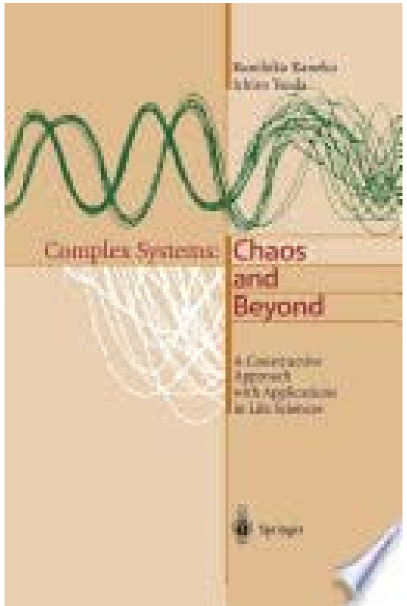
Revisit to Globally Coupled Maps after 30 year ; Hierarchical Clustering, Chaotic Griffith Phase, and High-dimensional-Torus-Chaos Transition

Kunihiko Kaneko, U Tokyo



Brief review: GCM,

- Clustering …? *Chimera?* 1989-90
- Chaotic Itinerancy 1989-90
- CI as Milnor Attractor Networks 1997-98
- Dominance of Milnor Attractors for N>5 2002

Chaotic Griffiths Phase in Coupled Map Network 2006

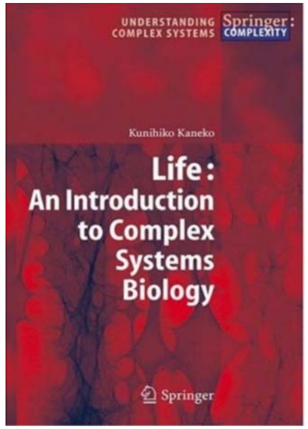
Chaos on/near High-dim Torus in Globally Coupled Circle Maps 2019

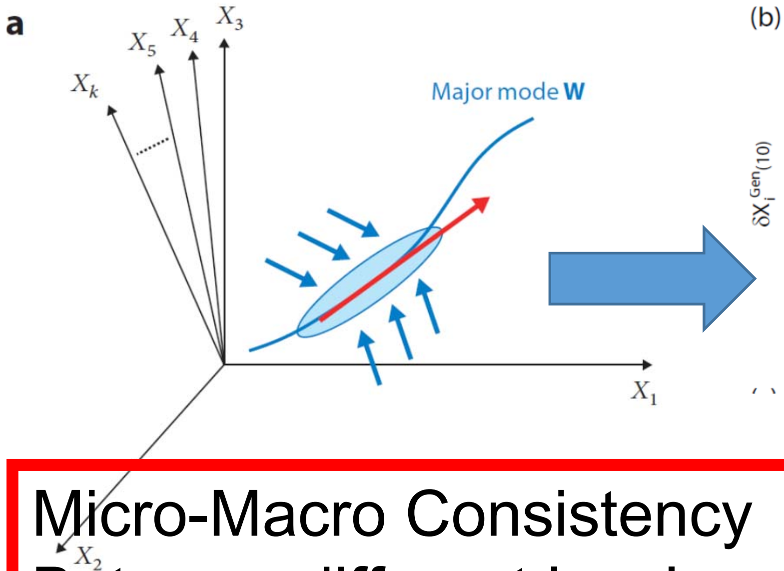
Beyond?

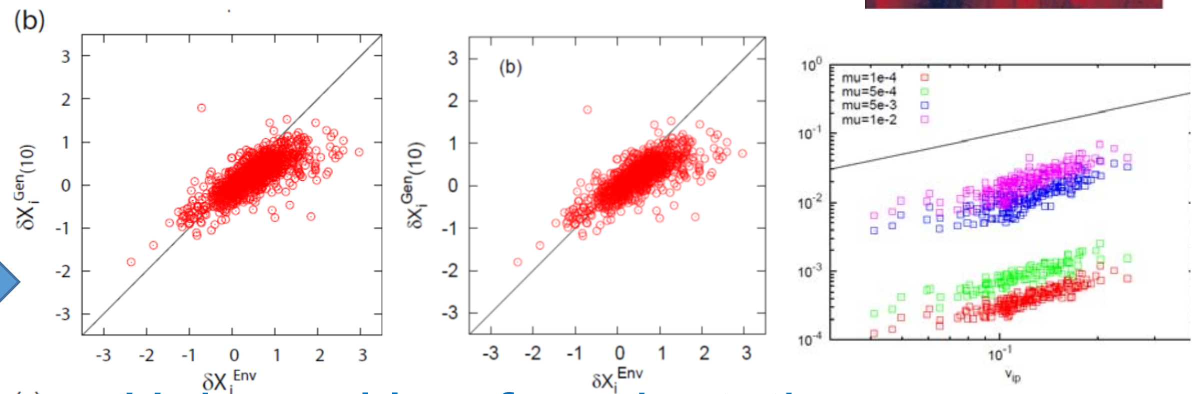
My current study: Universal Biology

Low-dimensional structure formed from high-dimensional phenotypic space ← robustness

(Furusawa, KK, Phys Rev E, 2018,
KK, Furusawa, Ann Rev Biophys 2018)







Micro-Macro Consistency
Between different levels
(molecule-cell-organism--)
(slow genetic change –
fast phenotypic dynamics
→ Universal law

Universal law for adaptation

(KK FurusawaYomo PRX2015)

Evolutionary LeChatelier Principle

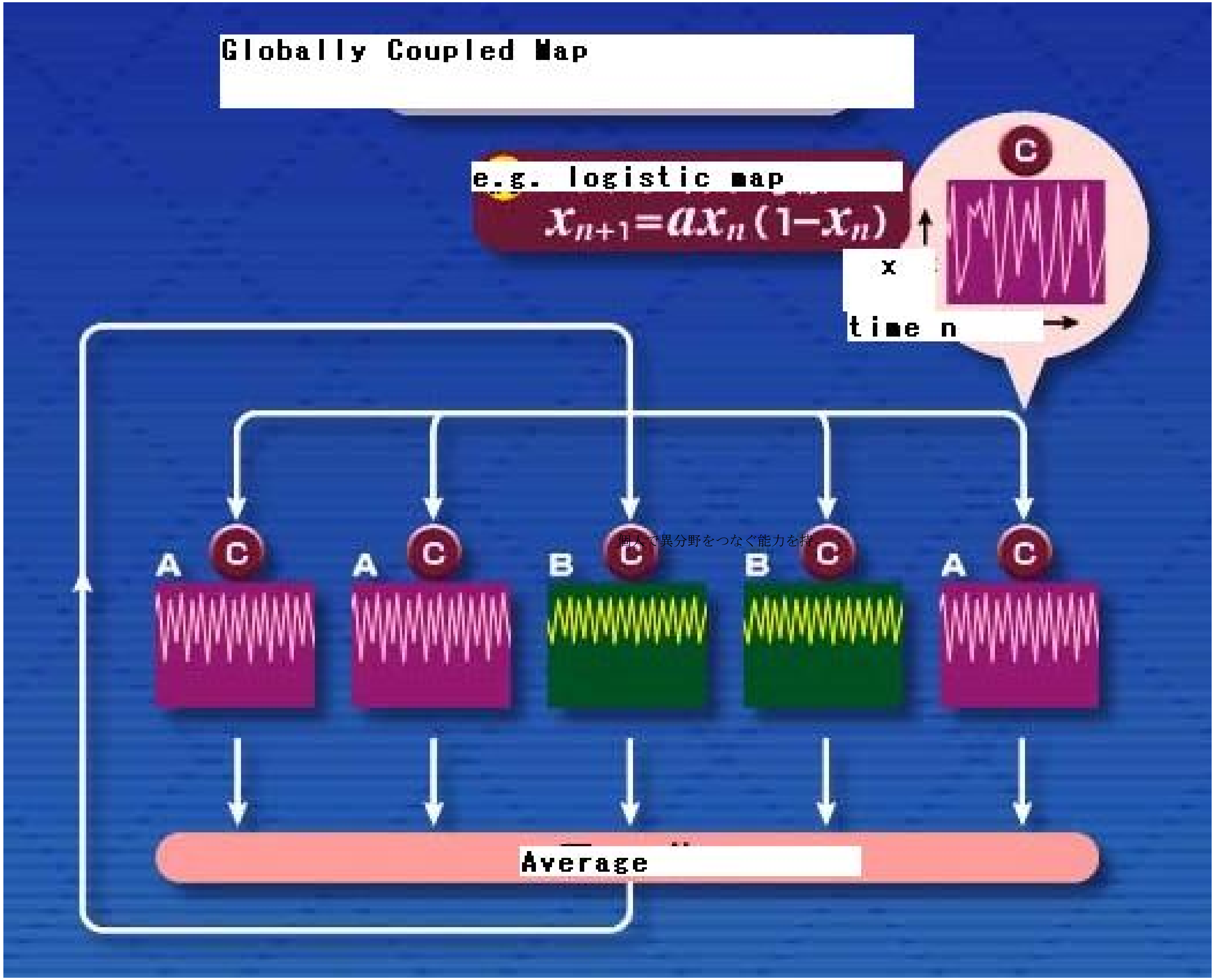
（Furusawa KK Interface 2015)

Evolutionary-Fluctuation-Response

+Vg-Vip Law (Sato et al2003,KK2006)

(→ direction in phenotypic evolution)

Complex System--- Multi-level Dynamics: a prototypic model



mean-field model for coupled map lattice

KK PRL1989 PhysicaD 1990

$$x_{n+1}(i) = (1-\epsilon) f(x_n(i)) + \frac{\epsilon}{N} \sum_{j=1}^{N} f(x_n(j)), \quad (1)$$

Globally coupled map (no spatial structure) (1989,KK)

logistic map $f(x) = 1 - ax^2$

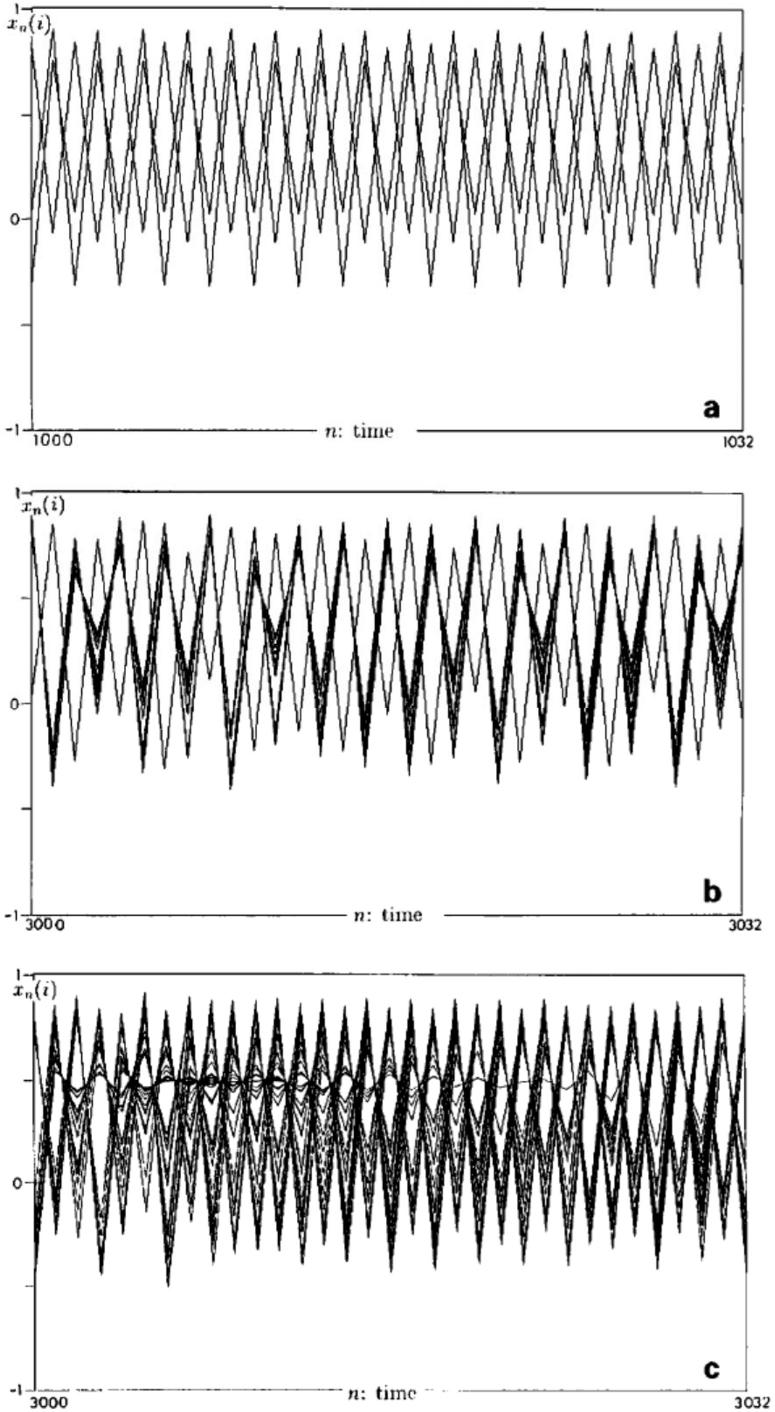
Equivalent with f(z)＝rz(1－z)

Cf Coupled map lattice → space-time chaos (1984,KK)

$$x_{n+1}(i) = (1-\epsilon) f(x_n(i)) + \tfrac{1}{2}\epsilon [f(x_n(i+1)) + f(x_n(i-1))], \quad (2)$$

Cf. synchronized state is stable if $\lambda_0 + \log(1-\epsilon) < 0.$

Synchronization of all elements with chaos is possible



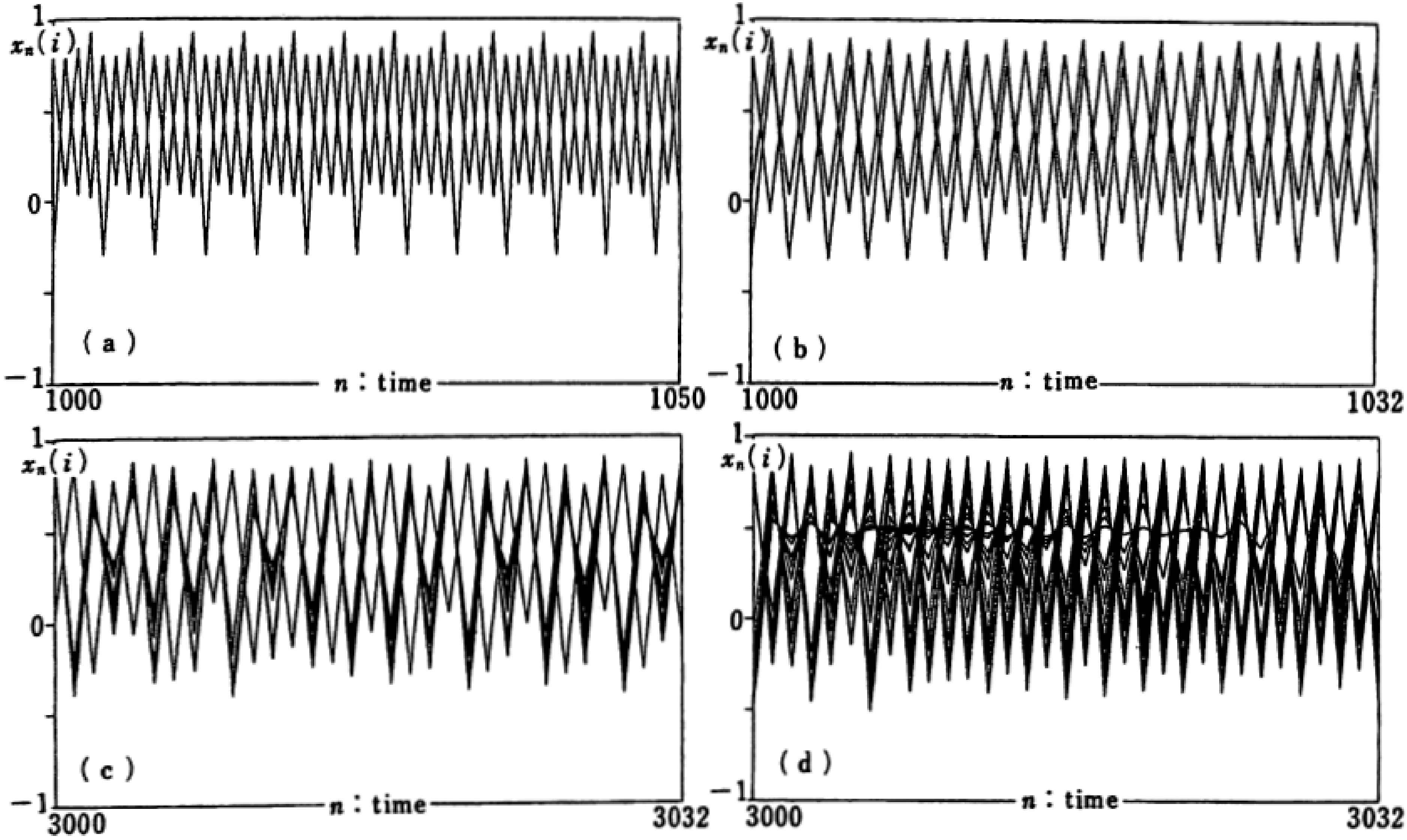
Clustering

3-clusters, with each synchronized oscillation

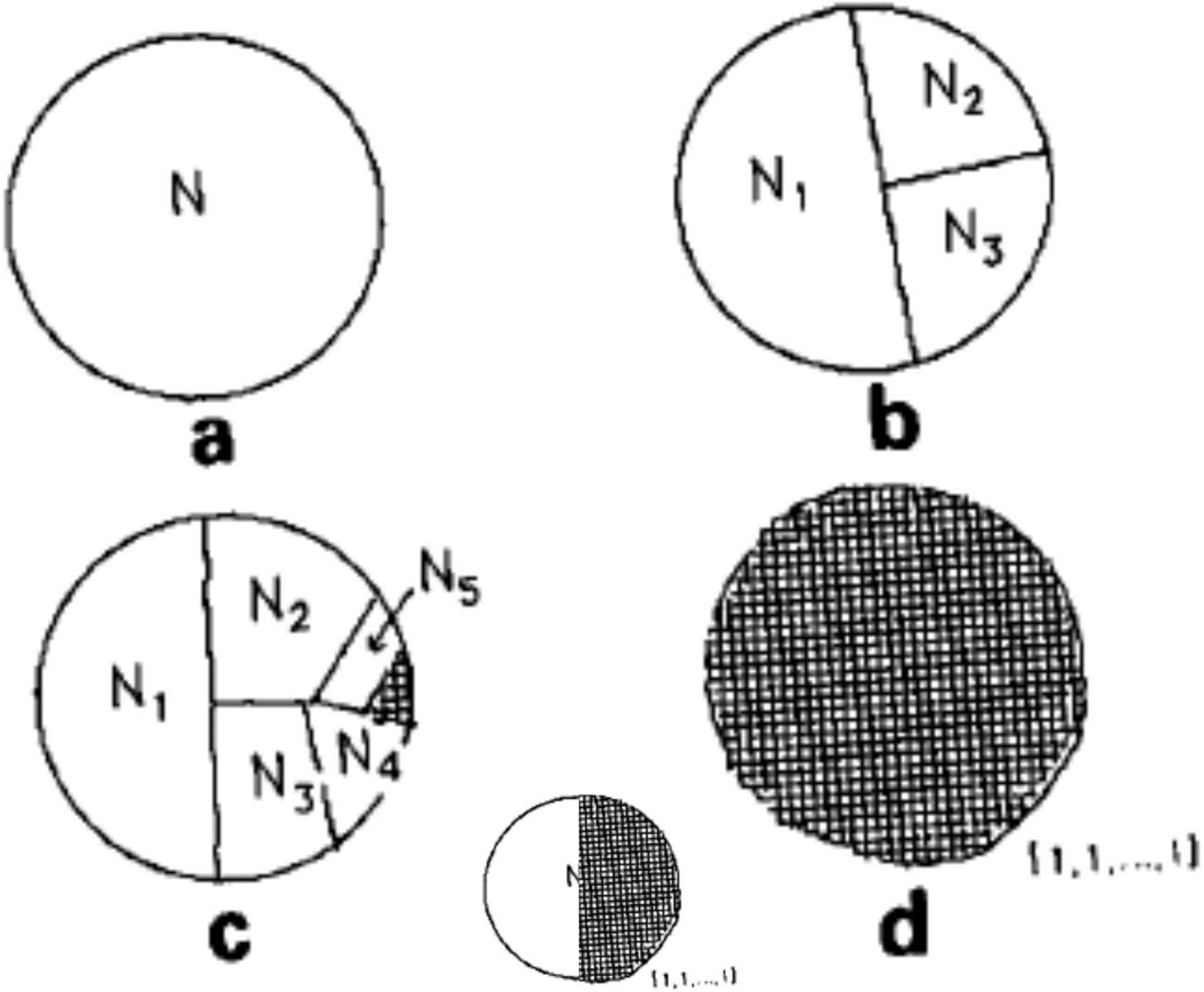
Differentiation of behavior of identical elements and identical interaction

Cluster of synchronized elements + non-synchronized elements

Desynchronized



1
$x_n(i)$
0
(a)
−1
1000
n : time
1050
1
$x_n(i)$
0
(b)
−1
1000
n : time
1032
1
$x_n(i)$
0
(c)
−1
3000
n : time
3032
1
$x_n(i)$
0
(d)
−1
3000
n : time
3032

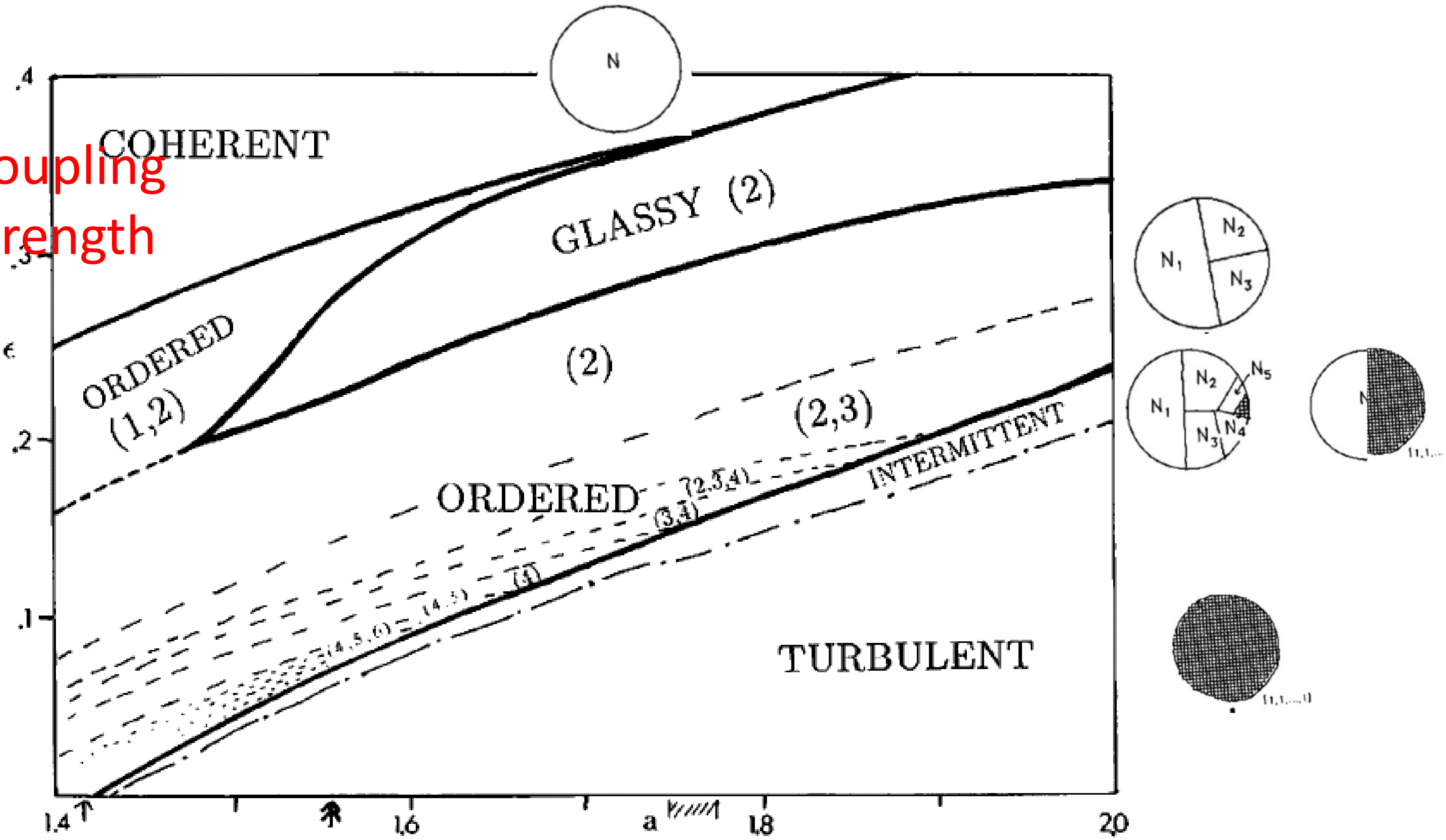


ig. 1. Schematic figure for clusterings: (a) Coherent attractor.
) Few clusters ($k = 3$). (c) Many-cluster attractor with un-
qual partition. (d) Many-cluster attractor with $k = N$.

Phase Diagram

ε

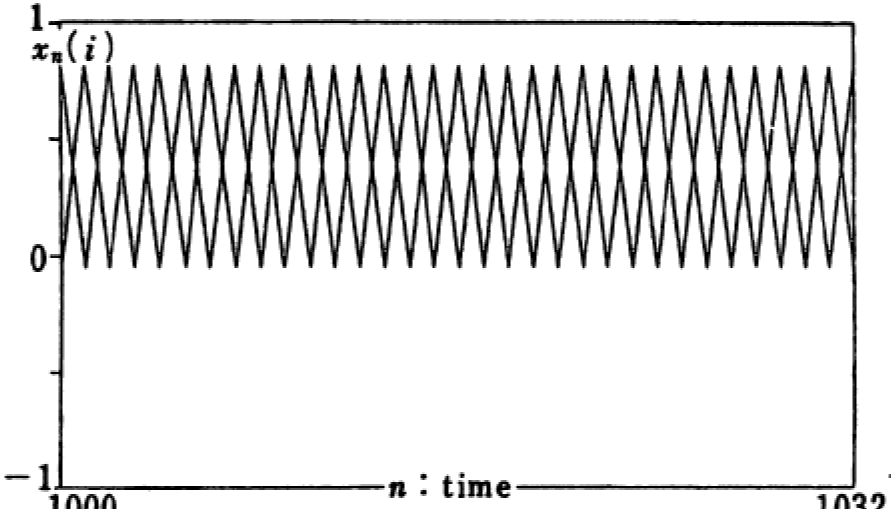
Coupling strength

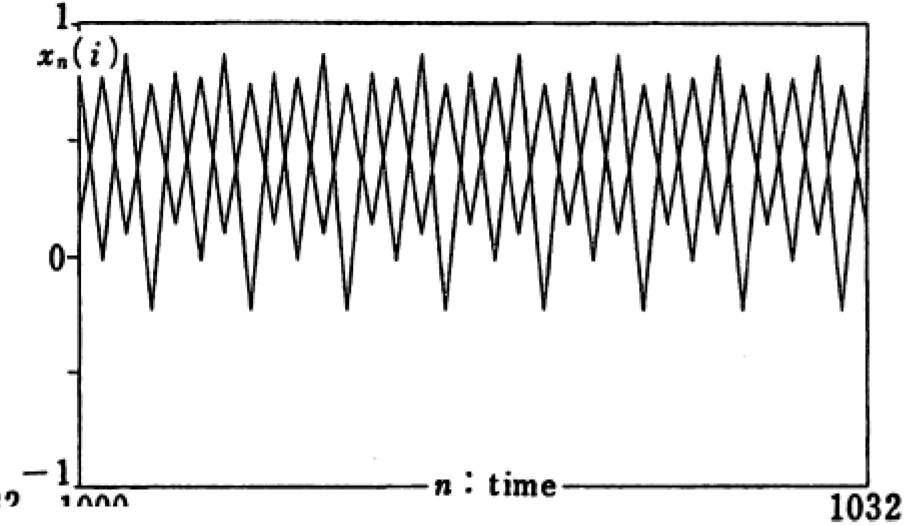


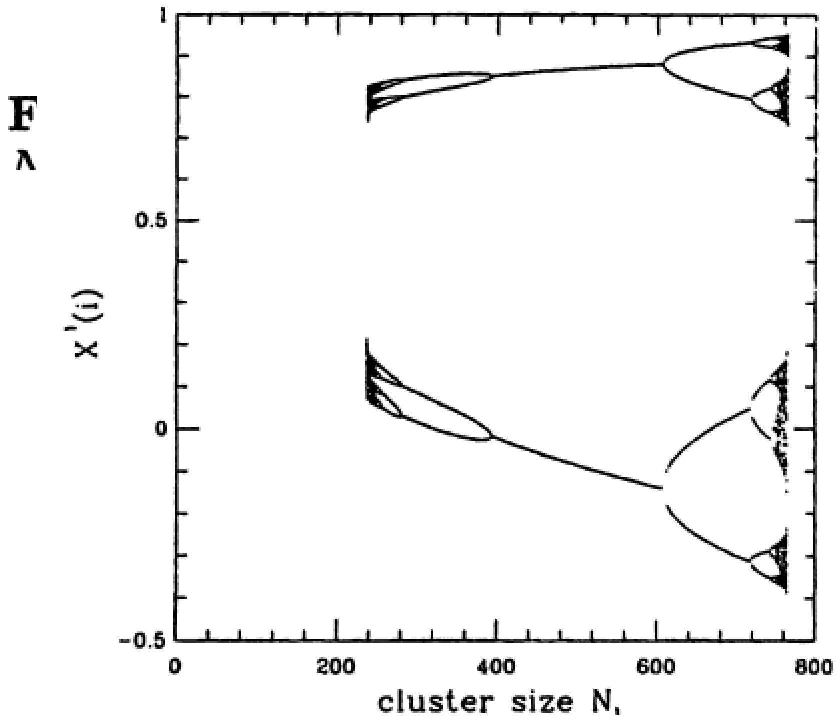
Onset of chaos

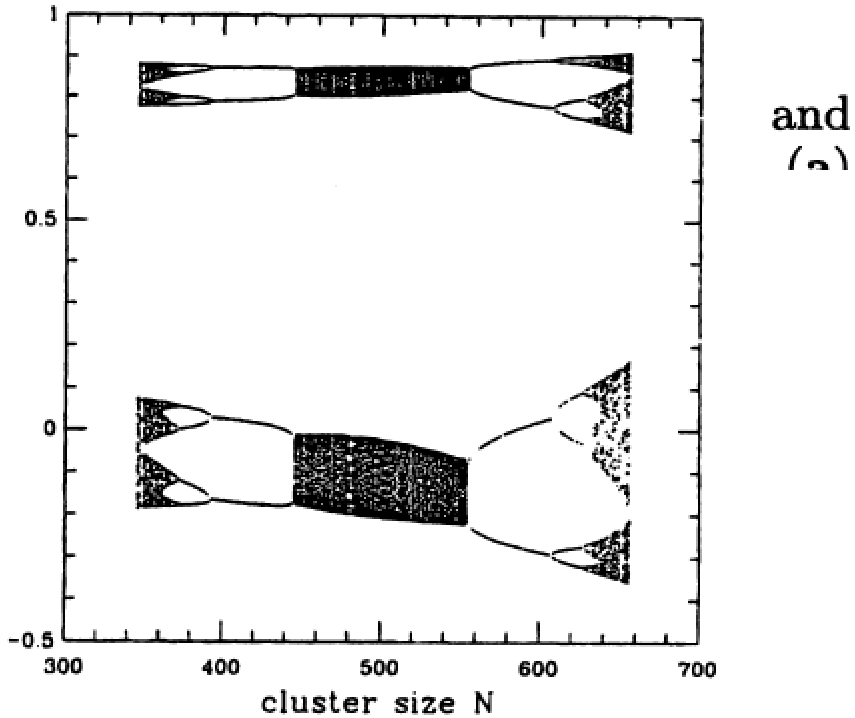
a: nonlinearity – strength of chaos

Many attractors : eg 2 cluster (N1,N2) ---
coded as ‘internal bifurcation’



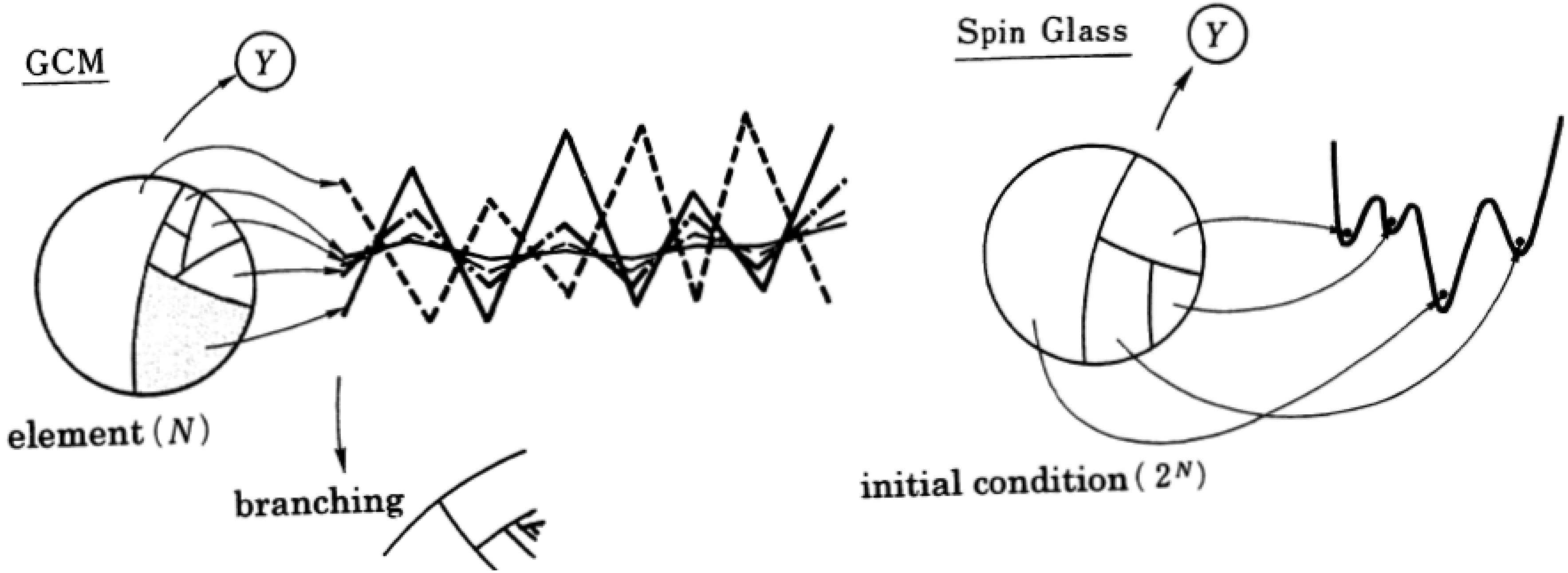






Partition Complexity in Hierarchically Clustered States

Similarity with spinglass



Fluctuation in partition remains even in N -> ∞
(KK, J. Phys 1992)

Remark 1:

clustering by

(i) ‘phase of oscillation’,

(ii) ‘amplitude’,

(iii) ‘frequency’

IN GCM mainly (i) (+ (ii),(iii))

(discussion with Walter Freeman around 1990, on the application to neuroscience) cf. clustering → (cell) differentiation (Furusawa,KK, Science 2012)

Remark 2 (kk,1989,90,.)

often large cluster + other desynchronized

e.g. (N-k, 1,1,1,…1) or (N-k’, 2,..2,1…,1)

Chimera? …. no spatial structure

(but global+ local can retain some spatial structure)

(Ouchi, KK 2000 Chaos)

Additional Remark:valid for continuous time (GC-Roessler,GCGL)

Chaotic Itinerancy:

effective degrees of freedom go down → stay at low-dimensional states ('attractor-ruin') → move back to high-dim chaotic state → come to another low-dim attractor-ruins *(in general)*

In GCM, formation/collapse of (almost) synchronized cluster

Np Number of Effective synchro clusters

s.t.x(i)∽ x(j) with precision P

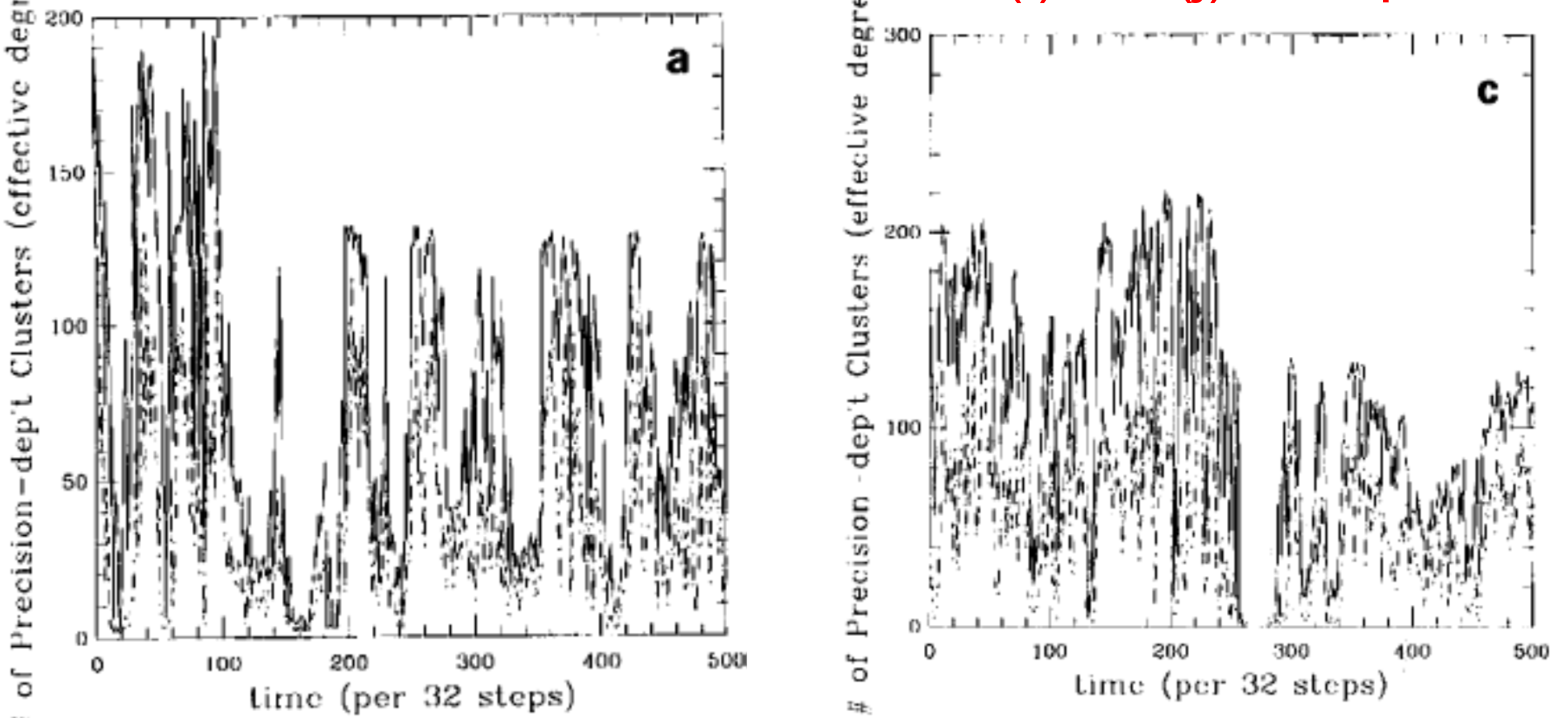
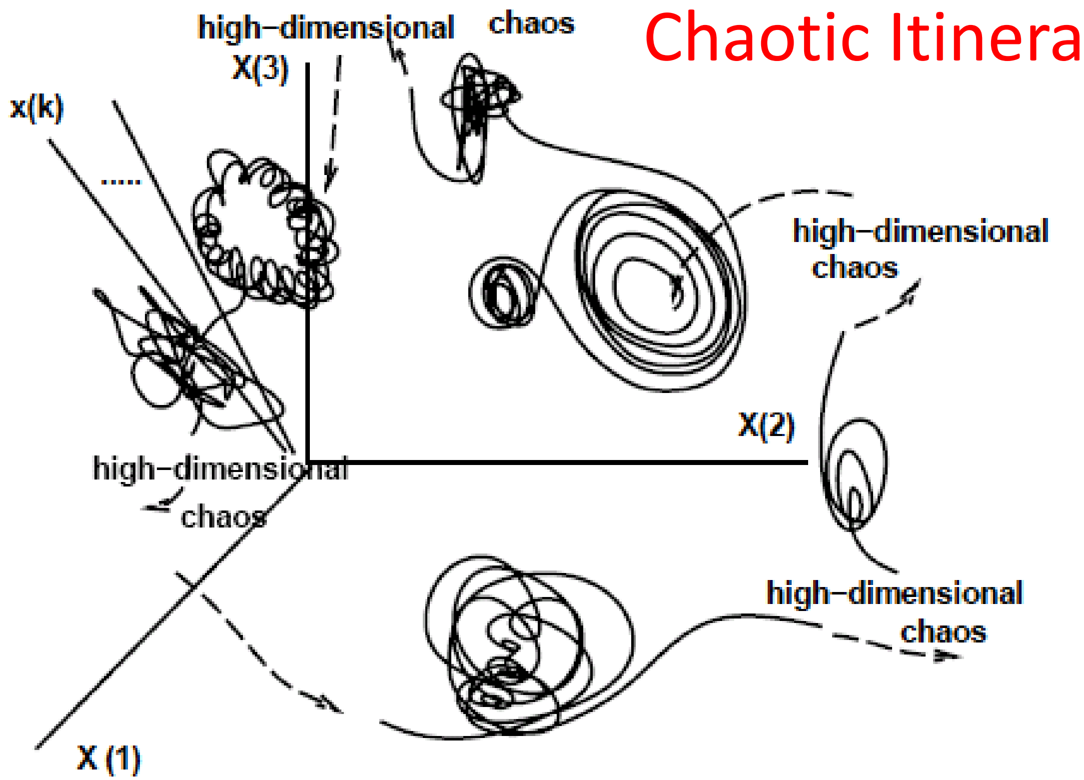


Fig. 23. Temporal evolution of number of precision-dependent clusters. k^P_{32n} is plotted as a function of time step n (per 32 steps).

Chaotic Itinerancy



If effectively 2-cluster → 2-dim ,

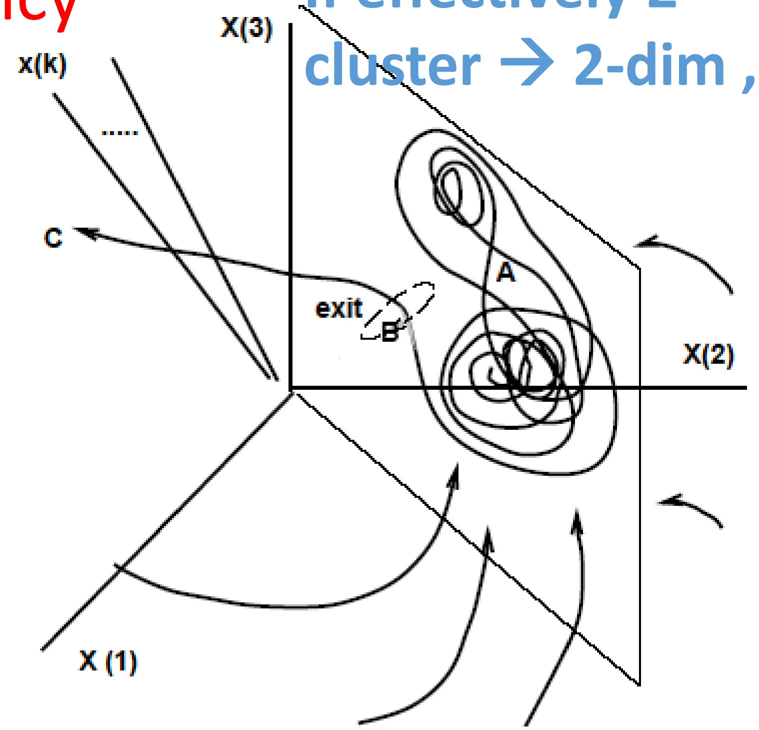


図 14: カオス的遍歴の模式

Commonly observed in high-dimensional dynamical system with (global or long-ranged interaction) GCM (89)

Neural network dynamics (Tsuda 1990)

Optical turbulence (Ikeda 1989)

KK Tsuda (Chaos 2003) - special issue with a variety of examples

currently actively studied in neuroscience

One possible interpretation of CI :

Network of 'Milnor-attractors ∽ attractor ruins'

Milnor attractor -- without asymptotic stability

(attractor and its basin boundary touches)

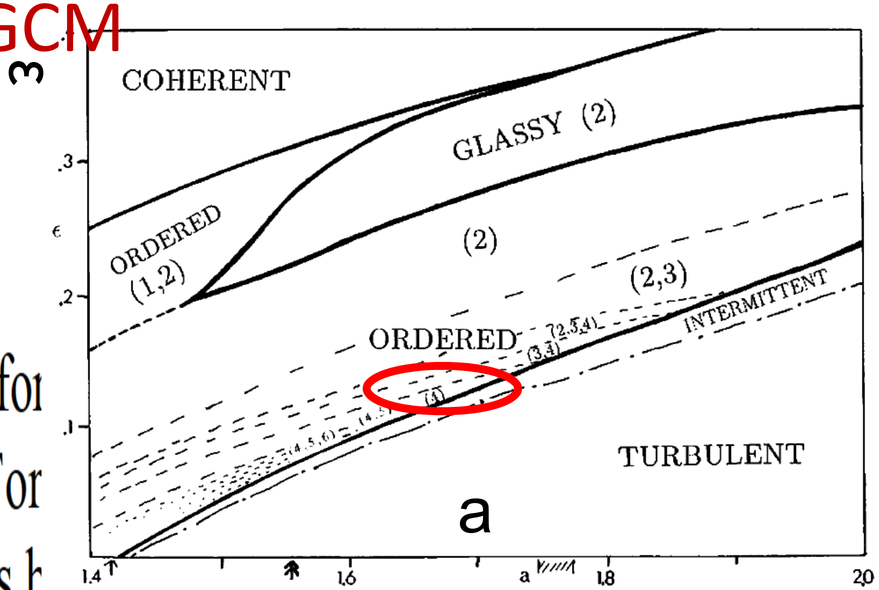
i.e., any small perturbation from it can kick the orbit out of the attractor, while it has a finite measure of basin

Observed; Milnor attractors large portion of basin for the partially ordered phase in GCM (kk,PRL97,PhysicaD98)

CI --- attraction to / leave from Milnor attractors

Combinatory many attractors in GCM

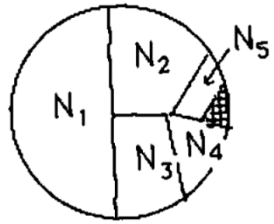
$$x_{n+1}(i)=(1-\epsilon)f(x_n(i))+\frac{\epsilon}{N}\sum_j f(x_n(j)),$$



where n is the discrete time and i being the index for elements ($i=1,2,\ldots,N=$ dimension of the system). For elements we choose $f(x)=1-ax^2$, since the model has b...

Cluster: group of elements such that x(i)=x(j);

Number of elements in each cluster; N1,N2,…,Nk



•at some parameter region many attractors with different clusterings

Due to the symmetry there are

$$M(N_1,\ldots,N_k)=(N!/\Pi_{i=1}^{k}N_i!)\Pi_{\text{oversets of } N_i=N_j}(1/m_\ell!)$$

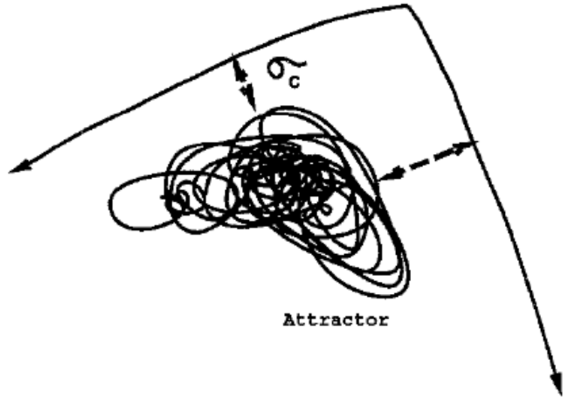
attractors of the same clusterings --

combinatorially many increase with the order of (N-1)!

or so (KK,PRL89)

Attractors that collide with their basin boundary (σ_c=0), yet have large basin volume

("Milnor Attractor")



Dominant at some parameter region

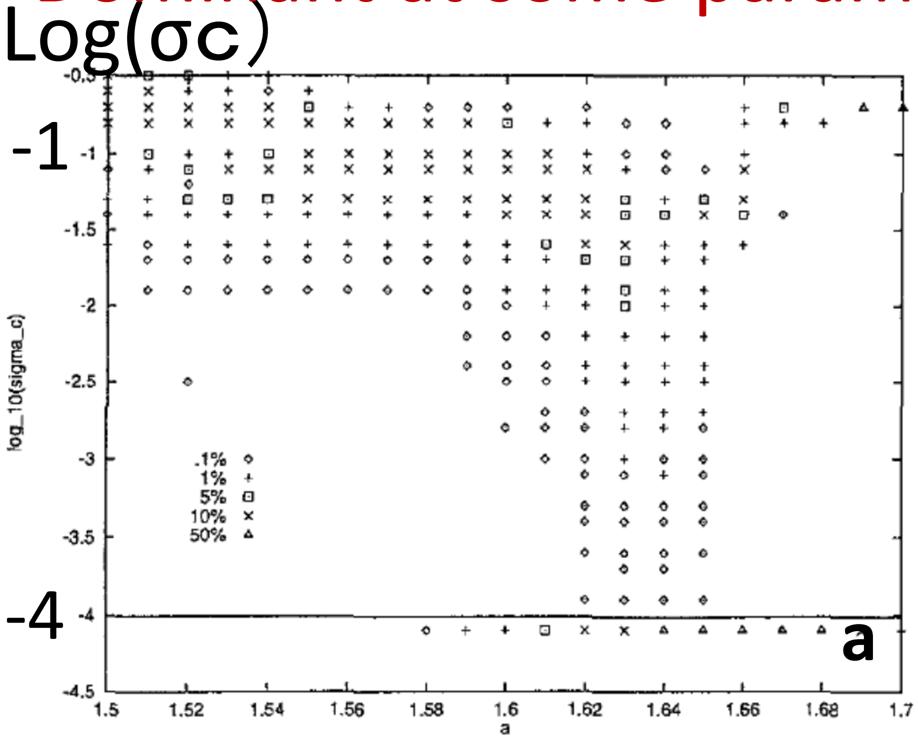
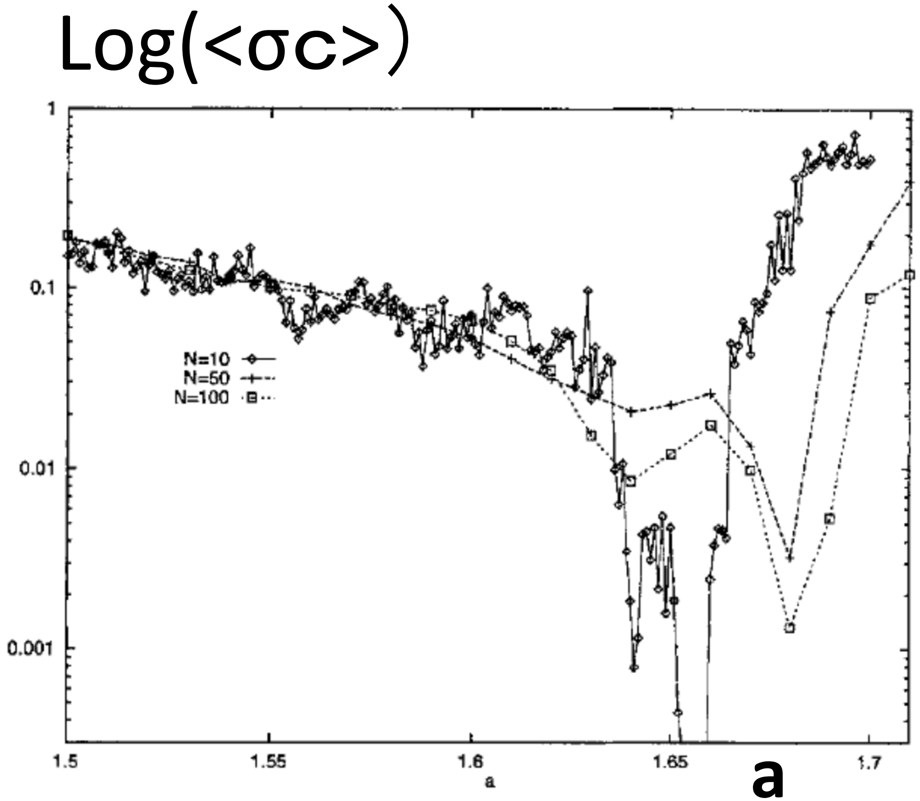


Fig. 7. Dependence of σ_c on the parameter a, for $N = 100$. By measuring σ_c for attractors fallen from 10^4 random initial conditions, a histogram of $\log_{10} \sigma_c$ is constructed with a bin size 0.1. The number of initial conditions leading to $\log_{10} \sigma_c$ within



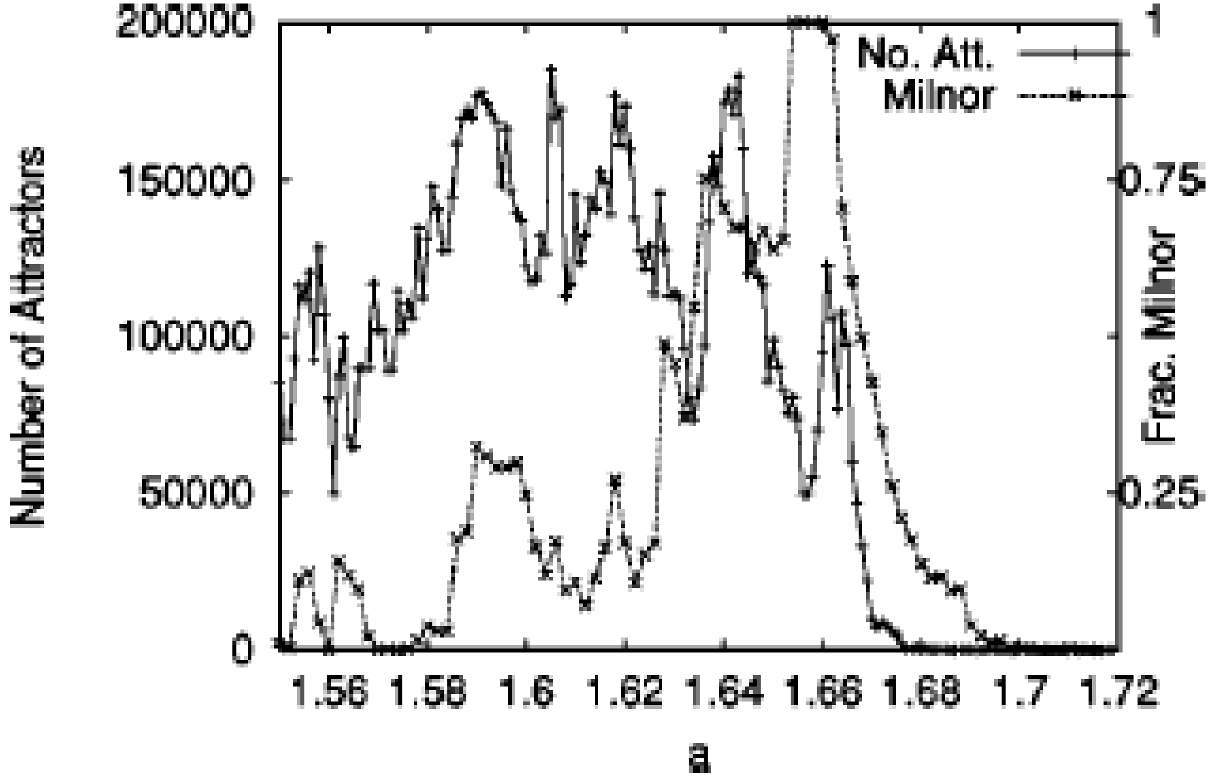


FIG. 3. The number of attractors $(+)$ estimated from simulations over 10^5 initial conditions. The estimated number of attractors is plotted as a function of a. $N = 10$. All attractors that are concluded to exist by the symmetry argument are also counted. The basin fraction of Milnor attractors obtained in the same way as in

The fraction of basin (i.e. initial values) for Milnor attractors,
Plotted as a function of Logistic map parameter

Note! Fraction is almost 1 for some region

Result for N=10,50,100
….

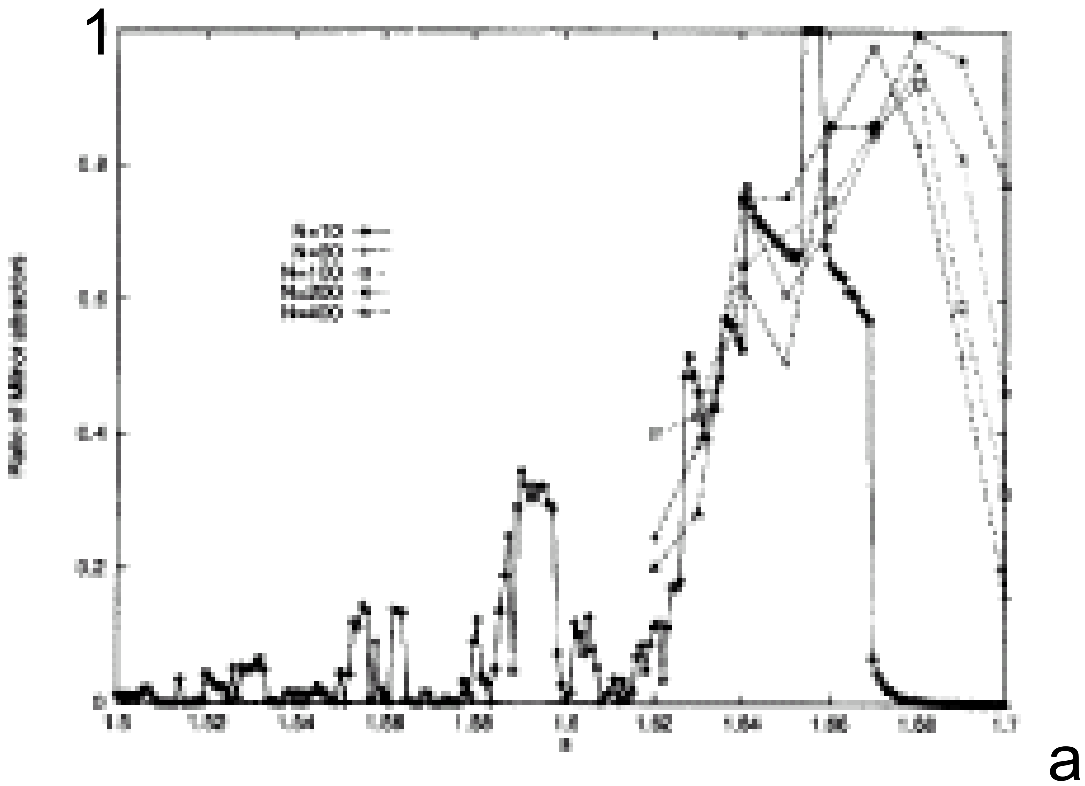


Fig. 9. The basin volume ratio of Milnor attractors with the change of a. For each a, we take 1000 initial conditions, and iterate the dynamics over 100 000 steps to get an attractor. We check if the orbit returns to the original attractor, by perturbing each attractor by $\sigma = 10^{-7}$ over 100 trails. If the orbit does not return at least for one of the trails, the attractor is counted as a Milnor one. For $N = 10$, the ratio is measured for $1.5 < a < 1.7$ with the increment 0.001, while for larger sizes it is measured only for $1.62 < a < 1.7$ with the increment 0.01.

Kk,97

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(attractor and its basin boundary touches)

i.e., any small perturbation from it can kick the orbit out of the attractor, while it has a finite measure of basin

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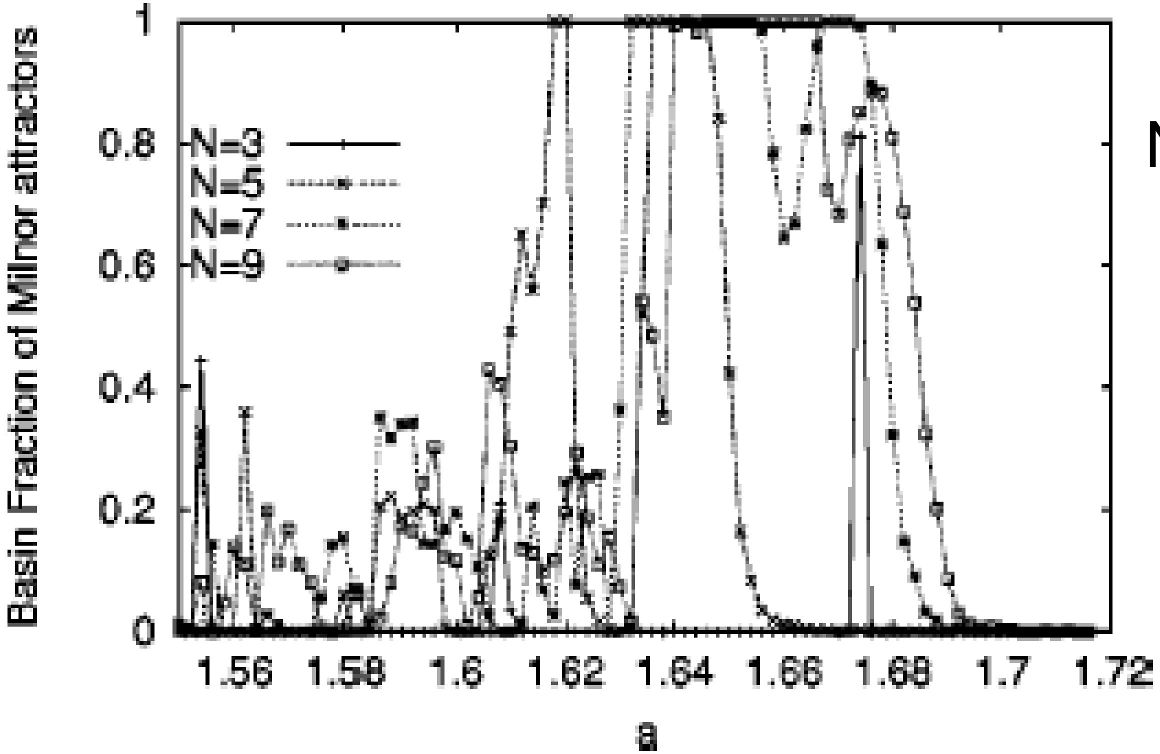
CI --- attraction to / leave from Milnor attractors

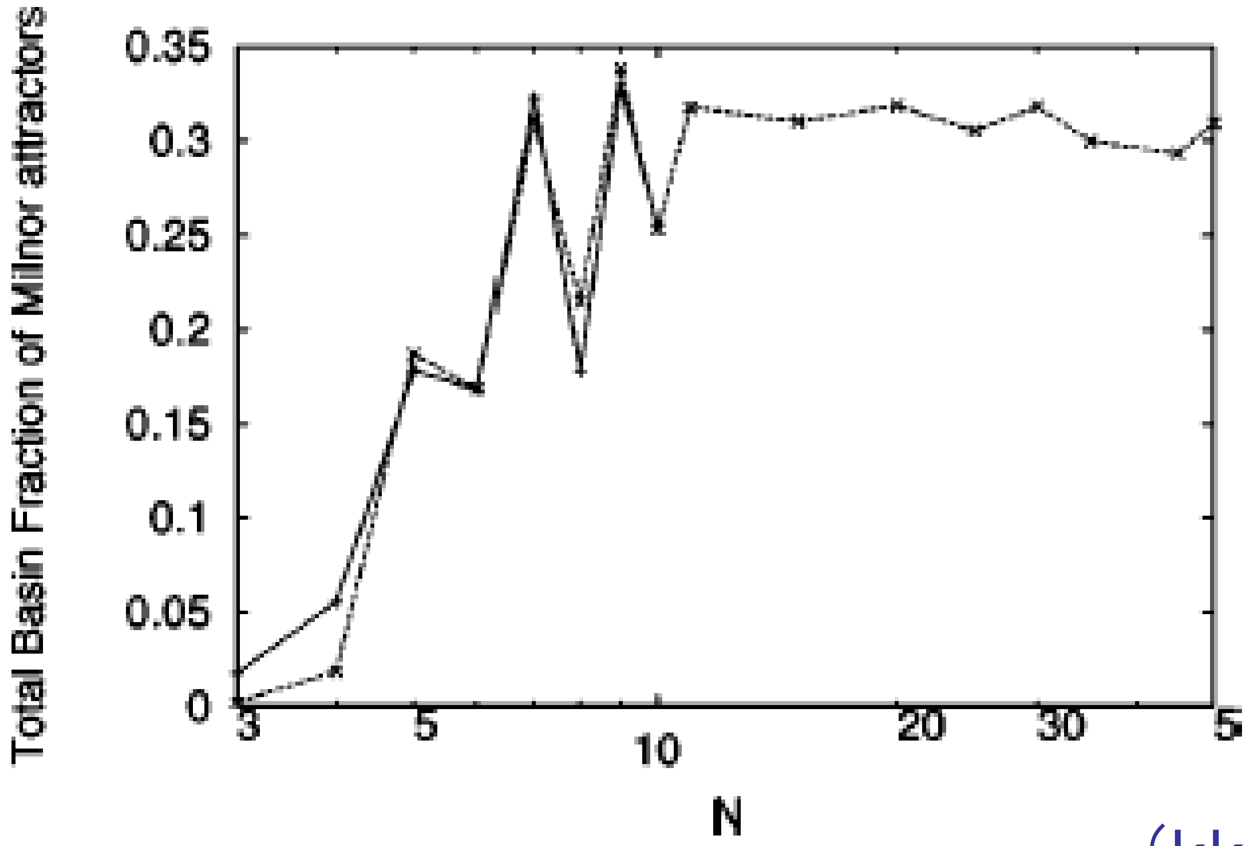
The Milnor attractors become dominant around N＞~（5－8）



N=3, almost 0
5, few cases
7,8,9,.. dominant

FIG. 1. The basin fraction of Milnor attractors plotted as a function of the parameter a, for $N=3$, 5, 7, and 9. For the present simulations, we take 1000 randomly chosen initial conditions, and iterate 10^5 steps. Then the orbit is perturbed as $x_n(i)+10^{-10}\sigma_i$,

The Milnor attractors become dominant around N ＞~（5－8）



Dependence
On the
Number of
Elements N

(accumulation
over
1.55<a<1.72)

（kk、PRE,2002)

FIG. 2. The average fraction of the basin ratio of Milnor attractors. After the basin fraction of Milnor attractor is computed as in Fig. 1, the average of the ratios for parameter values *a* =1.550,1.552,1.554, . . . ,1.72 is taken. This average fraction is

Magic No. 7 ± 2 (cf Ishihara, KK, PRL 2005)

- Why?

Conjecture by combinatorial explosion of basin boundaries

Simple separation x(i)>x* or x(i)<x*; one can separate 2 ^N attractors by N planes. In this case the distance between attractor and the basin boundary does not change with N

but **The boundary makes combinatorial explosion ----**

Order of (N-1)! ← many ways of partition

On the other hand, consider a boundary given by some condition for $[x(1),\ldots,x(N)]$. In the present system with global (all-to-all) couplings, many of permutational change of $x(i)$ in the condition give also basin boundaries. Here the condition for the basin can also have clustering $(N_1,\ldots,N_k)$, since the attractors are clustered as such. Then there are $M(N_1,\ldots,N_k)$ partitions by boundaries equivalent by permutations. The number of regions parti-

$$M(N_1,\ldots,N_k)=(N!/\Pi_{i=1}^{k}N_i!)\Pi_{\text{subsets of } N_i=N_j}(1/m_\ell!)$$

- The number of basin boundary planes has combinatorial explosion, as factorial wins over exponential ($(N-1)! > 2^N$ at N=6).
- Then, the basin boundary is ‘crowded’ in the phase space. Thus often attractors touch with basin boundaries

→ dominance of Milnor attractors

(complete symmetry is unnecessary)

When combinatorial variety wins over exponential increase of the phase space, ‘complex dynamics’

(also in neural net model, Ishihara,kk 2005,PRL).

If elements more than 7 are entangled, clear separation behavior is difficult

cf magic number 7±2 in psycology

Chaotic Griffiths Phase with Anomalous Lyapunov Spectra in Coupled Map Networks

PRL 117, 254101 (2016) PHYSICAL REVIEW LETTERS week ending 16 DECEMBER 2016

Chaotic Griffiths Phase with Anomalous Lyapunov Spectra in Coupled Map Networks

Kenji Shinoda* and Kunihiko Kaneko†

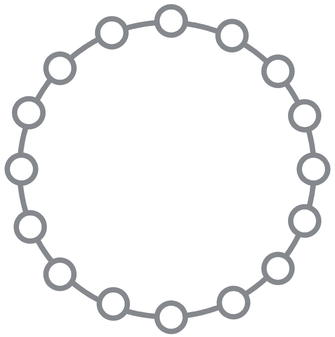
Shinoda, KK, PRL 2016

Randomly Coupled Map(RCM)

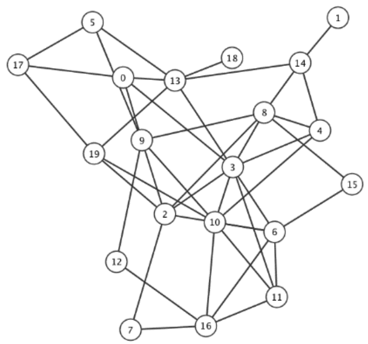
$$x^i_{n+1} = (1-\epsilon)f(x^i_n) + \frac{\epsilon}{K^i}\sum_{j=1}^{N} T^j_i f(x^i_n)$$

K:degree of a element, T: adjacency matrix

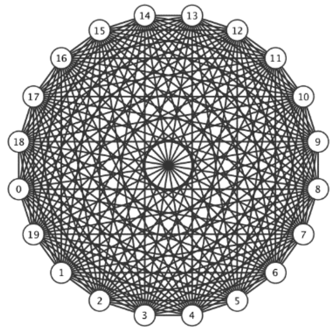
Sparse Limit ⟷ Dense Limit



ex. CML



RCM

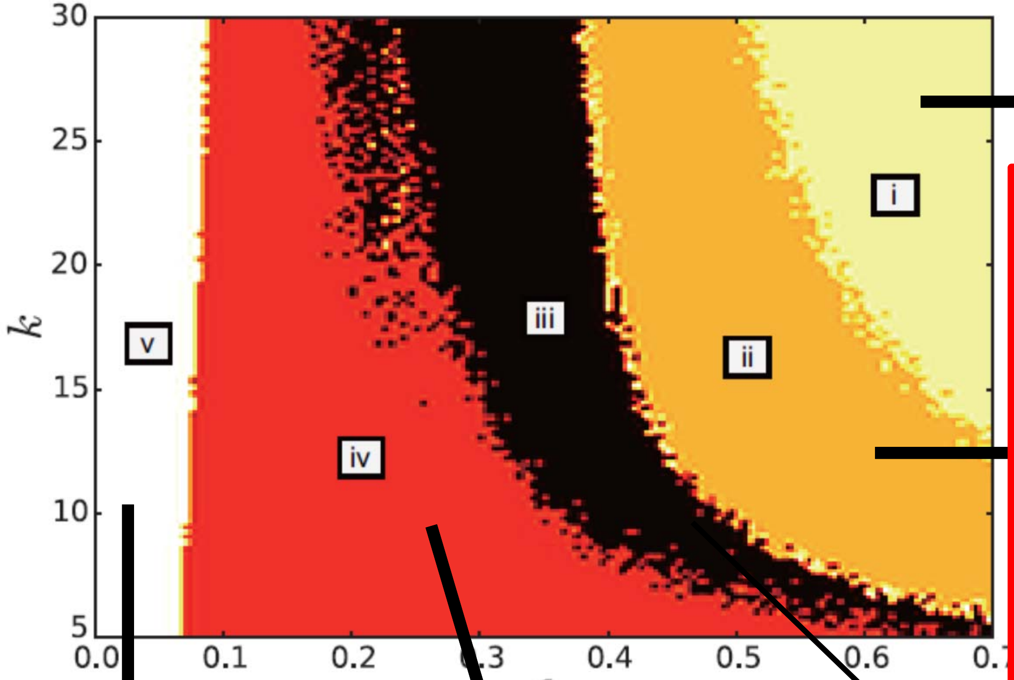


GCM

Studying RCM, the properties of the border between CML and GCM will become clear, and new effect which is dependent on its degree will be discovered.

Phase Diagram

Connectivity k

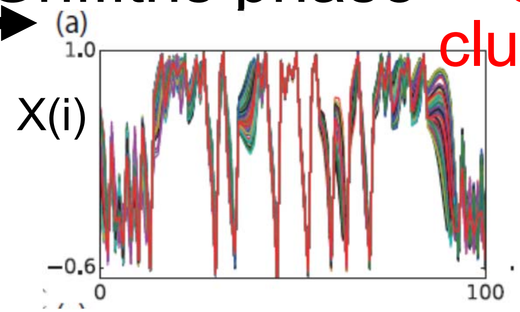


Coupling ε

Chaotic synchronization

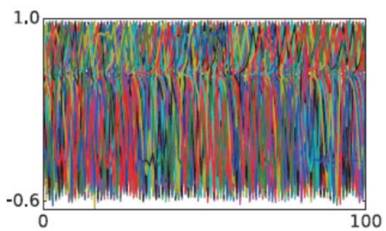
Chaotic Griffiths phase

Formation/ Collapse of large synchro cluster

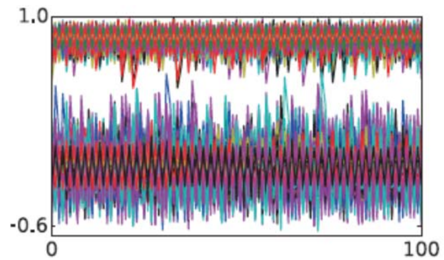


Time series per 2 steps

Fully chaotic



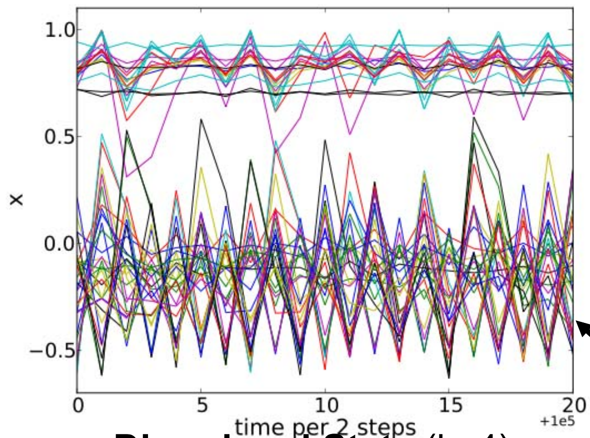
Frozen chaos with macro order



Ordered

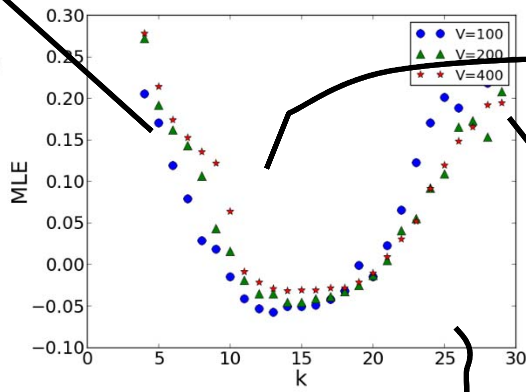
Order for optimal degrees of connection?
– to eliminate chaos

N=50, a=1.7, ε=0.38 (Coherent Phase@GCM)

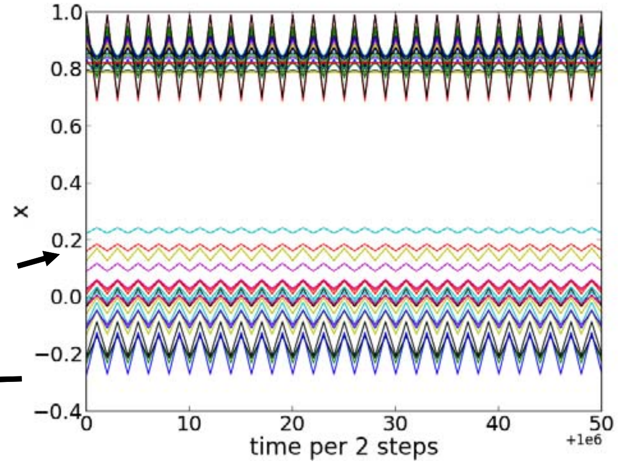


Disordered State (k=4)

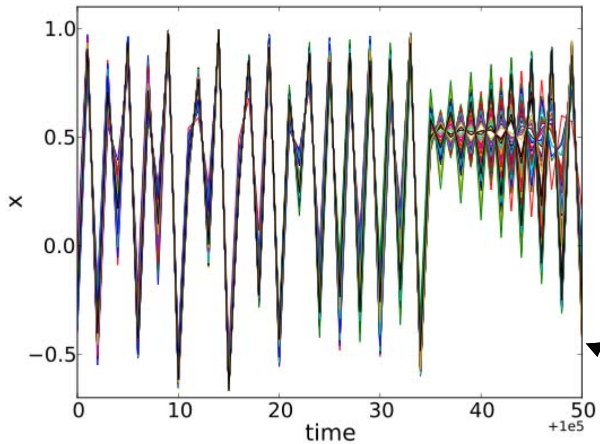
Maximum Lyapunov Exponent



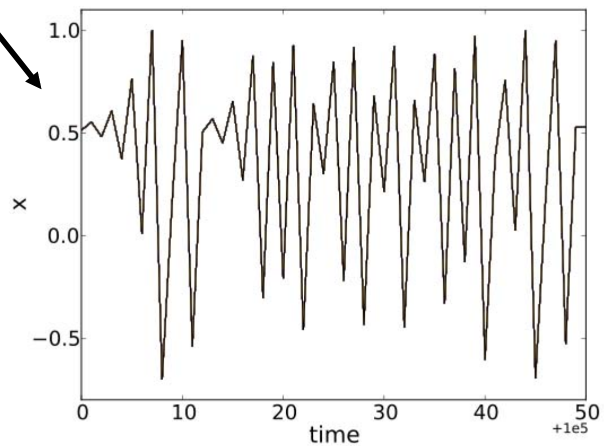
Degree k



Ordered State (k=10)



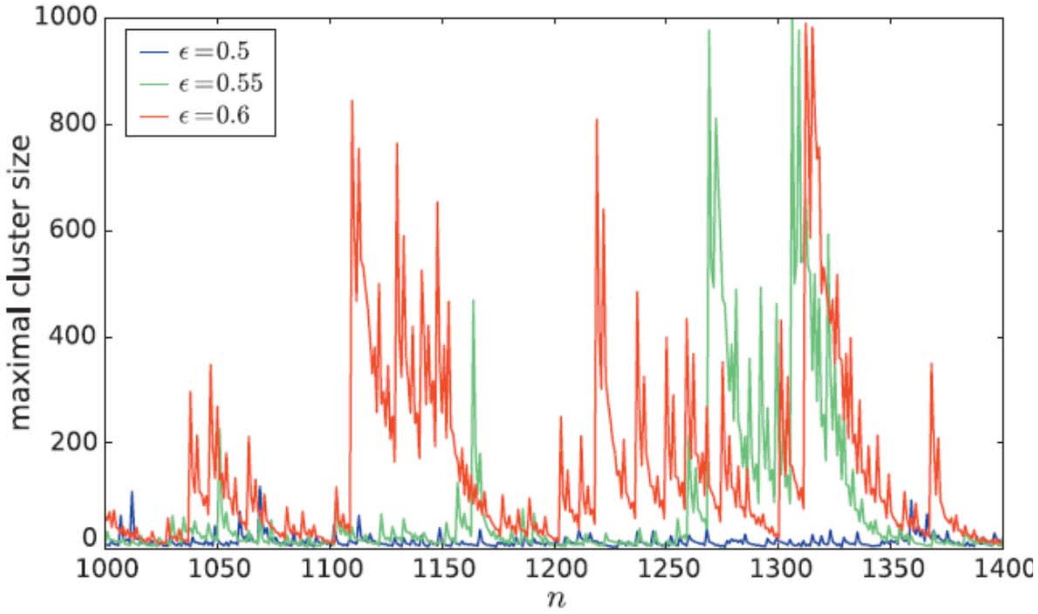
Chaotic Itinerancy (k=40)



Coherent State (k=49)
--GCM

Synchronization-Desynchronization process in Chaotic Griffiths Phase

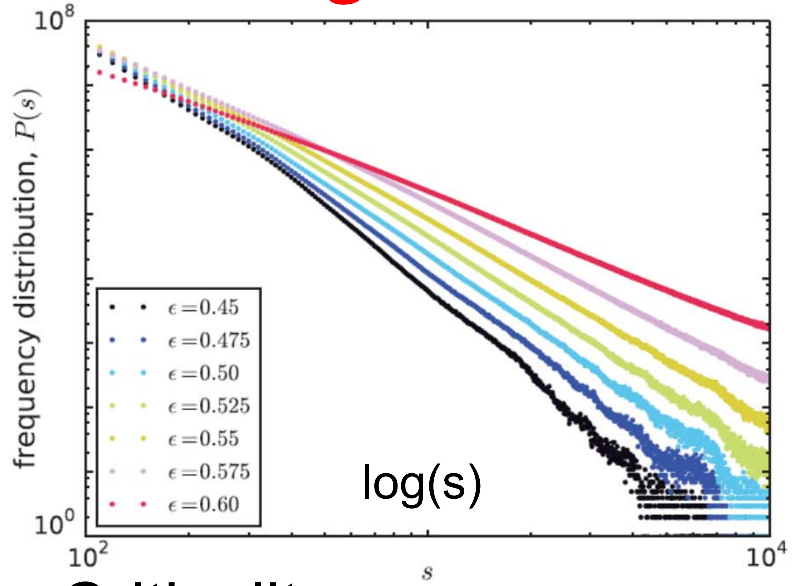
Temporal evolution of maximum synchro-cluster size s (N=1000)



Cluster=synchronized within the **resolution .001**

Power law distribution of synchro- cluster sizes

log P(S) $s^{-\alpha}$



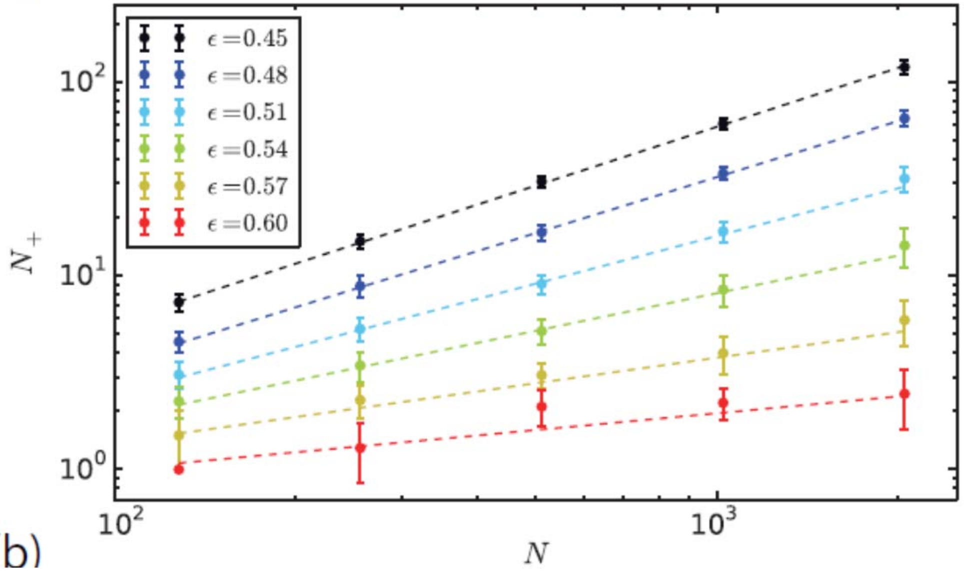
Criticality over a range of parameters

Chaotic Itinerancy (CI)

Exponent α changes with parameters

Number of positive Lyapunov exponents is scaled with anomalous power N^{β} Exponent β changes with parameters

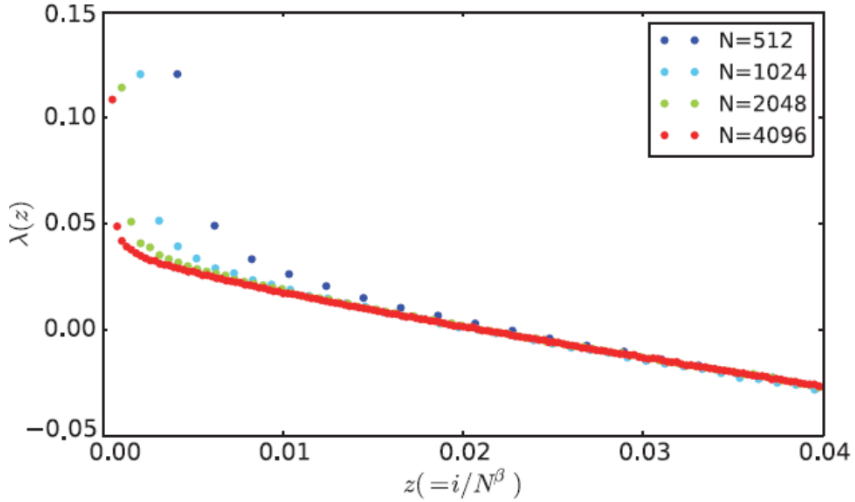
(a)



(b)

N:system size

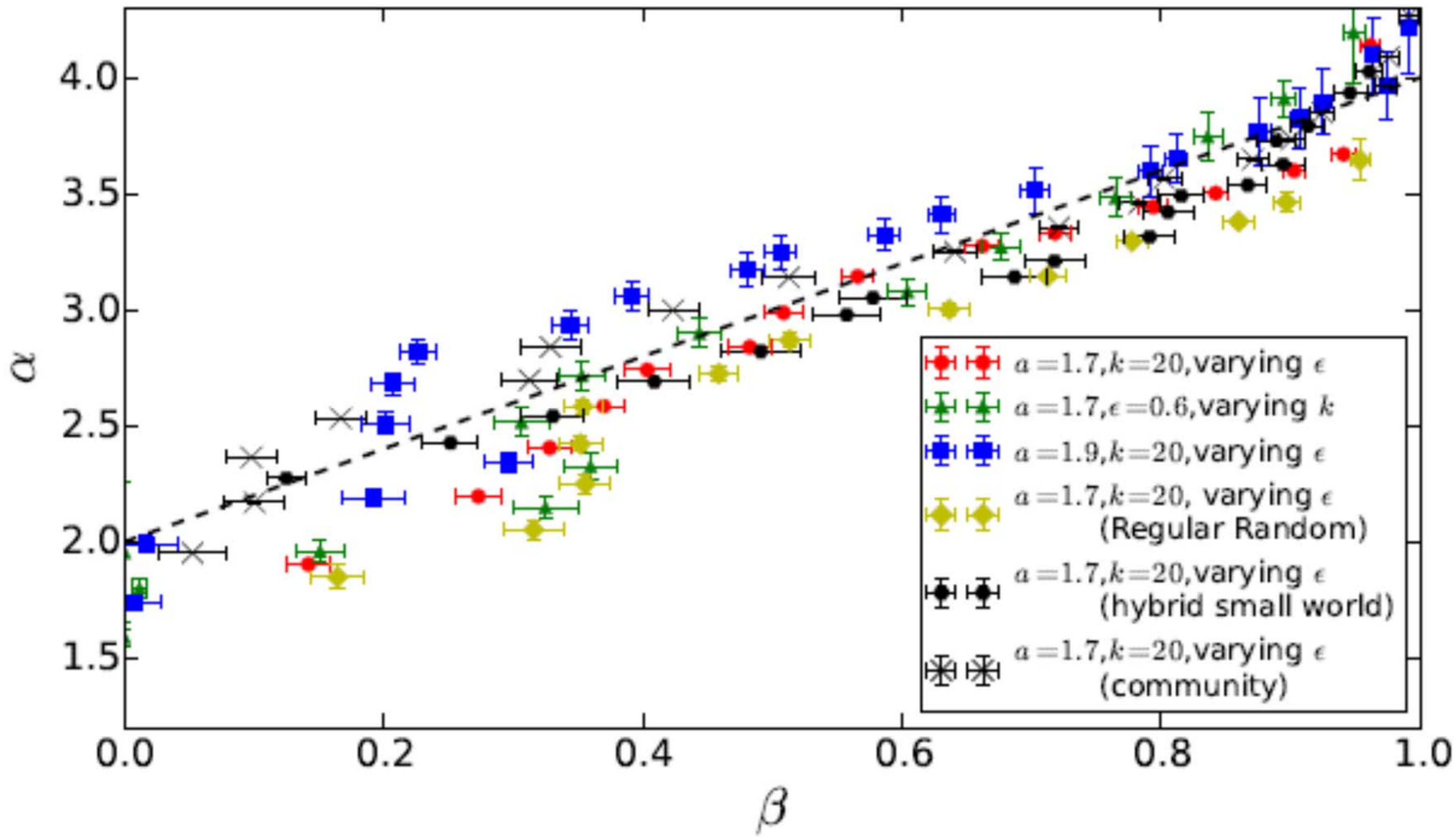
Lyapunov spectra are scaled anomolusly with the power β



Exponents for cluster distribution α and for anomalous Lyapunov spectra β satisfy

$\alpha \sim 2(1+\beta)$

universal in a class of random networks



Possible explanation butnnot yet an answer..

Size of coherent cluster s: random-walk approximation, but add an element or escape is proportional to s (normal case)

Distribution of cluster size P(s)

$$ds = s \circ dt$$

$$P(s) \sim s^{-2},$$

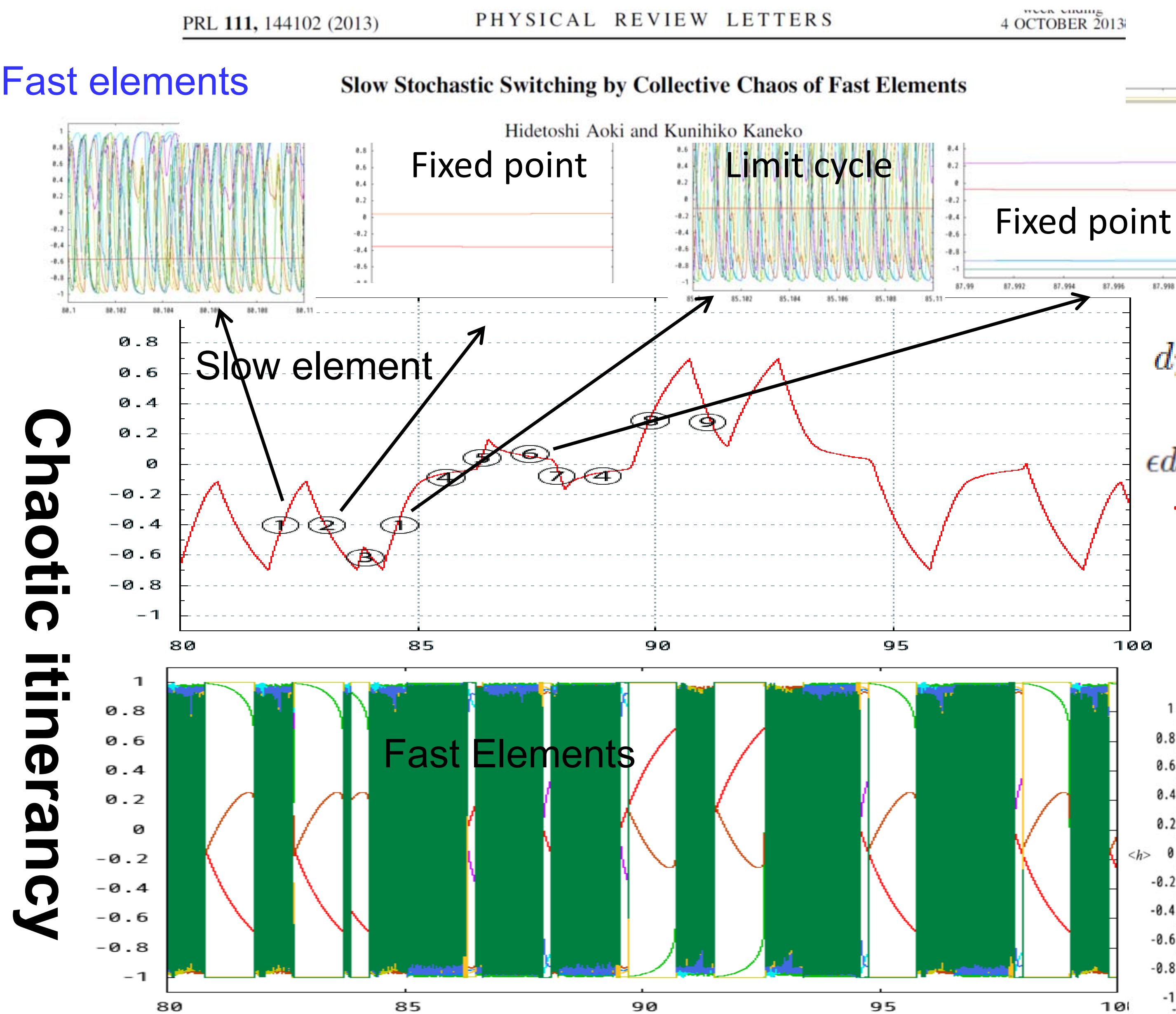
consider the degree of chaos increases anomalously with s with an exponent β

$$ds = s^{1+\beta} \circ dt.$$

$$s^{-2(1+\beta)}$$

$$\alpha = 2(1+\beta)$$

Another example in CI: slow-fast system



1 slow many fast elements, coupled globally (threshold dynamics, neural network)

Slightly beyond adiabatic elimination

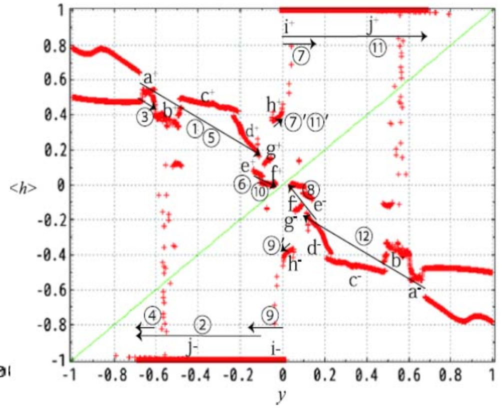
$$dy_i/dt = F_i(\{x_j, y_j\});$$

Slow (i=1)

$$\epsilon dx_i/dt = G_i(\{x_j, y_j\}).$$

fast $(i = 1 \sim N - 1)$

Multi-branced Slow Manifold



Stochastic switch over multistable states by collective chaos

Globally coupled circle maps, high-dimensional torus to chaos

Yamagishi, KK, 2019, in prep

- Heterogeneous (with different frequencies)

(I) $$\theta_{n+1}(i) = \theta_n(i) + \Omega(i) + \frac{a}{2\pi}\sin 2\pi\theta_n(i) + \frac{b}{2\pi N}\sum_{j=1}^{N}\sin 2\pi\theta_n(j)$$

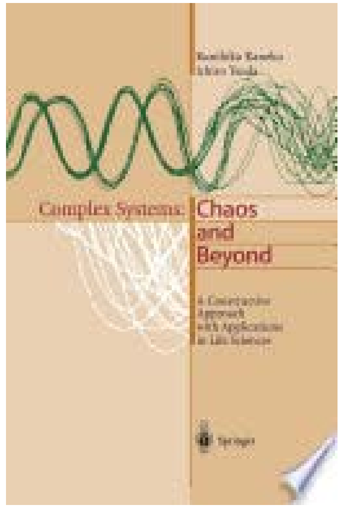
or $\sum \sin 2\pi\theta_n(j) \longrightarrow \sum J_{ij} \sin 2\pi\theta_n(j)$

or (II) $$\theta_{n+1}(i) = \theta_n(i) + \Omega(i) + \frac{a}{2\pi}\sin 2\pi\theta_n(i) + \frac{b}{2\pi N}\sum_j \sin 2\pi(\theta_n(j) - \theta_n(i))$$

Below, mostly the case (I), for (II) also valid, but probably lower-dim tori

N-dim in map ←→(N+1)-dim in flow

Brief partial review of GCM,



- Hierarchical Clustering…? Chimera?
- Chaotic Itinerancy over clusterings
- CI as Milnor Attractor Networks
- Dominance of Milnor Attractors for N>5

Chaotic Griffiths Phase in Coupled Map Network

Formation-Collapse of Synchro clusters, power law, anomalous Lyapunov spectra; universal scaling with Kenji Shinoda

Chaos on/near High-dim Torus in Coupled Oscillators (Maps)

Chaos on high-dim tori, transition via fractalization? with Jumpei F Yamagishi