Effect of intrinsic noise on chimera states in populations of hierarchically coupled oscillators: beyond Ott–Antonsen theory

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Systems of the type:

$$\dot{\varphi}_k = \Omega(t) + \mathrm{Im}(2h(t)e^{-i\varphi_k}), \qquad k = 1,\ldots,N.$$

Kuramoto-Sakaguchi ensemble:

$$\dot{\varphi}_k = \Omega + \frac{\mu}{N}\sum_{j=1}^{N}\sin(\varphi_j - \varphi_k - \alpha), \qquad h(t) = \mu\left\langle e^{i\varphi_j}\right\rangle_j e^{-i\alpha}.$$

Chain of superconducting (Josephson) junctions in parallel with a resistive load:

$$\frac{1}{\gamma}\ddot{\varphi}_k + \dot{\varphi}_k = I_{\mathrm{inp}}(t) + I_0\sin\varphi_k - \frac{\mu}{N}\sum_{j=1}^{N}\sin\varphi_j,$$

$$\Omega = I_{\mathrm{inp}}(t) + \mu\,\mathrm{Im}\left\langle e^{i\varphi_j}\right\rangle_j, \qquad h(t) = I_0/2, \quad \gamma \gg 1.$$

S. A. Marvel, S. H. Strogatz, Chaos **19**, 013132 (2009)

Active rotators (and theta-neurons):

$$\dot{\varphi}_k = \Omega + \frac{\mu}{N}\sum_{j=1}^{N}K_j\sin(\varphi_j - \varphi_k) - B\sin\varphi_k, \quad h(t) = \frac{1}{2}\left(B + \left\langle K_j e^{i\varphi_j}\right\rangle_j\right).$$

Sh. Shinomoto, Y. Kuramoto, Prog. Theor. Phys. **75**(5), 1105 (1986)

Advance: Watanabe–Strogatz theory

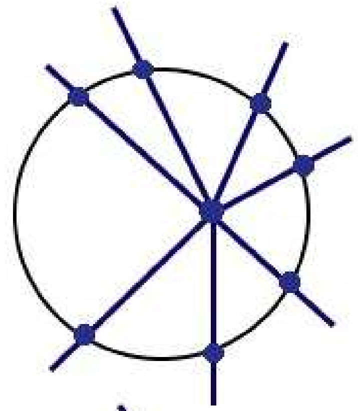
$$\dot{\varphi}_k = \Omega(t) + \mathrm{Im}(2h(t)e^{-i\varphi_k}), \qquad k = 1,\ldots,N.$$

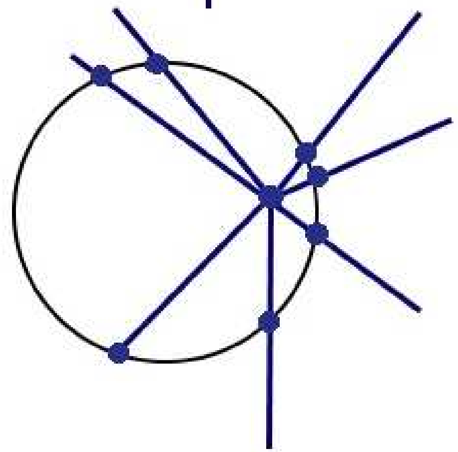
In terms of ψ_k and z :

$$e^{i\varphi_k} = \frac{z + e^{i\psi_k}}{1 + z^* e^{i\psi_k}}, \qquad z: \quad \sum_{k=1}^{N} e^{i\psi_k} = 0,$$

ψ_k evolve with identical rate;

$$\dot{z} = i\Omega z + h - h^* z^2, \quad \dot{\psi}_k = \Omega(t) + \mathrm{Im}(2h(t)z^*).$$





Dynamics is partially integrable:

3 variables + (N–3) integrals of motion

S. Watanabe, S. H. Strogatz, Phys. D **74**, 197 (1994)

A. Pikovsky, M. Rosenblum, Phys. Rev. Lett. **101**, 2264103 (2008)

S. A. Marvel, R. E. Mirollo, S. H. Strogatz, Chaos **19**, 043104 (2009)

Interpretation in terms of order parameters?

Ott–Antonsen theory (has simple interpretation)

$$\dot{\varphi}_k = \Omega(t) + \mathrm{Im}(2h(t)e^{-i\varphi_k}), \qquad k = 1,\ldots,N.$$

E. Ott, T. M. Antonsen, Chaos **18**, 037113 (2008)

The Master-equation for the probability density $w(\varphi, t)$:

$$\frac{\partial w}{\partial t} + \frac{\partial}{\partial \varphi}((\Omega - ihe^{-i\varphi} + ih^* e^{i\varphi})w) = 0.$$

Particular solution $w(\varphi,t) = \frac{1}{2\pi}\left[1 + \sum_{m=1}^{\infty}(a^m e^{-im\varphi} + c.c.)\right],$

where complex amplitude a is the order parameter $a = \langle e^{i\varphi} \rangle$:

$$\dot{a} = i\Omega a + h - h^* a^2$$

- According to W–S theory, **OA manifold is not attracting in *ideal* situations**
- **In *real* systems,** small detuning of parameters makes the **OA manifold attracting**

Powerful Tool & Expectation of Troubles

$$\dot{\varphi}_k = \Omega(t) + \mathrm{Im}(2h(t)e^{-i\varphi_k}), \qquad k = 1,\ldots,N$$

- **We can describe and understand the collective dynamics reliably and in great detail.**
- **Our main sources of intuition for the theory of collective phenomena are very specific systems. Here many of collective phenomena (e.g., clustering) are forbidden by conservation laws.**
- **In reality, the OA conditions are (slightly?) violated.**
- **The perturbation theory cannot be constructed in a regular way.**

A perturbation theory is needed.
Preferably, for OA ansatz (not for WS theory).

V. Vlasov, M. Rosenblum, A. Pikovsky, J. Phys. A Math. Theor. **49**, 31LT02 (2016)

Populations of Oscillators with Intrinsic Noise

$$\dot{\varphi}_k = \Omega(t) + \mathrm{Im}(2h(t)e^{-i\varphi_k}) + f_k$$

$f_k = \sigma\xi_k(t)$, $\xi_k(t)$: Gaussian noise signals, $\langle \xi_k(t)\xi_m(t') \rangle = 2\delta_{km}\delta(t - t')$.

Fokker–Planck equation:

$$\frac{\partial w}{\partial t} + \frac{\partial}{\partial \varphi}[(\Omega - ihe^{-i\varphi} + ih^* e^{i\varphi})w] = \sigma \frac{\partial^2 w}{\partial \varphi^2}$$

In Fourier space, $w(\varphi, t) = (2\pi)^{-1}[1 + \sum_{m=1}^{\infty}(a_m e^{-im\varphi} + c.c.)]$:

$$\dot{a}_m = im\Omega a_m + mha_{m-1} - mh^* a_{m+1} - \sigma^2 m^2 a_m.$$

With Lorentzian distribution of frequencies $g(\Omega) = \dfrac{\gamma}{\pi\left[(\Omega - \Omega_0)^2 - \gamma^2\right]}$:

$$\dot{a}_m = m(i\Omega_0 - \gamma)a_m + mha_{m-1} - mh^* a_{m+1} - \sigma^2 m^2 a_m.$$

‘Circular’ Cumulants

I.V.T., D.S.G., L.S.K., A.P., Phys. Rev. Lett. **120**, 264101 (2018)

Perturbation to O-A manifold $a_m = a_1^m$? $\begin{cases} a_1 = a \\ a_2 = a^2 + [smthg.] \\ a_3 = a^3 + G([smthg.], a) + [smthg.2] \\ \quad \ldots \end{cases}$

For $a_m = \left\langle (e^{i\varphi_j})^m \right\rangle$, moment-generating function $F(k) = \left\langle e^{ke^{\varphi_j}} \right\rangle = \sum_{m=0}^{\infty} a_m \dfrac{k_m}{m!}$

Cumulant-generating function:

$$\Phi(k) = k\frac{\partial}{\partial k}\ln F(k) = \sum_{m=1}^{\infty} \kappa_m k^m .$$

$$\kappa_1 = a_1, \quad \kappa_2 = a_2 - a_1^2, \quad \kappa_3 = \frac{1}{2}(a_3 - 3a_2a_1 + 2a_1^3).$$

Cumulant Expansion

$$k\frac{\partial}{\partial k}\ln F(k) = \sum_{m=1}^{\infty}\kappa_m k^m$$

$$\dot{\kappa}_m = m(i\Omega_0 - \gamma)\kappa_m + h\delta_{m1} - h^*\left(m^2\kappa_{m+1} + m\sum_{j=0}^{m-1}\kappa_{m-j}\kappa_{j+1}\right)$$
$$- \sigma^2\left(m^2\kappa_m + m\sum_{j=0}^{m-2}\kappa_{m-1}\kappa_{j+1}\right)$$

instead of

$$\dot{a}_m = m(i\Omega_0 - \gamma)a_m + mha_{m-1} - mh^*a_{m+1} - \sigma^2 m^2 a_m.$$

OA solution = wrapped Cauchy distribution of phases:

$$\kappa_1 = a_1, \quad \kappa_{m>1} = 0$$

Expansion

$\kappa_1 \sim \varepsilon \ll 1$	$\kappa_1 \sim 1$
$\kappa_{m>1} \sim \varepsilon^m$	$\kappa_{m>1} \sim \sigma^{2(m-1)}$

Conventional Cumulants *versus* 'Circular' Cumulants

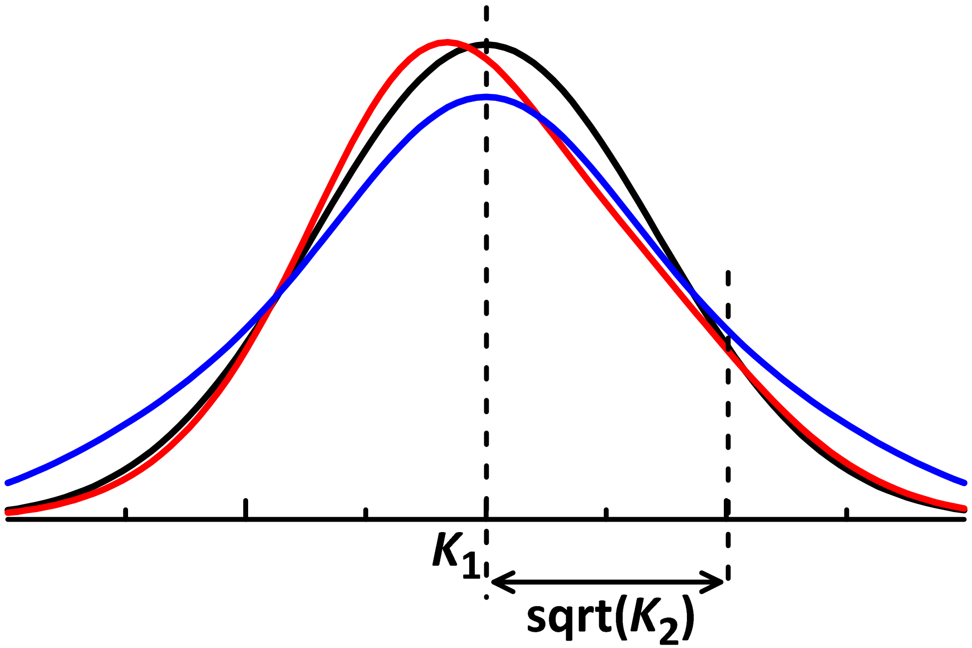
Gaussian distribution

$K_j \in \mathbb{R}, \quad K_1, K_2, K_{j>2} = 0$

K_1: centered (mean), K_2: width

Deviation from Gaussian distribution:

K_3: **skewness** K_4: **kurtosis**



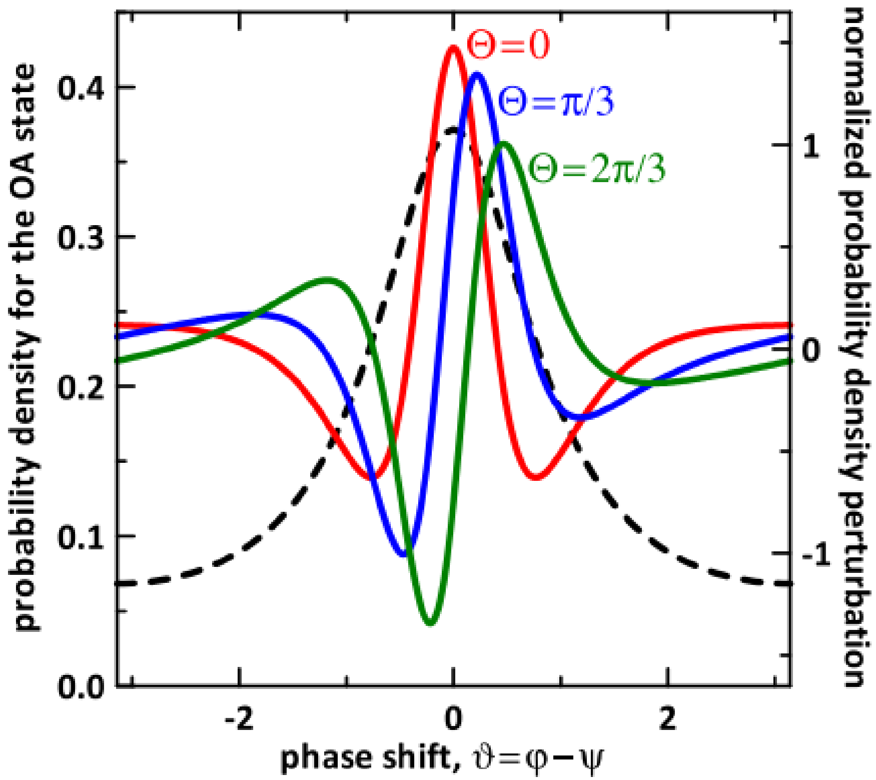
wrapped Cauchy distribution on circle

$\kappa_j \in \mathbb{C}, \quad \kappa_1 = \langle e^{i\varphi} \rangle, \kappa_{j>1} = 0$

$\arg(\kappa_1)$: centered, $|\kappa_1| = e^{-\text{[width]}}$

Deviation from wrap. Cauchy distr.:

κ_2: $|\kappa_2|, \quad \arg(\kappa_2 / \kappa_1^2) = \Theta$



$$\dot{\kappa}_m = m(i\Omega_0 - \gamma)\kappa_m + h\delta_{m1} - h^*(m^2\kappa_{m+1} + m\sum_{j=0}^{m-1}\kappa_{m-j}\kappa_{j+1})$$
$$- \sigma^2(m^2\kappa_m + m\sum_{j=0}^{m-2}\kappa_{m-1}\kappa_{j+1})$$

Model Reductions: Series Truncations

Expansion

$$\kappa_1 \sim \varepsilon \ll 1$$
$$\kappa_{m>1} \sim \varepsilon^m$$

$$\kappa_1 \sim 1$$
$$\kappa_{m>1} \sim \sigma^{2(m-1)}$$

Two-cumulant truncations: $\kappa_3 = 0,\ \kappa_3 = \frac{3}{2}\frac{\kappa_2^2}{\kappa_1},\ \kappa_3 = \frac{3}{2}\kappa_2^2\kappa_1^*$

$$\dot{\kappa}_1 = (i\Omega_0 - \gamma)\kappa_1 + h - h^*(\kappa_1^2 + \kappa_2) - \sigma^2\kappa_1,$$
$$\dot{\kappa}_2 = (i2\Omega_0 - 2\gamma - 4h^*\kappa_1)\kappa_2 - 4h^*\kappa_3 - \sigma^2(4\kappa_2 + 2\kappa_1^2).$$

D.S.G., I.V.T., L.S.K., A.P., Chaos **28**, 101101 (2018)

$$\dot{\kappa}_m = m(i\Omega_0 - \gamma)\kappa_m + h\delta_{m1} - h^*\left(m^2\kappa_{m+1} + m\sum_{j=0}^{m-1}\kappa_{m-j}\kappa_{j+1}\right)$$
$$- \sigma^2\left(m^2\kappa_m + m\sum_{j=0}^{m-2}\kappa_{m-1}\kappa_{j+1}\right)$$

Two-cumulant truncations: $\kappa_3 = 0,\ \kappa_3 = \frac{3}{2}\frac{\kappa_2^2}{\kappa_1},\ \kappa_3 = \frac{3}{2}\kappa_2^2\kappa_1^*$:

$$\dot{\kappa}_1 = (i\Omega_0 - \gamma)\kappa_1 + h - h^*(\kappa_1^2 + \kappa_2) - \sigma^2\kappa_1,$$
$$\dot{\kappa}_2 = (i2\Omega_0 - 2\gamma - 4h^*\kappa_1)\kappa_2 - 4h^*\kappa_3 - \sigma^2(4\kappa_2 + 2\kappa_1^2).$$

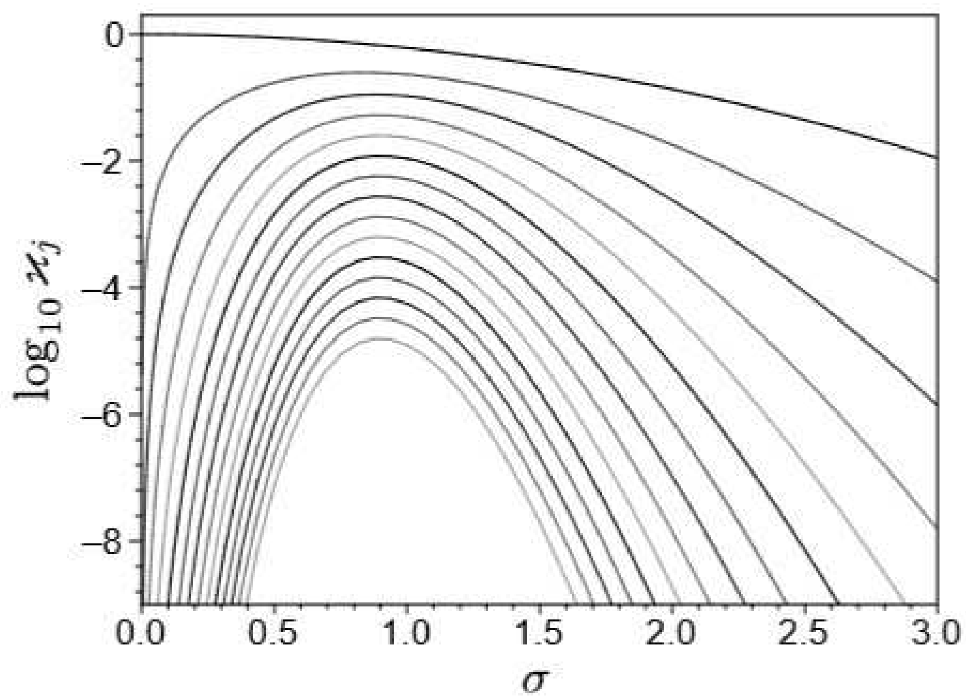
Wrapped Gaussian distribution:

$$a_m = R_1^{m^2} e^{im\psi} = e^{-\frac{1}{2}m^2\sigma^2} e^{im\psi}$$

Zaks *et al.*, Phys. Rev. E **68**, 066206 (2003);
Hannay *et al.*, Sci. Adv. **4**, e1701047 (2018)

$\kappa_1 \sim \varepsilon \ll 1$ $\qquad$ $\kappa_1 \sim 1$

$\kappa_{m>1} \sim \varepsilon^m$ $\qquad$ $\kappa_{m>1} \sim \varepsilon^{m-1}$



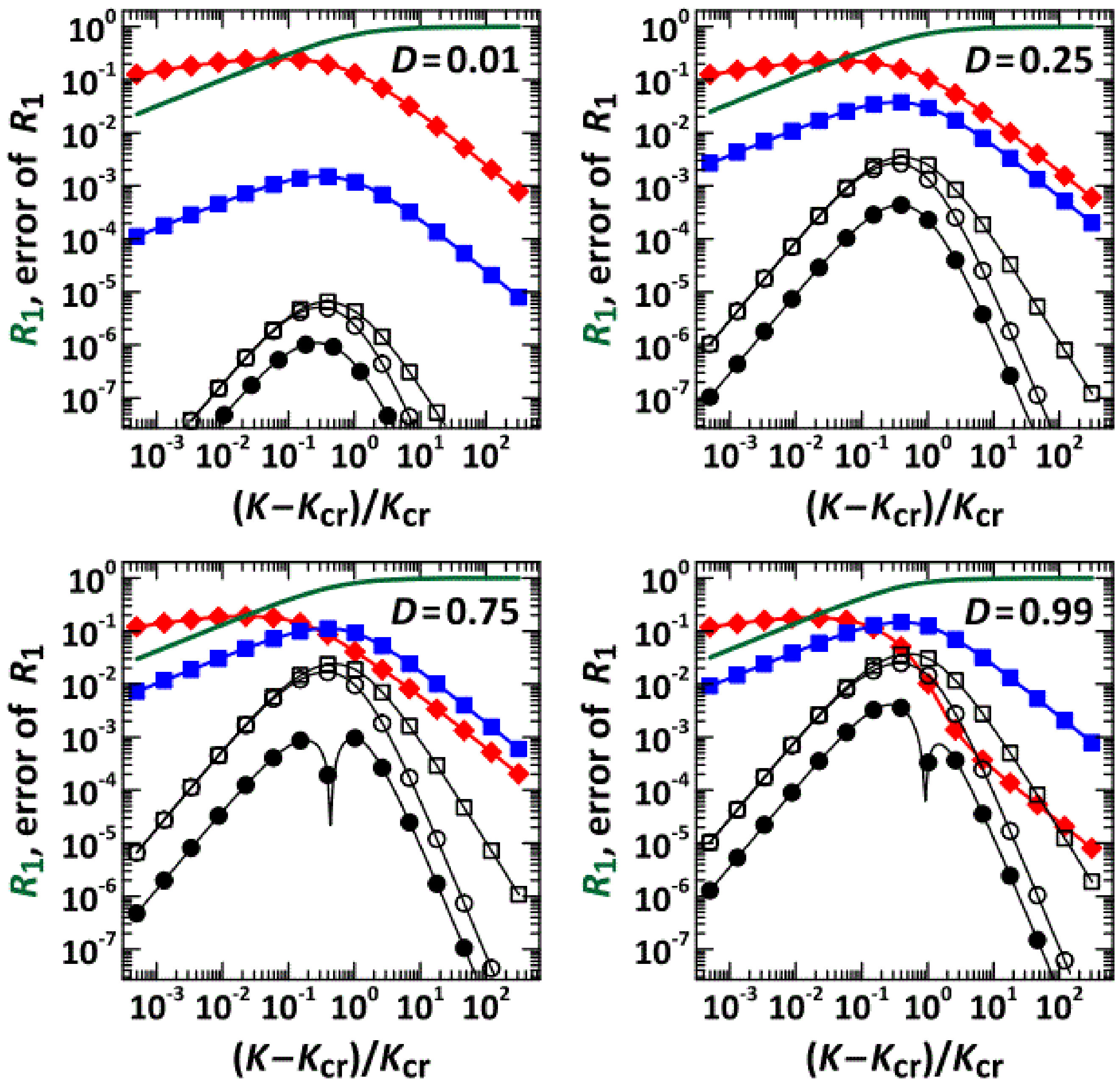


Fig: Kuramoto model with intrinsic noise

OA ansatz: **blue squares**

Gauss. ans.: **red diamonds**

$\kappa_3 = \mathbf{0}$: **open squares**

$\kappa_3 = \dfrac{3}{2}\dfrac{\kappa_2^2}{\kappa_1}$: **filled circles**

$\kappa_3 = \dfrac{3}{2}\kappa_2^2\kappa_1^*$: **open circles**

D.S.G., I.V.T., L.S.K., A.P., Chaos **28**, 101101 (2018)

$$\dot{\kappa}_m = m(i\Omega_0 - \gamma)\kappa_m + h\delta_{m1} - h^*\left(m^2\kappa_{m+1} + m\sum_{j=0}^{m-1}\kappa_{m-j}\kappa_{j+1}\right)$$
$$- \sigma^2\left(m^2\kappa_m + m\sum_{j=0}^{m-2}\kappa_{m-1}\kappa_{j+1}\right)$$

For $\sigma = 0$, extension of Ott–Antonsen solution $\kappa_1 \neq 0, \kappa_{2n} \neq 0, \kappa_{2n+1} = 0$:

$$\dot{\kappa}_1 = (i\Omega_0 - \gamma)\kappa_1 + h - h^*(\kappa_1^2 + \kappa_2)$$
$$\dot{\kappa}_2 = (i2\Omega_0 - 2\gamma - 4h^*\kappa_1)\kappa_2.$$
$$\kappa_{2n} = C_n(\kappa_2(t))^n, \ \{C_n\} = \left\{1, -\frac{1}{3}, \frac{2}{15}, -\frac{17}{315}, \ldots\right\}$$

This is a **two-bunch solution** with equipartition of elements between bunches.

IVT, DSG, LSK, AP, Radiophys. Quantum Electron. **61**, no.8–9, 640–649 (2019)

Analytical study of some problems without expansion truncation

Chimera States in Coupled Kuramoto Ensembles

$$\dot{\varphi}_k = \Omega + \frac{1+A}{2N}\sum_{j=1}^{N}\sin(\varphi_j - \varphi_k - \alpha) + \frac{1-A}{2N}\sum_{j=1}^{N}\sin(\phi_j - \phi_k - \alpha) + \sigma\xi_k(t),$$

$$\dot{\phi}_k = \Omega + \frac{1+A}{2N}\sum_{j=1}^{N}\sin(\phi_j - \phi_k - \alpha) + \frac{1-A}{2N}\sum_{j=1}^{N}\sin(\varphi_j - \varphi_k - \alpha) + \sigma\zeta_k(t).$$

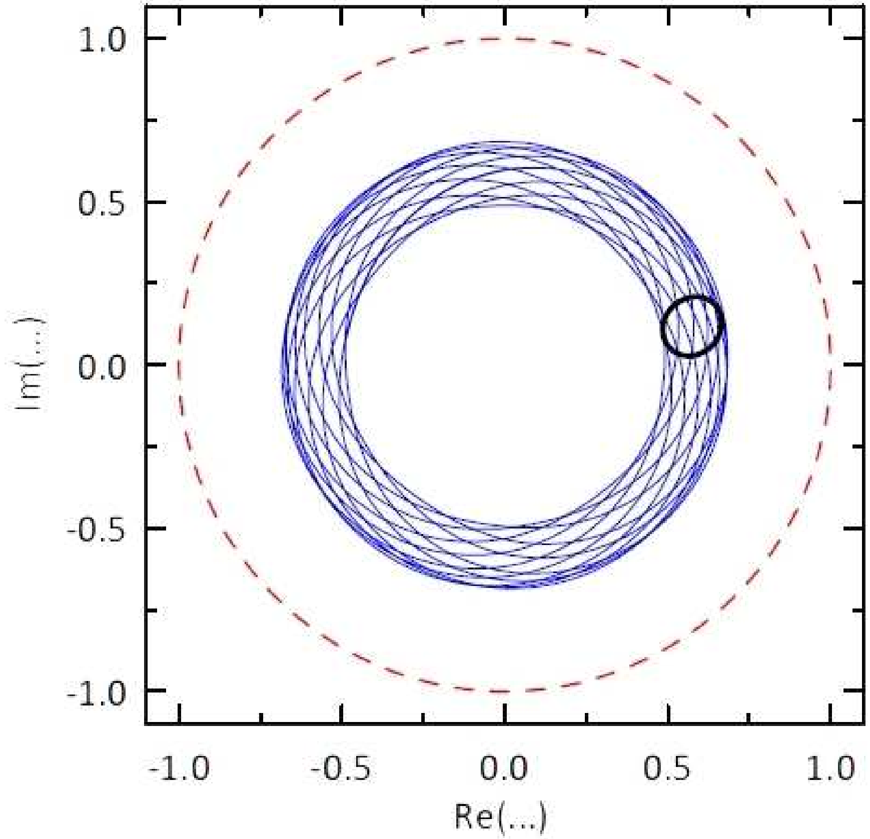
$\sigma = 0$:

D. M. Abrams, R. Mirollo, S. H. Strogatz, D. A. Wiley, PRL **101**, 084103 (2008)

$$\phi_1 = \phi_2 = ... = \phi_N = \Phi,$$

$$Z = \langle e^{i\varphi} \rangle, \quad 0 < |Z| < 1$$

$Ze^{-i\Phi}$: (i) static or (ii) periodic in time

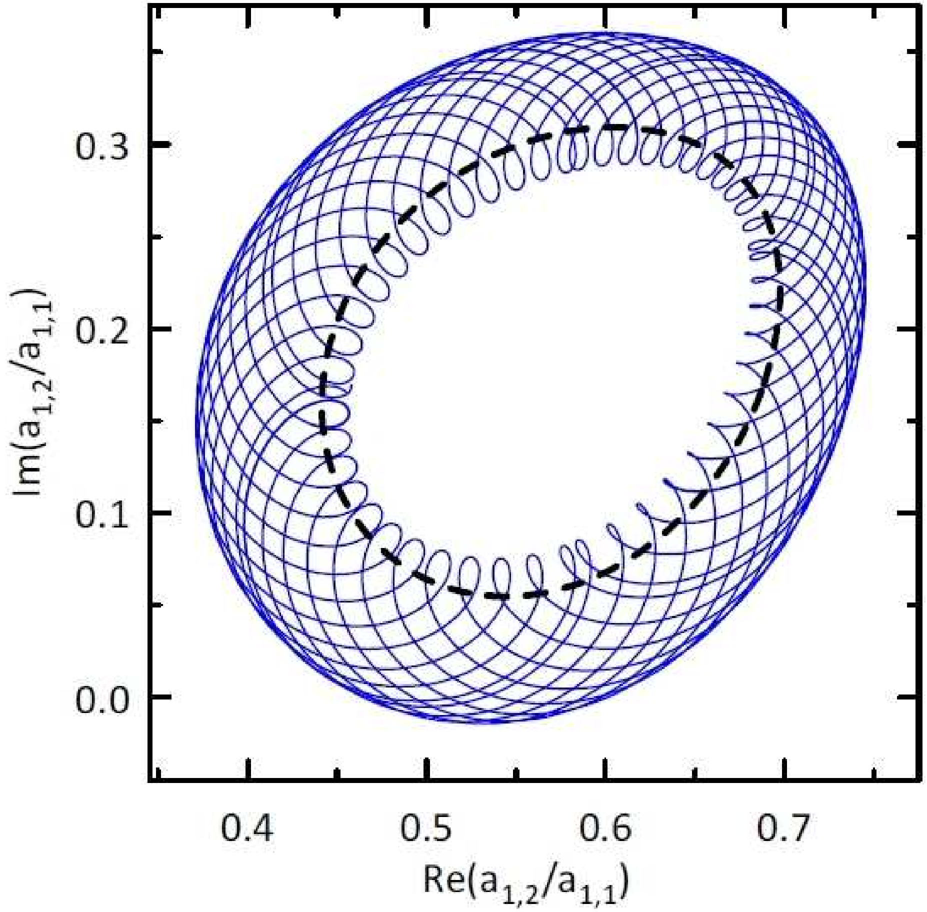


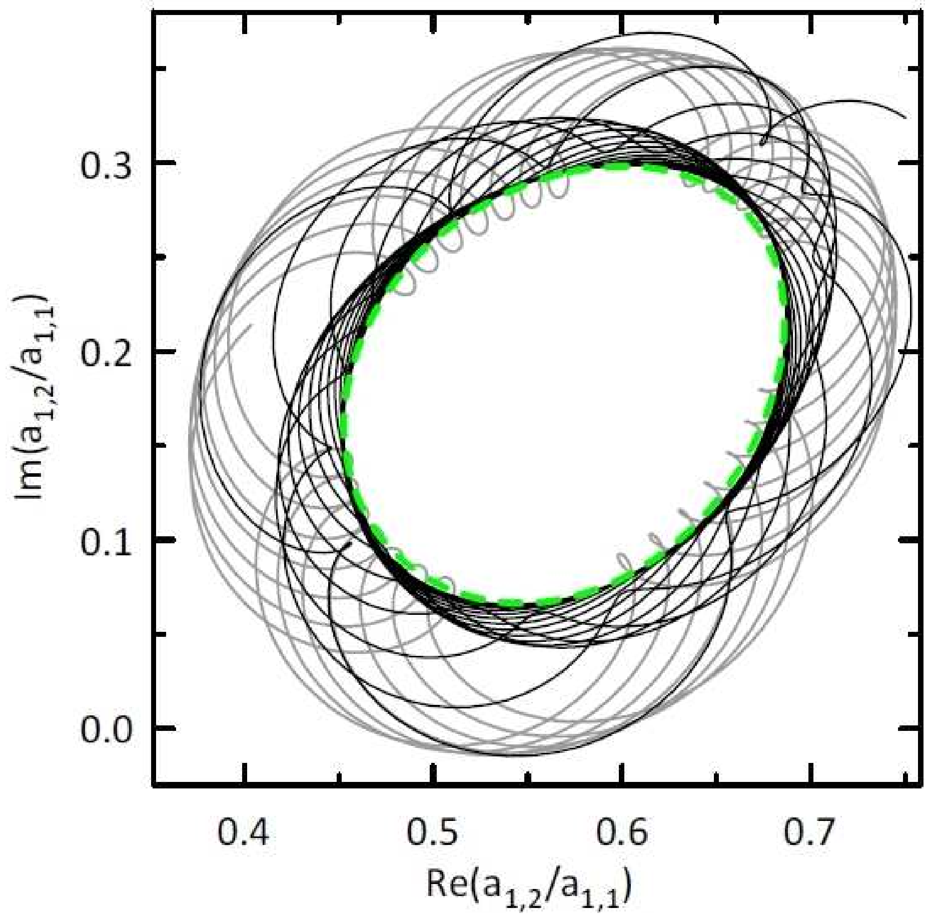
I.V.T. *et al.*, PRL **120**, 264101 (2018)

A. Pikovsky, M. Rosenblum, PRL **101**, 264103 (2008)

if φ_j are not on the OA manifold, + 1 frequency:

static -> periodic; periodic -> quasiperiodic





$A = \mathbf{0.3}, \alpha = \pi / \mathbf{2} - \mathbf{0.15}$. **Left** panel: Two-bunch state for $\sigma = \mathbf{0}$.
Right panel: Two-bunch state for $\sigma = \mathbf{0.01}$.

Conclusion

- **'Circular' cumulant approach for the extension of OA theory**
- **Hierarchies of cumulants**

  PRL **120**, 264101 (2018); Chaos **28**, 101101 (2018);
  Radiophys. Quantum Electron. **61**, no.8–9, 640–649 (2019).

Open questions:

- **Weak intrinsic noise makes the heir of OA manifold attracting. Or?**
- **Numerical instability of truncation with more than 2 cumulants.**
- **Relationships between the higher-order cumulants and $W(\psi,t)$.**

  D.S. Goldobin, Fluct. Noise Lett. **18**, no.2, 1940002 (2019).
- **Perturbation theory for other situations (small inertia, 'deterministic' perturbations)?**
- **Constrains on cumulant values; e.g., $|\kappa_2| \leq 1 - |Z_1|^2$.**