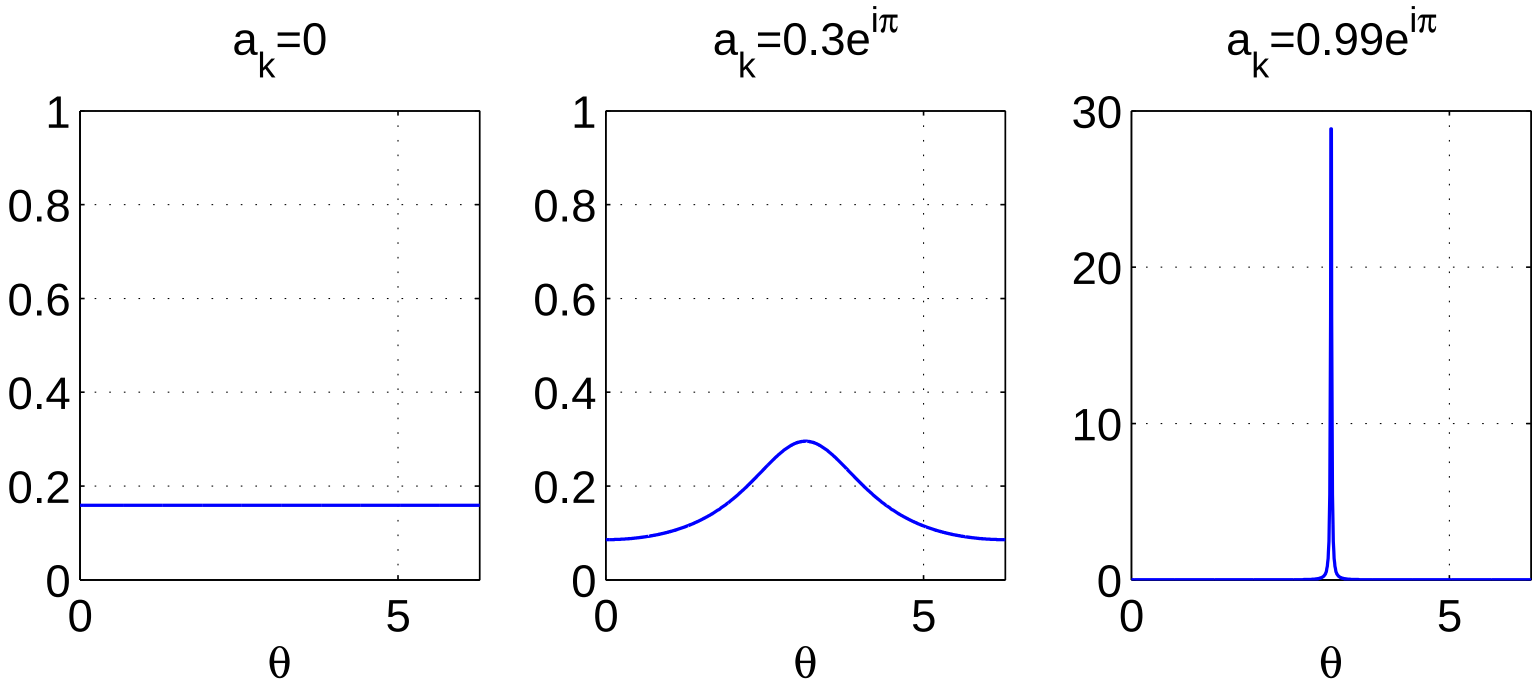
$$A_j = \frac{1}{N}\sum_{k=1}^{N} A_{jk}\overline{a}_k \qquad B_j = \frac{1}{N}\sum_{k=1}^{N} B_{jk}\overline{b}_k$$

$$C_j = \frac{1}{N}\sum_{k=1}^{N} C_{jk}\overline{b}_k \qquad D_j = \frac{1}{N}\sum_{k=1}^{N} D_{jk}\overline{a}_k$$

• $a_k = \langle e^{i\theta_k} \rangle$. Magnitude gives "peaked-ness" of phase distribution at node k in population 1 (over ensemble).

• Phase of a_k gives angle about which distribution is peaked.

• Similarly for b_k and population 2.



Sparse connectivity — Erdös-Rényi

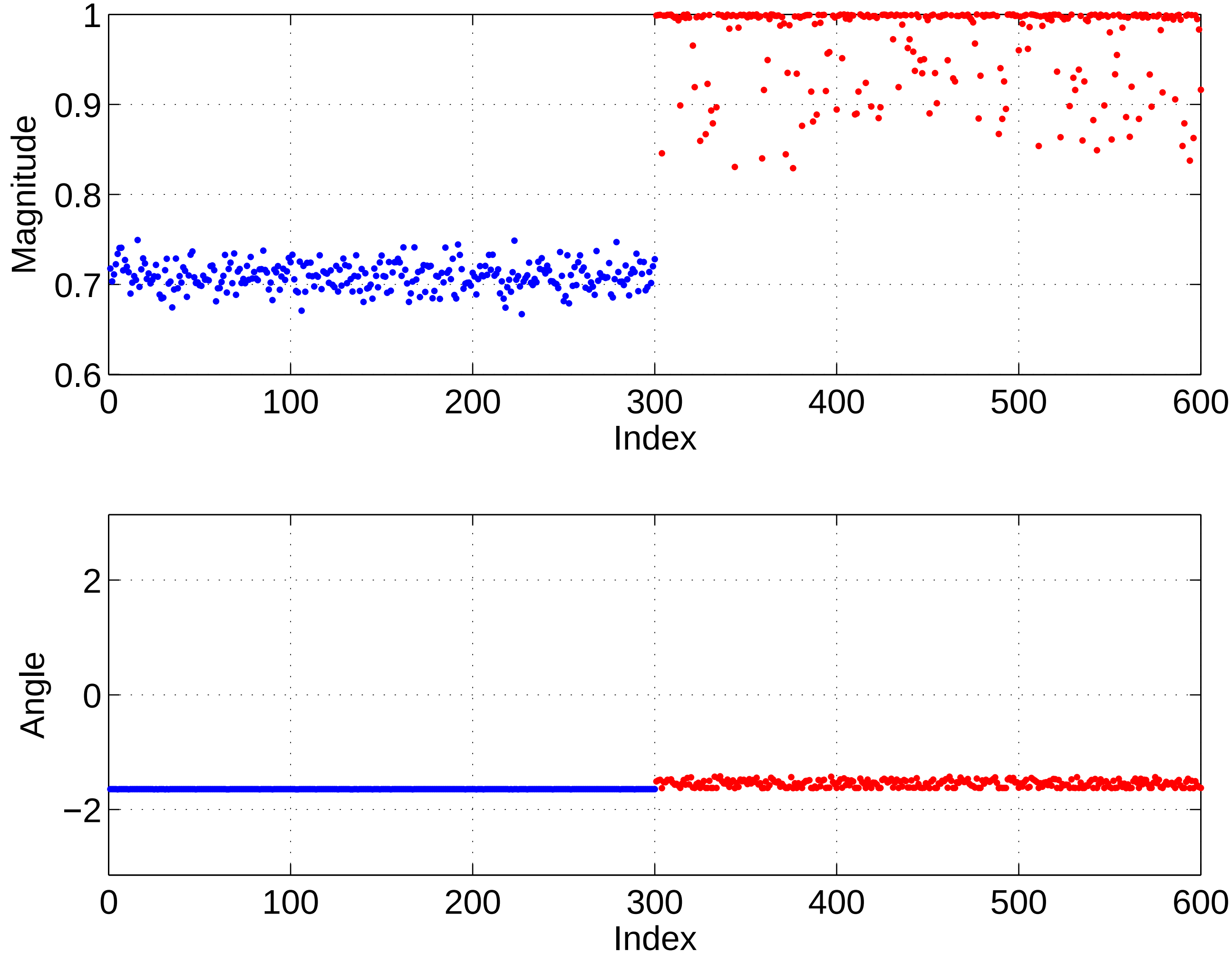
• Randomly delete entries from the connectivity matrices $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ and $\boldsymbol{D}$, and increase remaining values of the weights to compensate.

$$\boldsymbol{A}_{jk} = \begin{cases} 1/p & \text{with probability } p \\ 0 & \text{with probability } 1-p \end{cases}$$

and similarly for $\boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}$.

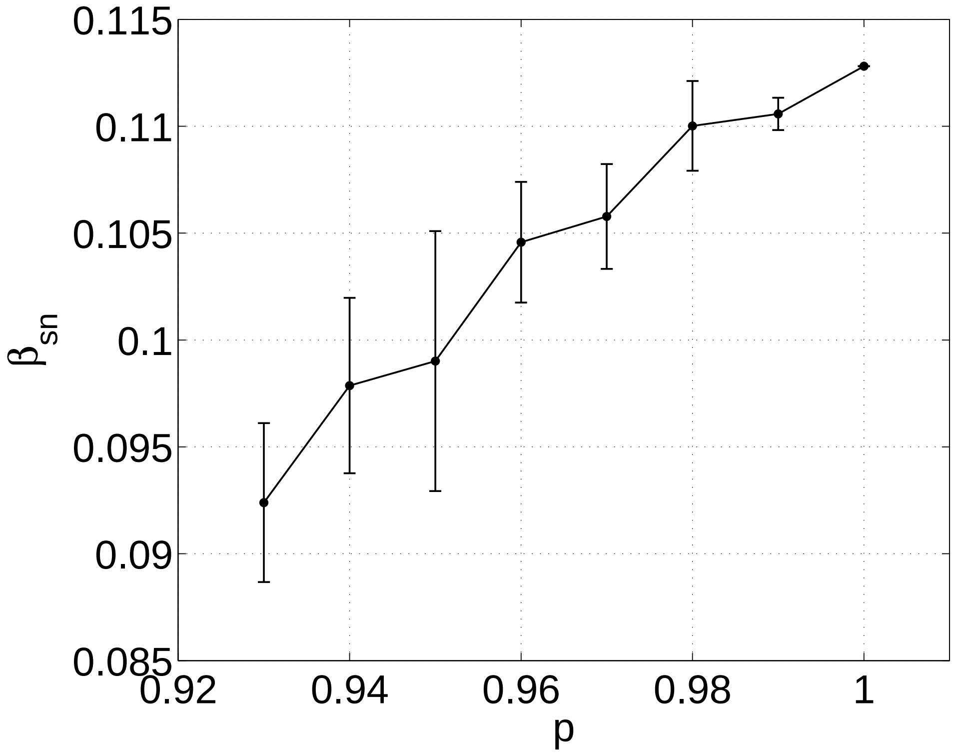
• $p = 1 \Rightarrow$ full connectivity.

Steady state: $p = 0.97$. Corresponding chimera in phase network



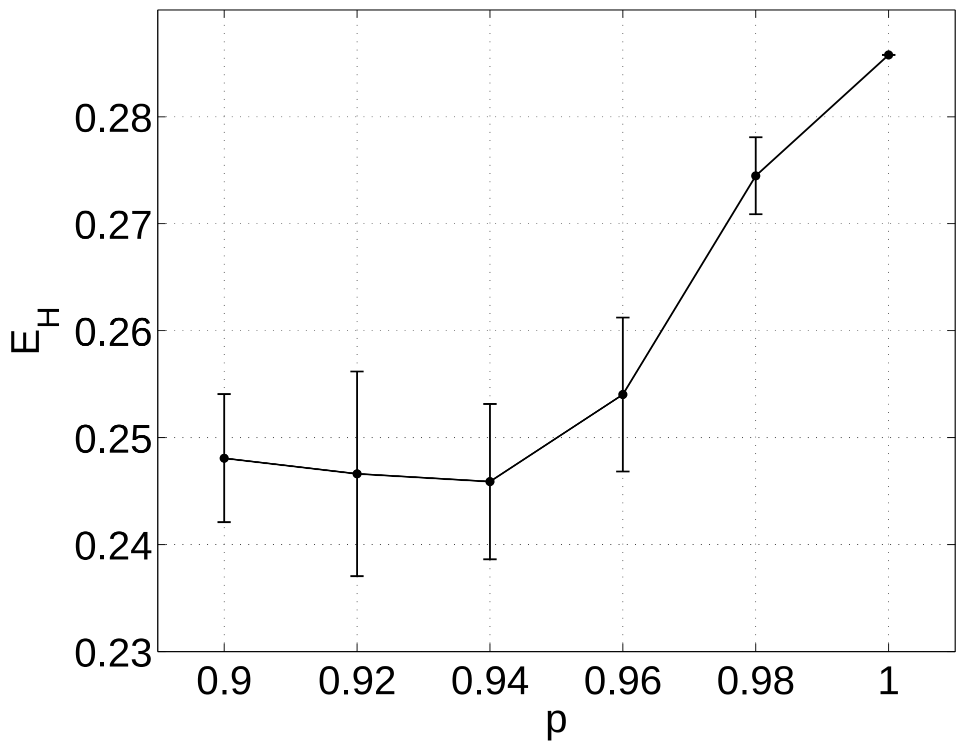
• Defining $\boldsymbol{\beta = \pi/2 - \alpha}$, stable chimera states undergo a saddle-node bifurcation as $\boldsymbol{\beta}$ is increased.

• This value decreases as the networks are made more sparse.

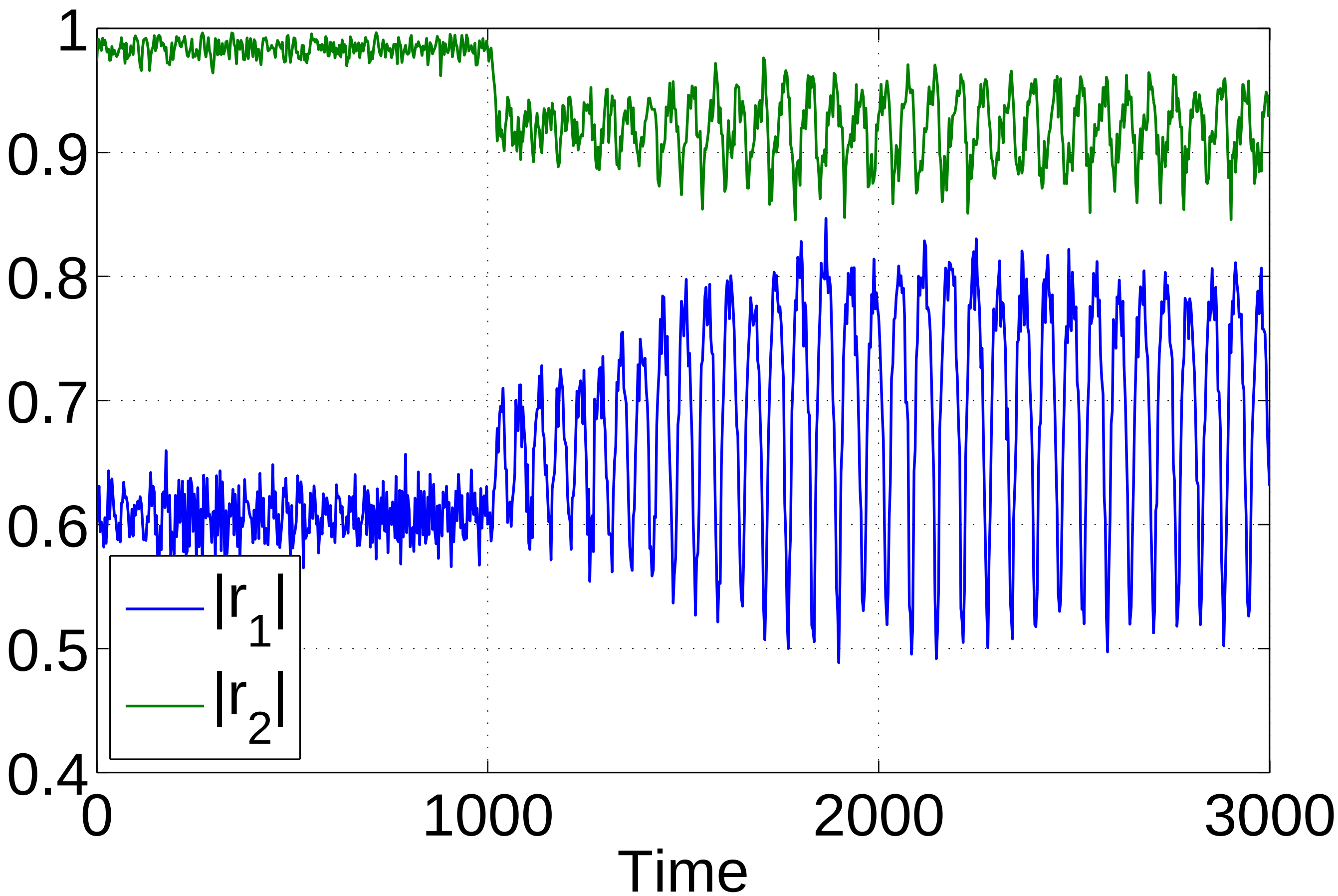


• Writing $\mu = (1 + E)/2$ and $\nu = (1 - E)/2$, stable chimeras undergo a Hopf bifurcation as E is increased.

• This value also decreases as the networks are made more sparse.



- We can induce oscillations by uniformly (and randomly) removing connections within and between populations.



Sparse connectivity — Chung-Lu-type networks

- Assign each oscillator in a population a weight $w_i = N(i/N)^r$

- Probability p_{ij} of a connection between oscillators i and j is

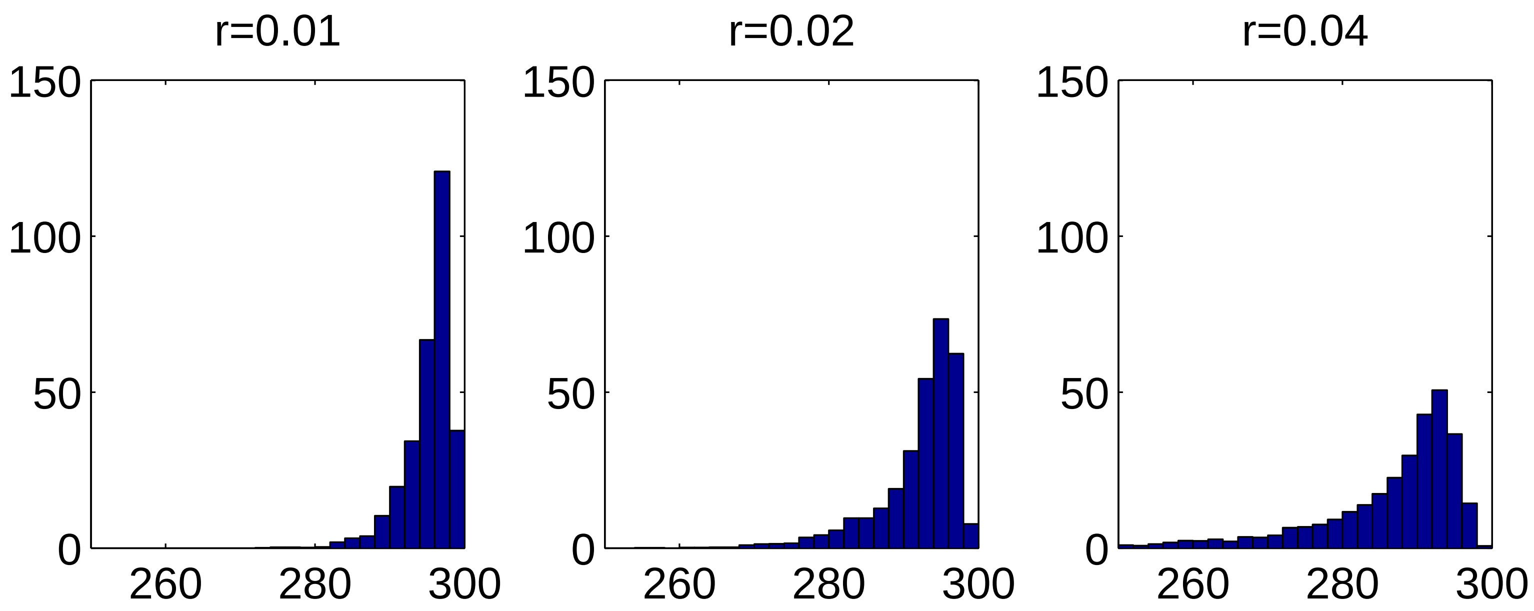
$$p_{ij} = \min\left(\frac{w_i w_j}{\sum_k w_k}, 1\right)$$

- Normalise connection strengths so that sum of matrix entries $= N^2$.

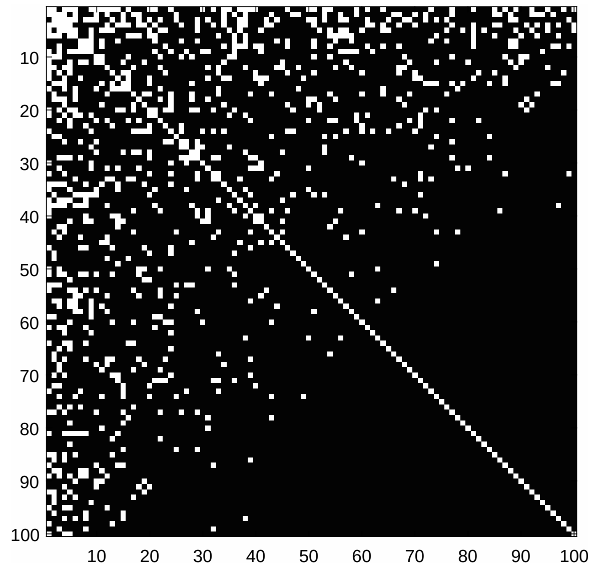
- $r = 0 \Rightarrow$ full connectivity.

- Increasing r gives long tail towards lower degrees.

Degree distributions:



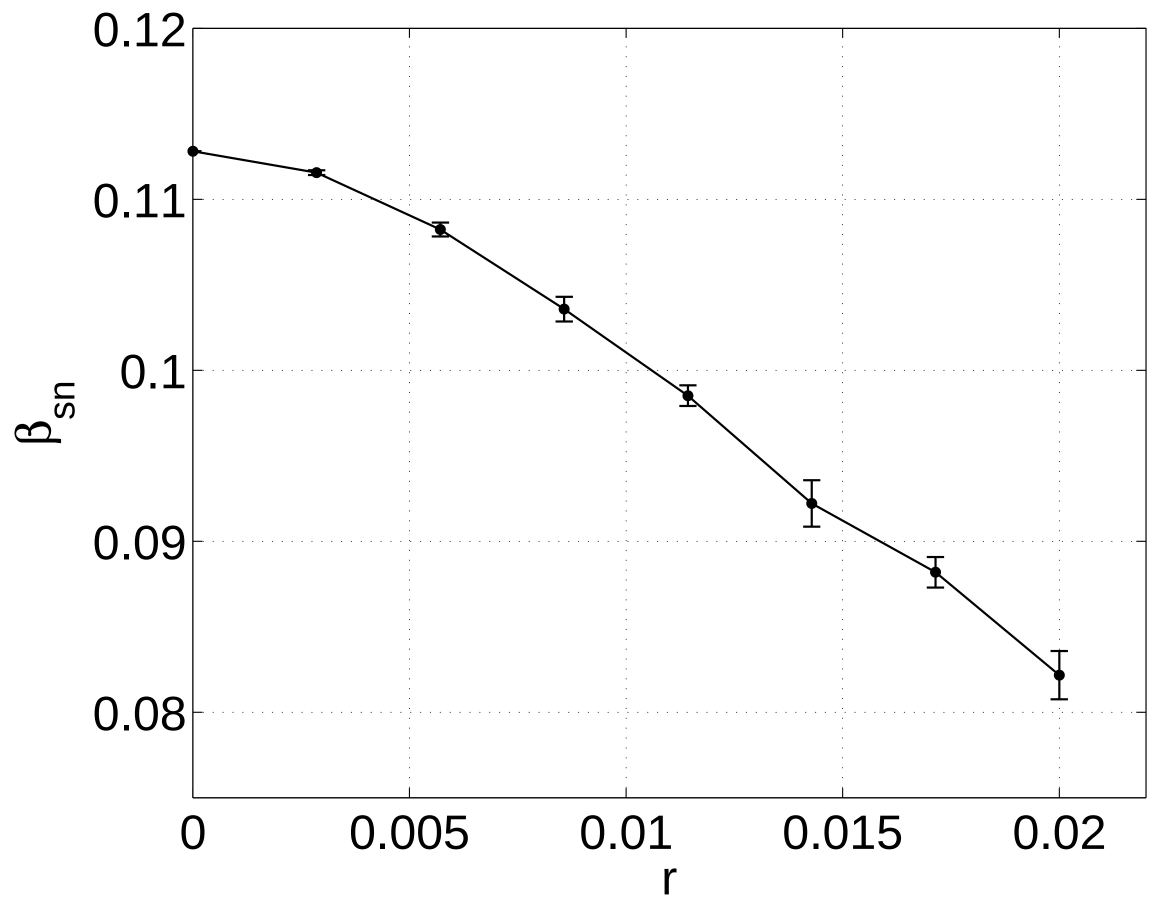
• Typical connectivity matrix:



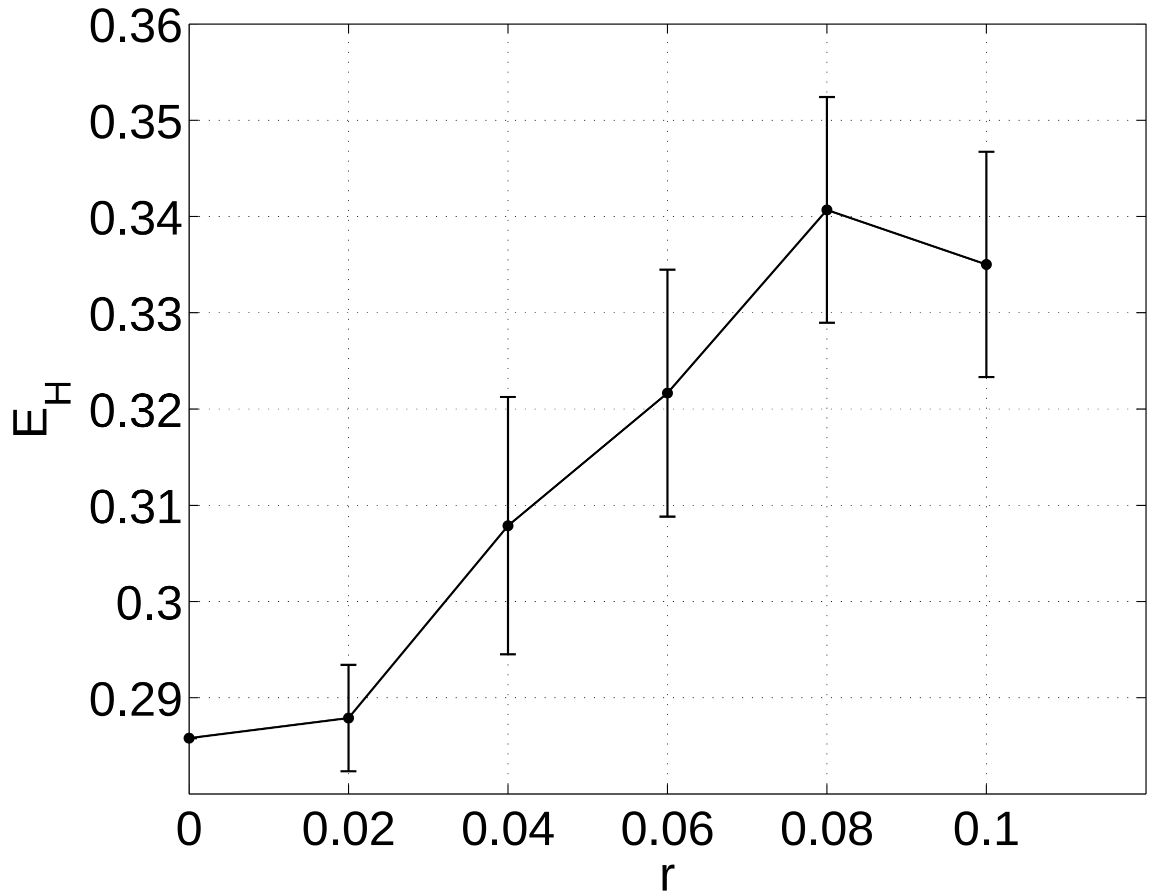
• Symmetric (by construction) so in-degree = out-degree.

• Number of connections removed **within** network ≈ number removed **between** networks.

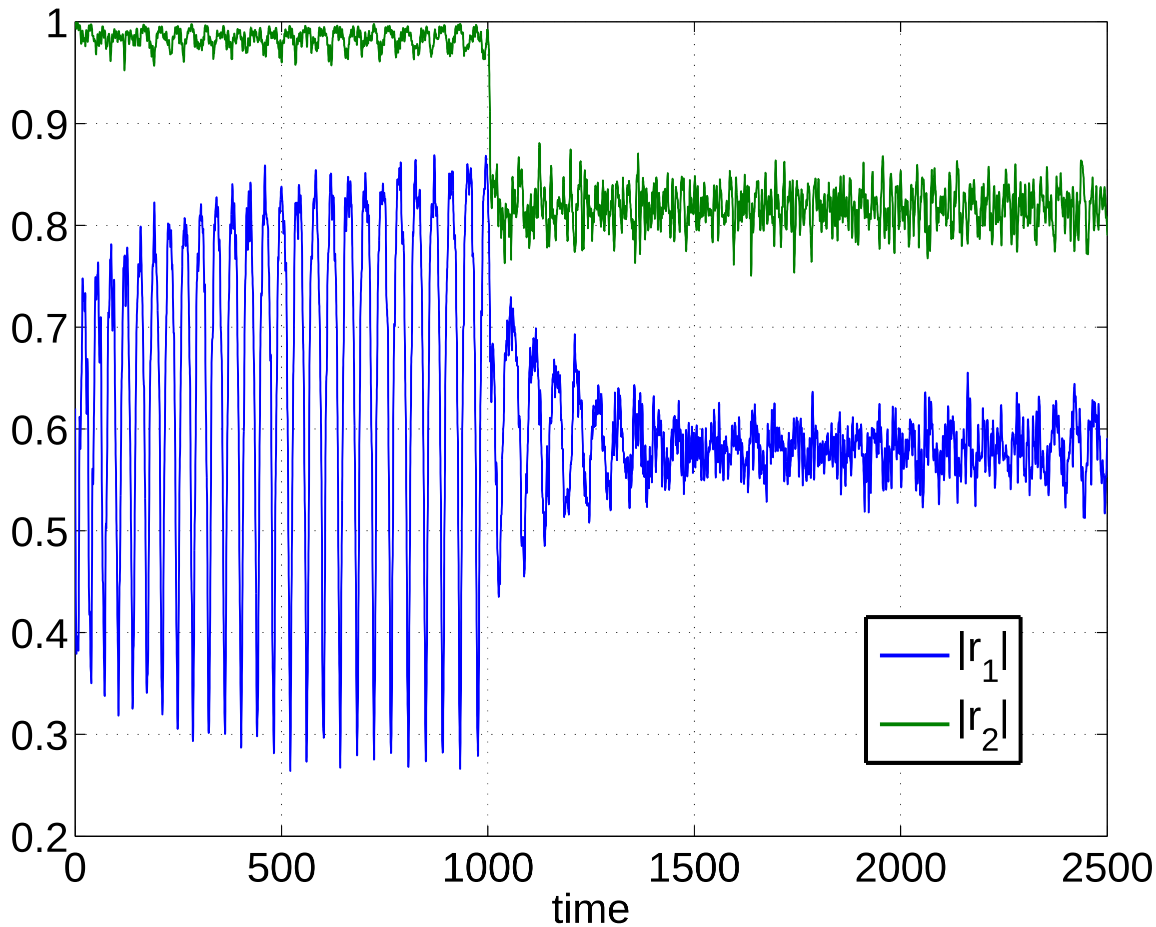
• Increasing r decreases value of β at which saddle-node bifurcation occurs:



- Increasing r **increases** value of E at which Hopf bifurcation occurs:



• We can **supress** oscillations by randomly removing connections within and between populations (in this prescribed way). Opposite of E-R.



Summary

• Averaged over ensemble of networks with **same** connectivity but different intrinsic frequencies to obtain ODEs for "order parameter" at each node.

• Chimera states are quite fragile: significant shifts in saddle-node bifurcations for $p \approx 0.9$ and $r \approx 0.02$.

• Oscillating chimeras due to Hopf bifurcation can be created (E-R) or destroyed (C-L) by randomly removing connections (in a specified way).

• Qualitatively similar results for Gaussian rather than Lorenztian frequency distribution.

• "Chimeras in random non-complete networks of phase oscillators." C. R. Laing, K. Rajendran and I. G. Kevrekidis. Chaos, (vol. 22. 013132, 2012).

Overall summary

- Chimeras occur in systems of non-identical oscillators (not shown).
- Non-local coupling not necessary.
- Chimeras occur in networks of planar (Stuart-Landau) oscillators.
- Somewhat robust to deletion of connections in two sub-network case.

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