Statistical Physics of Synchronization

Shamik Gupta

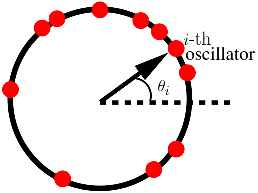
Ramakrishna Mission Vivekananda
Educational and Research Institute, Kolkata, INDIA
&
ICTP, Trieste, ITALY
(Regular Associate, Quantitative Life Sciences Section)

Collaborators:

- Stefano Ruffo (Trieste)
- Alessandro Campa (Rome)
- Debraj Das (Kolkata)

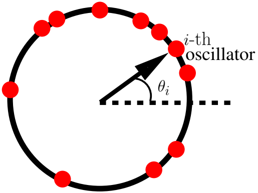
The Kuramoto model

1. N globally coupled oscillators with distributed natural frequencies.
2. θ_i: Phase.
3. $\frac{d\theta_i}{dt} = \omega_i + \frac{\widetilde{K}}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i)$.
4. $\widetilde{K}$: Coupling constant, ω_i's: Natural frequencies, Unimodal distr. $g(\omega)$.

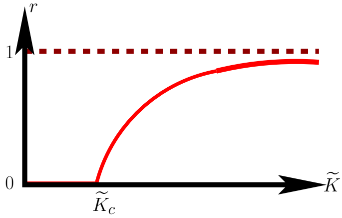


The Kuramoto model

1. N globally coupled oscillators with distributed natural frequencies.
2. θ_i: Phase.
3. $\frac{d\theta_i}{dt} = \omega_i + \frac{\widetilde{K}}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i)$.
4. $\widetilde{K}$: Coupling constant, ω_i's: Natural frequencies, Unimodal distr. $g(\omega)$.



- $N \to \infty, t \to \infty$ limit:
  1. Define $r = |\frac{1}{N} \sum_{j=1}^{N} e^{i\theta_j}|$.
  2. High $\widetilde{K}$: Synchronized phase, $r \neq 0$.
  3. Low $\widetilde{K}$: Incoherent phase, $r = 0$.
  4. "Phase transition" (Bifurcation) on tuning $\widetilde{K}$.
  5. $\widetilde{K}_c = \frac{2}{\pi g(\langle \omega \rangle)}$.



Nonlinear Dynamics *vis-à-vis* Statistical Physics approach

Nonlinear dynamics:

1. Deterministic equations of motion $\frac{\mathrm{d}x}{\mathrm{d}t} = f(x)$.
2. Study, on a case-by-case basis, x as a function of t; Interest: limit $t \to \infty$.

Nonlinear Dynamics *vis-à-vis* Statistical Physics approach

Nonlinear dynamics:

1. Deterministic equations of motion $\frac{\mathrm{d}x}{\mathrm{d}t} = f(x)$.
2. Study, on a case-by-case basis, x as a function of t; Interest: limit $t \to \infty$.

Statistical Physics:

1. Stochastic equations of motion, e.g., a Hamiltonian system with no external drive + environment (heat bath at temp. T):
   $\frac{\mathrm{d}q}{\mathrm{d}t} = p$,
   $\frac{\mathrm{d}p}{\mathrm{d}t} = \text{Force derived from Hamiltonian} \underbrace{-\gamma p + \eta(t).}_{\textit{Effect of environment}}$
   Langevin: Gaussian white noise $\eta(t)$:
   $\langle \eta(t) \rangle = 0, \ \langle \eta(t)\eta(t') \rangle = 2D\delta(t - t')$.

Nonlinear Dynamics *vis-à-vis* Statistical Physics approach

Nonlinear dynamics:

1. Deterministic equations of motion $\frac{\mathrm{d}x}{\mathrm{d}t} = f(x)$.
2. Study, on a case-by-case basis, x as a function of t; Interest: limit $t \to \infty$.

Statistical Physics:

1. Stochastic equations of motion, e.g., a Hamiltonian system with no external drive + environment (heat bath at temp. T):
   $\frac{\mathrm{d}q}{\mathrm{d}t} = p$,
   $\frac{\mathrm{d}p}{\mathrm{d}t} = \text{Force derived from Hamiltonian} \underbrace{-\gamma p + \eta(t).}_{\textit{Effect of environment}}$
   Langevin: Gaussian white noise $\eta(t)$:
   $\langle \eta(t) \rangle = 0, \ \langle \eta(t)\eta(t') \rangle = 2D\delta(t - t')$.
2. Study distribution $P(x, t) \equiv P(q, p, t)$ as a function of t.

Nonlinear Dynamics *vis-à-vis* Statistical Physics approach

Nonlinear dynamics:

1. Deterministic equations of motion $\frac{\mathrm{d}x}{\mathrm{d}t} = f(x)$.
2. Study, on a case-by-case basis, x as a function of t; Interest: limit $t \to \infty$.

Statistical Physics:

1. Stochastic equations of motion, e.g., a Hamiltonian system with no external drive + environment (heat bath at temp. T):
   $\frac{\mathrm{d}q}{\mathrm{d}t} = p$,
   $\frac{\mathrm{d}p}{\mathrm{d}t} = \text{Force derived from Hamiltonian} \underbrace{-\gamma p + \eta(t).}_{\textit{Effect of environment}}$
   Langevin: Gaussian white noise $\eta(t)$:
   $\langle \eta(t) \rangle = 0, \ \langle \eta(t)\eta(t') \rangle = 2D\delta(t - t')$.
2. Study distribution $P(x, t) \equiv P(q, p, t)$ as a function of t.
3. $D = \gamma k_B T$ (Fluc.-Dissipation Th.) $\Rightarrow P(x, t \to \infty) \propto \exp(-H/T)$ (no need to solve the dynamics).

Nonlinear Dynamics *vis-à-vis* Statistical Physics approach

Nonlinear dynamics:

1. Deterministic equations of motion $\frac{\mathrm{d}x}{\mathrm{d}t} = f(x)$.
2. Study, on a case-by-case basis, x as a function of t; Interest: limit $t \to \infty$.

Statistical Physics:

1. Stochastic equations of motion, e.g., a Hamiltonian system with no external drive + environment (heat bath at temp. T):
   $\frac{\mathrm{d}q}{\mathrm{d}t} = p$,
   $\frac{\mathrm{d}p}{\mathrm{d}t} = \text{Force derived from Hamiltonian} \underbrace{-\gamma p + \eta(t).}_{\textit{Effect of environment}}$
   Langevin: Gaussian white noise $\eta(t)$:
   $\langle \eta(t) \rangle = 0, \ \langle \eta(t)\eta(t') \rangle = 2D\delta(t - t')$.
2. $D = \gamma k_B T$ (Fluc.-Dissipation Th.)
   $\Rightarrow P(x, t \to \infty) \propto \exp(-H/T)$
   (no need to solve the dynamics).
3. Useful concepts like free energy whose minimization yields equilibrium phases.

Our contributions from statphys perspective

- The key steps taken:
  1. Including noise in the Kuramoto dynamics.
  2. Interpreting the model as a long-range interacting system of particles with quenched disorder, driven out of equilibrium.
  3. Employing tools of statistical mechanics and kinetic theory to study statics and dynamics.

Our contributions from statphys perspective

- The key steps taken:
  1. Including noise in the Kuramoto dynamics.
  2. Interpreting the model as a long-range interacting system of particles with quenched disorder, driven out of equilibrium.
  3. Employing tools of statistical mechanics and kinetic theory to study statics and dynamics.

Our contributions from statphys perspective

- The key steps taken:
  1. Including noise in the Kuramoto dynamics.
  2. Interpreting the model as a long-range interacting system of particles with quenched disorder, driven out of equilibrium.
  3. Employing tools of statistical mechanics and kinetic theory to study statics and dynamics.

- ...and the main results:
  1. Proving that the system relaxes to a nonequilibrium steady state (NESS) at long times.
  2. Developing an exact analytical approach to compute the steady state distr.
  3. By considering generalized Kuramoto dynamics, demonstrating with exact results a very rich phase diagram with eqlbm. and noneqlbm. transitions.

Our setting: The generalized Kuramoto model

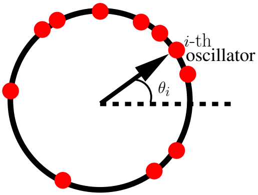
1. N globally coupled rotors.

Our setting: The generalized Kuramoto model

1. N globally coupled rotors.

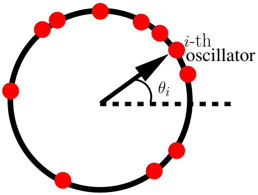
Our setting: The generalized Kuramoto model

1. N globally coupled rotors.
2. 2 dynamical variables:
   θ_i: Phase,
   v_i: Angular velocity.



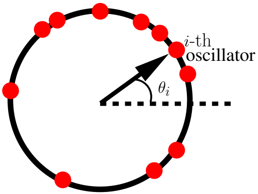
Our setting: The generalized Kuramoto model

1. N globally coupled rotors.
2. 2 dynamical variables:
   θ_i: Phase,
   v_i: Angular velocity.
3. $\frac{d\theta_i}{dt} = v_i$.



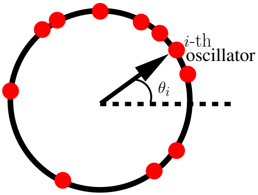
Our setting: The generalized Kuramoto model

1. N globally coupled rotors.
2. 2 dynamical variables:
   θ_i: Phase,
   v_i: Angular velocity.
3. $\frac{d\theta_i}{dt} = v_i$.
4. $m\frac{dv_i}{dt} = \underbrace{\frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i)}_{\text{Long-range interaction}}$.



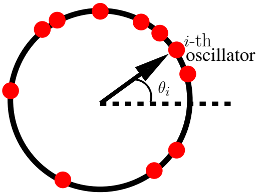
Our setting: The generalized Kuramoto model

1. N globally coupled rotors.
2. 2 dynamical variables:
   θ_i: Phase,
   v_i: Angular velocity.
3. $\frac{d\theta_i}{dt} = v_i$.
4. $m\frac{dv_i}{dt} = \underbrace{\frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i)}_{\text{Long-range interaction}}$.



- m: Moment of inertia, K: Coupling constant.

Our setting: The generalized Kuramoto model

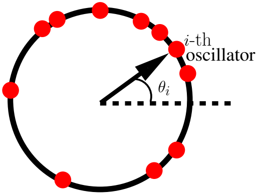


1. N globally coupled rotors.
2. 2 dynamical variables:
   θ_i: Phase,
   v_i: Angular velocity.
3. $\frac{d\theta_i}{dt} = v_i$.
4. $m\frac{dv_i}{dt} = \underbrace{\frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i)}_{\text{Long-range interaction}}$.

- m: Moment of inertia, K: Coupling constant.
- Hamiltonian dynamics:
  $H = \sum_{i=1}^{N} \frac{p_i^2}{2m} + \frac{K}{2N}\sum_{i,j=1}^{N}[1-\cos(\theta_i - \theta_j)]$, $p_i = mv_i$.

Our setting: The generalized Kuramoto model

1. N globally coupled rotors.
2. 2 dynamical variables:
   θ_i: Phase,
   v_i: Angular velocity.
3. $\frac{d\theta_i}{dt} = v_i$.
4. $m\frac{dv_i}{dt} =$

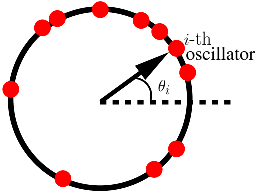


$$\underbrace{-\gamma v_i}_{\text{Damping}} + \underbrace{\frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i)}_{\text{Long-range interaction}} + \underbrace{\eta_i(t)}_{\text{Noise}}.$$

- m: Moment of inertia, K: Coupling constant, γ: Friction constant.
- Hamiltonian + heat bath.
- Gaussian white noise:
  $\langle \eta_i(t) \rangle = 0, \langle \eta_i(t)\eta_j(t') \rangle = 2\gamma T\delta_{ij}\delta(t-t')$
  T: Temperature of the heat bath.

Our setting: The generalized Kuramoto model

1. N globally coupled rotors.
2. 2 dynamical variables:
   θ_i: Phase,
   v_i: Angular velocity.
3. $\frac{d\theta_i}{dt} = v_i$.
4. $m\frac{dv_i}{dt} =$



$$\underbrace{-\gamma v_i}_{\text{Damping}} + \underbrace{\frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i)}_{\text{Long-range interaction}} + \underbrace{\gamma\omega_i}_{\text{Drive (Quenched disorder)}} + \underbrace{\eta_i(t)}_{\text{Noise}}.$$

- m: Moment of inertia, γ: Friction constant,
  K: Coupling constant.
- ω_i's: Quenched random variables from a common distr. $g(\omega)$.
- Gaussian white noise:
  $\langle \eta_i(t) \rangle = 0, \langle \eta_i(t)\eta_j(t') \rangle = 2\gamma T\delta_{ij}\delta(t - t')$.

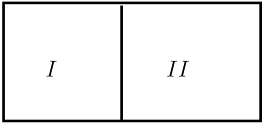
Long-range interactions: A one-slide summary

$$V(r) \sim \frac{1}{r^\alpha}; \quad 0 \leq \alpha \leq d.$$

Examples: Gravitation, Coulomb interaction,...

Main distinguishing feature: Non-additivity.

$$E_{\mathrm{Total}} \neq E_I + E_{II}.$$



Consequences:

- Statics: Ensemble inequivalence.
- Dynamics:
  Slow relaxation over times diverging with the system size.

Physics of Long-Range Interacting Systems,
Campa, Dauxois, Fanelli, Ruffo (Oxford, 2014)

Our setting: The generalized Kuramoto model

- $\frac{d\theta_i}{dt} = v_i$.
- $m\frac{dv_i}{dt} =$

$$\underbrace{-\gamma v_i}_{\text{Damping}} + \underbrace{\frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i)}_{\text{Long-range interaction}} + \underbrace{\gamma\omega_i}_{\text{Drive (Quenched disorder)}} + \underbrace{\eta_i(t)}_{\text{Noise}}.$$

1. $g(\omega)$ unimodal, with mean $\widetilde{\omega}$ and width σ.

Our setting: The generalized Kuramoto model

- $\frac{d\theta_i}{dt} = v_i$.
- $m\frac{dv_i}{dt} =$

$$\underbrace{-\gamma v_i}_{\text{Damping}} + \underbrace{\frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i)}_{\text{Long-range interaction}} + \underbrace{\gamma\omega_i}_{\text{Drive (Quenched disorder)}} + \underbrace{\eta_i(t)}_{\text{Noise}}.$$

1. $g(\omega)$ unimodal, with mean $\widetilde{\omega}$ and width σ.
2. Dynamics invariant under
   $\theta_i \to \theta_i + \widetilde{\omega}t, v_i \to v_i + \widetilde{\omega}, \omega_i \to \omega_i + \widetilde{\omega}$
   (Go to the comoving frame)
   $\Rightarrow$ consider $g(\omega)$ with zero mean.

Our setting: The generalized Kuramoto model

- $\frac{d\theta_i}{dt} = v_i$.
- $m\frac{dv_i}{dt} =$

$$\underbrace{-\gamma v_i}_{\text{Damping}} + \underbrace{\frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i)}_{\text{Long-range interaction}} + \underbrace{\gamma\omega_i}_{\text{Drive (Quenched disorder)}} + \underbrace{\eta_i(t)}_{\text{Noise}}.$$

1. $g(\omega)$ unimodal, with mean $\widetilde{\omega}$ and width σ.
2. Dynamics invariant under
   $\theta_i \to \theta_i + \widetilde{\omega}t, v_i \to v_i + \widetilde{\omega}, \omega_i \to \omega_i + \widetilde{\omega}$
   (Go to the comoving frame)
   $\Rightarrow$ consider $g(\omega)$ with zero mean.
3. Take $\omega_i \to \sigma\omega_i$,
   where $g(\omega)$ now has unit width.

Dynamics in a reduced parameter space

- $\frac{d\theta_i}{dt} = v_i$.
- $m\frac{dv_i}{dt} =$

$$\underbrace{-\gamma v_i}_{\text{Damping}} + \underbrace{\frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i)}_{\text{Long-range interaction}} + \underbrace{\gamma\sigma\omega_i}_{\text{Drive (Quenched disorder)}} + \underbrace{\eta_i(t)}_{\text{Noise}}.$$

Dynamics in a reduced parameter space

- $\frac{d\theta_i}{dt} = v_i$.
- $m\frac{dv_i}{dt} =$

$$\underbrace{-\gamma v_i}_{\text{Damping}} + \underbrace{\frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i)}_{\text{Long-range interaction}} + \underbrace{\gamma\sigma\omega_i}_{\text{Drive (Quenched disorder)}} + \underbrace{\eta_i(t)}_{\text{Noise}}.$$

- **The dimensionless dynamics:**

$$\frac{d\theta_i}{dt} = v_i,$$

$$\frac{dv_i}{dt} = -\frac{1}{\sqrt{m}}v_i + \frac{1}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i) + \sigma\omega_i + \eta_i(t).$$

- $\langle\eta_i(t)\rangle = 0, \langle\eta_i(t)\eta_j(t')\rangle = 2(T/\sqrt{m})\delta_{ij}\delta(t-t')$.
- Only 3 dimensionless parameters: m, T, σ.

$\sigma = 0 \Rightarrow$ No external drive $\Rightarrow$ Equilibrium (Hamiltonian system + heat bath)

$$\frac{d\theta_i}{dt} = v_i,$$

$$\frac{dv_i}{dt} = -\frac{1}{\sqrt{m}} v_i + \frac{1}{N}\sum_{j=1}^{N} \sin(\theta_j - \theta_i) + \eta_i(t).$$

$$\langle \eta_i(t) \rangle = 0, \langle \eta_i(t)\eta_j(t') \rangle = 2(T/\sqrt{m})\delta_{ij}\delta(t - t').$$

$\sigma = 0 \Rightarrow$ No external drive $\Rightarrow$ Equilibrium (Hamiltonian system + heat bath)

$$\frac{d\theta_i}{dt} = v_i,$$

$$\frac{dv_i}{dt} = -\frac{1}{\sqrt{m}} v_i + \frac{1}{N}\sum_{j=1}^{N} \sin(\theta_j - \theta_i) + \eta_i(t).$$

$$\langle \eta_i(t) \rangle = 0, \langle \eta_i(t)\eta_j(t') \rangle = 2(T/\sqrt{m})\delta_{ij}\delta(t-t').$$

- Hamiltonian $H = \sum_{i=1}^{N} \frac{v_i^2}{2} + \frac{1}{2N}\sum_{i,j=1}^{N} \Big[1 - \cos(\theta_i - \theta_j)\Big]$.
- Mean-field XY model.
- Steady state: Canonical equilibrium

  $P_{\mathrm{eq}}(\{\theta_i, v_i\}) \propto \exp(-H/T)$.

$\sigma = 0 \Rightarrow$ No external drive $\Rightarrow$ Equilibrium (Hamiltonian system + heat bath)

$$\frac{d\theta_i}{dt} = v_i,$$

$$\frac{dv_i}{dt} = -\frac{1}{\sqrt{m}} v_i + \frac{1}{N}\sum_{j=1}^{N} \sin(\theta_j - \theta_i) + \sigma\omega_i + \eta_i(t).$$

$$\langle \eta_i(t) \rangle = 0, \langle \eta_i(t)\eta_j(t') \rangle = 2(T/\sqrt{m})\delta_{ij}\delta(t - t').$$

- Hamiltonian $H = \sum_{i=1}^{N} \frac{v_i^2}{2} + \frac{1}{2N}\sum_{i,j=1}^{N} \Big[1 - \cos(\theta_i - \theta_j)\Big].$
- Motion of a single particle in a mean field $\Rightarrow$

  single-particle Hamiltonian $H_{\mathrm{single}} = \frac{v^2}{2} - r_x \cos\theta - r_y \sin\theta$,

  single-particle equilibrium $P_{\mathrm{eq}}(\theta, v) \propto \exp(-H_{\mathrm{single}}/T)$.

$\sigma = 0 \Rightarrow$ No external drive $\Rightarrow$ Equilibrium (Hamiltonian system + heat bath)

$$\frac{d\theta_i}{dt} = v_i,$$

$$\frac{dv_i}{dt} = -\frac{1}{\sqrt{m}}v_i + \frac{1}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i) + \sigma\omega_i + \eta_i(t).$$

$$\langle \eta_i(t)\rangle = 0, \langle \eta_i(t)\eta_j(t')\rangle = 2(T/\sqrt{m})\delta_{ij}\delta(t-t').$$

- Single-particle Hamiltonian $H_{\mathrm{single}} = \frac{v^2}{2} - r_x \cos\theta - r_y \sin\theta$,

  single-particle equilibrium $P_{\mathrm{eq}}(\theta, v) \propto \exp(-H_{\mathrm{single}}/T)$.

- $O(2)$ symmetry $\Rightarrow r_y = 0$,

  $r_x = \frac{\int \mathrm{d}\theta \mathrm{d}v \cos\theta \exp(-H_{\mathrm{single}}/T)}{\int \mathrm{d}\theta \mathrm{d}v \; \exp(-H_{\mathrm{single}}/T)} = \frac{\int_0^{2\pi} \mathrm{d}\theta \; \cos\theta \exp(r_x/T \; \cos\theta)}{\int_0^{2\pi} \mathrm{d}\theta \; \exp(r_x/T \; \cos\theta)}$

  $\Rightarrow$ continuous transition between unsynchronized and synchronized phase, critical temperature $T_c = 1/2$ independent of m.

Dynamics in a reduced parameter space

$$\frac{d\theta_i}{dt} = v_i,$$

$$\frac{dv_i}{dt} = -\frac{1}{\sqrt{m}} v_i + \frac{1}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i) + \sigma\omega_i + \eta_i(t).$$

- $\sigma = 0 \rightarrow$ continuous transition between unsynchronized and synchronized phase, critical line $T_c = 1/2$.
- $\sigma \neq 0 \Rightarrow$ Nonequilibrium stationary state.
- Kuramoto dyn.: $m = T = 0$, $\sigma \neq 0$: continuous "transition", critical point.

Dynamics in a reduced parameter space

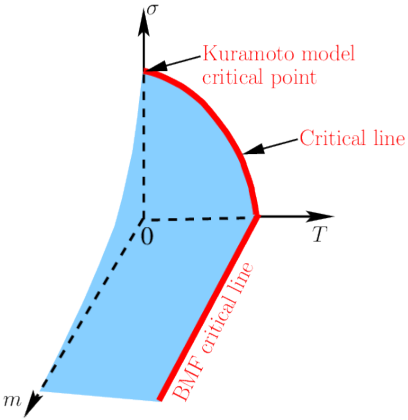
$$\frac{d\theta_i}{dt} = v_i,$$

$$\frac{dv_i}{dt} = -\frac{1}{\sqrt{m}} v_i + \frac{1}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i) + \sigma\omega_i + \eta_i(t).$$

- $\sigma = 0 \to$ continuous transition between unsynchronized and synchronized phase, critical line $T_c = 1/2$.
- $\sigma \neq 0 \Rightarrow$ Nonequilibrium stationary state.
- Kuramoto dyn.: $m = T = 0$, $\sigma \neq 0$: continuous "transition", critical point.

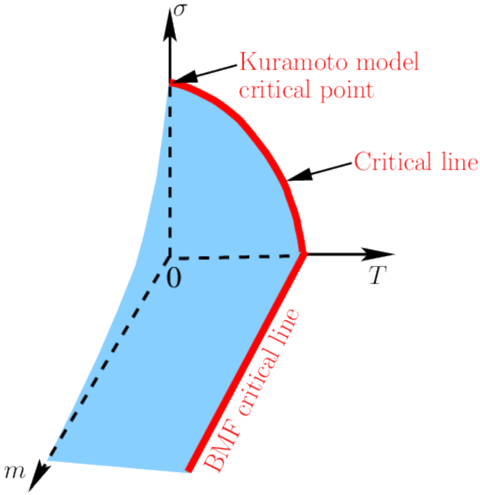
QUESTION: For the generalized model,
STEADY STATE ?? SYNCHRONY ?? TRANSITIONS ??

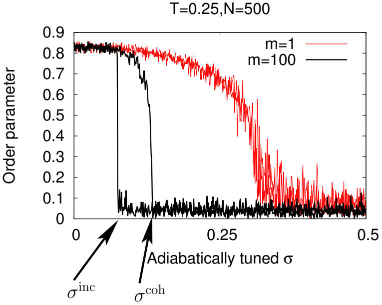
The complete phase diagram

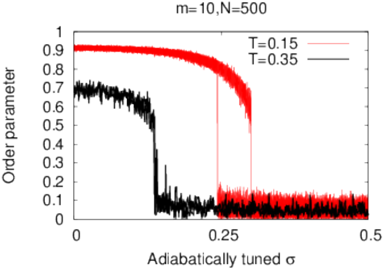


Synchrony within the region bounded by the blue surface.

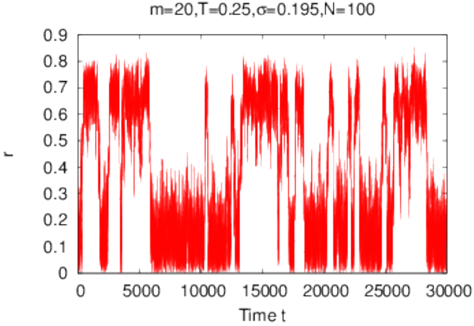
The complete phase diagram

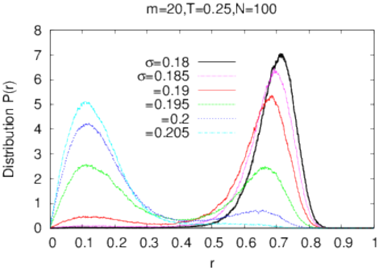






Bistability close to the first-order transition





Continuum limit ($N \to \infty$) analysis: The main steps

1. Fokker-Planck eqn. for the $2N$-dim. phase space density.
2. Reduced distribution functions.
3. Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy.
4. $N \to \infty$: Closure by neglecting two-particle correlations.
5. Single-particle distribution $f(\theta, v, \omega, t)$: Evolution by "Kramers" equation,
   $\frac{\partial f}{\partial t} = -v\frac{\partial f}{\partial \theta} + \frac{\partial}{\partial v}\Big(\frac{v}{\sqrt{m}} - \sigma\omega - r\sin(\psi - \theta)\Big)f + \frac{T}{\sqrt{m}}\frac{\partial^2 f}{\partial v^2}$.
6. r determined self-consistently:
   $re^{i\psi} = \int d\theta dv d\omega \; g(\omega)e^{i\theta}f(\theta, v, \omega, t)$.

The unsynchronized steady state

$$0 = -v\frac{\partial f}{\partial \theta} + \frac{\partial}{\partial v}\Big(\frac{v}{\sqrt{m}} - \sigma\omega - r\sin(\psi-\theta)\Big)f + \frac{T}{\sqrt{m}}\frac{\partial^2 f}{\partial v^2},$$

$$re^{i\psi} = \int d\theta dv d\omega \; g(\omega)e^{i\theta}f(\theta, v, \omega).$$

1. Unsynchronized ($r = 0$) solution:

   $f_{\rm st}^{\rm inc}(\theta, v, \omega) = \frac{1}{2\pi}\frac{1}{\sqrt{2\pi T}}e^{-(v-\sigma\omega\sqrt{m})^2/(2T)}.$
2. Stability analysis gives $\sigma^{\rm inc}$.
   $\Rightarrow$ Linear stability analysis:
   $f(\theta, v, \omega, t) = f_{\rm st}^{\rm inc}(\theta, v, \omega) + e^{\lambda t}\delta f(\theta, v, \omega).$
3. λ satisfies *(Acebrón, Bonilla and Spigler (2000))*

   $1 = \frac{e^{mT}}{2T}\sum_{p=0}^{\infty}\frac{(-mT)^p(1+\frac{p}{mT})}{p!}\int_{-\infty}^{+\infty}\frac{g(\omega)d\omega}{1+\frac{p}{mT}+i\frac{\sigma\omega}{T}+\frac{\lambda}{T\sqrt{m}}}.$

Linear stability of the unsynchronized phase

$$1 = \frac{e^{mT}}{2T} \sum_{p=0}^{\infty} \frac{(-mT)^p (1+\frac{p}{mT})}{p!} \int_{-\infty}^{+\infty} \frac{g(\omega)d\omega}{1+\frac{p}{mT}+i\frac{\sigma\omega}{T}+\frac{\lambda}{T\sqrt{m}}}.$$

1. We proved that
   (1) the equation has at most one solution for λ with a positive real part, and (2) when the solution exists, it is necessarily real.

Linear stability of the unsynchronized phase

$$1 = \frac{e^{mT}}{2T} \sum_{p=0}^{\infty} \frac{(-mT)^p(1+\frac{p}{mT})}{p!} \int_{-\infty}^{+\infty} \frac{g(\omega)d\omega}{1+\frac{p}{mT}+i\frac{\sigma\omega}{T}+\frac{\lambda}{T\sqrt{m}}}.$$

1. We proved that
   (1) the equation has at most one solution for λ with a positive real part, and (2) when the solution exists, it is necessarily real.
2. Neutral stability $\Rightarrow \lambda = 0$ gives
   $1 = \frac{e^{mT}}{2T} \sum_{p=0}^{\infty} \frac{(-mT)^p(1+\frac{p}{mT})^2}{p!} \int_{-\infty}^{+\infty} \frac{g(\omega)d\omega}{(1+\frac{p}{mT})^2+\frac{(\sigma^{\mathrm{inc}})^2\omega^2}{T^2}}$
   $\Rightarrow$ Stability surface $\sigma^{\mathrm{inc}}(m, T)$.

The synchronized steady state

$$0 = -v\frac{\partial f}{\partial \theta} + \frac{\partial}{\partial v}\Big(\frac{v}{\sqrt{m}} - \sigma\omega - r\sin(\psi - \theta)\Big)f + \frac{T}{\sqrt{m}}\frac{\partial^2 f}{\partial v^2},$$

$$re^{i\psi} = \int d\theta dv d\omega \; g(\omega)e^{i\theta}f(\theta, v, \omega)$$

- Exact steady state distr. for the sync. phase: Main steps
  1. $f_{\rm st}^{\rm coh}(\theta, v, \omega) = \Phi_0\Big(\frac{v}{\sqrt{2T}}\Big)\sum_{n=0}^{\infty} b_n\Phi_n\Big(\frac{v}{\sqrt{2T}}\Big)$;

     Φ_n: Hermite functions

The synchronized steady state

$$0 = -v\frac{\partial f}{\partial \theta} + \frac{\partial}{\partial v}\Big(\frac{v}{\sqrt{m}} - \sigma\omega - r\sin(\psi - \theta)\Big)f + \frac{T}{\sqrt{m}}\frac{\partial^2 f}{\partial v^2},$$

$$re^{i\psi} = \int d\theta dv d\omega \; g(\omega)e^{i\theta}f(\theta, v, \omega)$$

- Exact steady state distr. for the sync. phase: Main steps
  1. $f_{\rm st}^{\rm coh}(\theta, v, \omega) = \Phi_0\Big(\frac{v}{\sqrt{2T}}\Big)\sum_{n=0}^{\infty} b_n \Phi_n\Big(\frac{v}{\sqrt{2T}}\Big)$;

     Φ_n: Hermite functions
  2. $b_p(\theta) = \sum_{k=0}^{\infty}(\sqrt{m})^k c_{p,k}(\theta)$

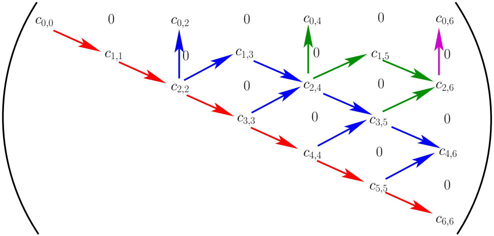
The synchronized steady state

$$0 = -v\frac{\partial f}{\partial \theta} + \frac{\partial}{\partial v}\Big(\frac{v}{\sqrt{m}} - \sigma\omega - r\sin(\psi-\theta)\Big)f + \frac{T}{\sqrt{m}}\frac{\partial^2 f}{\partial v^2},$$

$$re^{i\psi} = \int d\theta dv d\omega \; g(\omega)e^{i\theta}f(\theta,v,\omega)$$

- Exact steady state distr. for the sync. phase: Main steps
  1. $f_{\rm st}^{\rm coh}(\theta,v,\omega) = \Phi_0\Big(\frac{v}{\sqrt{2T}}\Big)\sum_{n=0}^{\infty} b_n\Phi_n\Big(\frac{v}{\sqrt{2T}}\Big)$;

     Φ_n: Hermite functions
  2. $b_p(\theta) = \sum_{k=0}^{\infty}(\sqrt{m})^k c_{p,k}(\theta)$
  3. Recursion relations for $c_{p,k}$:



Sparse matrix, computationally efficient

The synchronized steady state

$$0 = -v\frac{\partial f}{\partial \theta} + \frac{\partial}{\partial v}\Big(\frac{v}{\sqrt{m}} - \sigma\omega - r\sin(\psi - \theta)\Big)f + \frac{T}{\sqrt{m}}\frac{\partial^2 f}{\partial v^2},$$

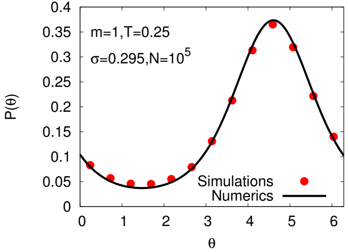
$$re^{i\psi} = \int d\theta dv d\omega \; g(\omega)e^{i\theta}f(\theta, v, \omega)$$

- Exact steady state distr. for the sync. phase: Main steps
  1. $f_{\rm st}^{\rm coh}(\theta, v, \omega) = \Phi_0\Big(\frac{v}{\sqrt{2T}}\Big)\sum_{n=0}^{\infty} b_n \Phi_n\Big(\frac{v}{\sqrt{2T}}\Big)$;

     Φ_n: Hermite functions
  2. $b_p(\theta) = \sum_{k=0}^{\infty}(\sqrt{m})^k c_{p,k}(\theta)$
- Key features of the analytic approach:
  1. Exact expansion—No small parameter
  2. Generalizable to any periodic mean-field potential

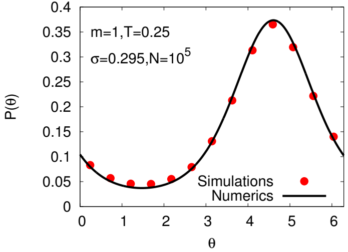
Comparison with N-body simulations: Gaussian $g(\omega)$

1. The θ distribution:

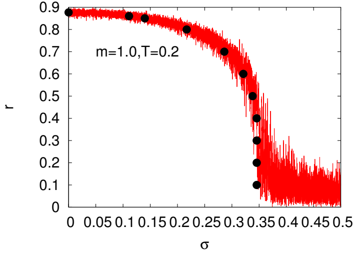


Comparison with N-body simulations: Gaussian $g(\omega)$

1. The θ distribution:



2. r vs. σ:

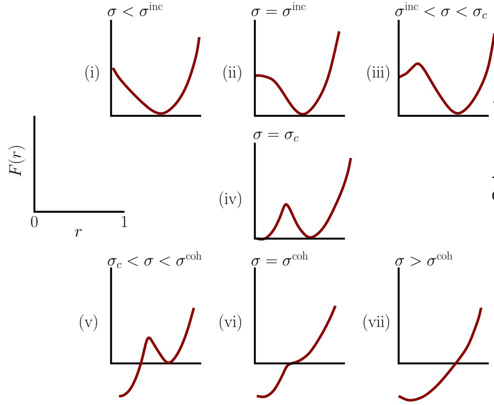


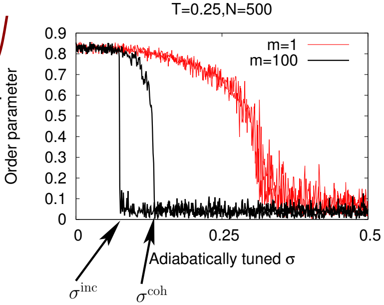
...and now the dynamics (relaxation to steady state)

(for a representative $g(\omega)$, namely, a Gaussian).

Schematic Landau "free energy" landscapes

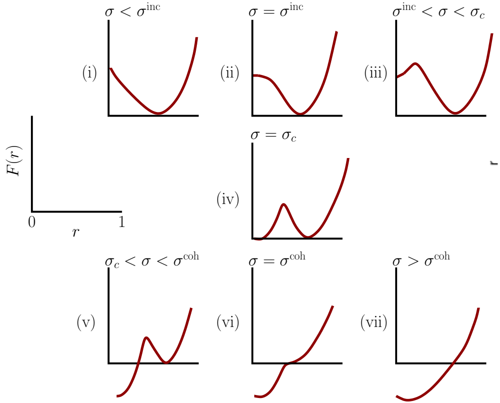
Landau "free energy" landscape:

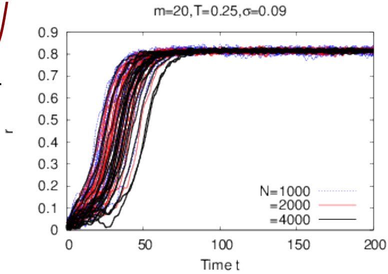




Relaxation dynamics (Gaussian $g(\omega)$)

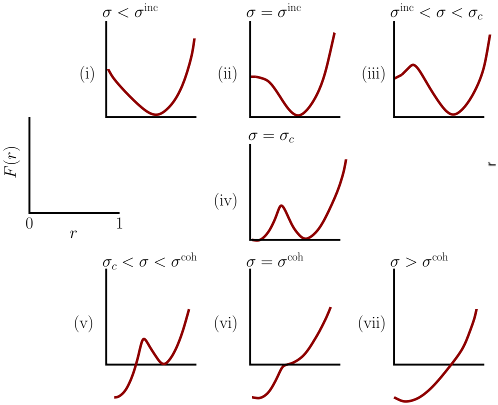
For $m = 20$, $T = 0.25$, $\sigma^{\text{inc}} \approx 0.10076\ldots$
Let us choose $\sigma < \sigma^{\text{inc}}$.

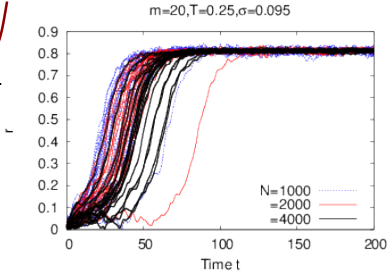




Relaxation dynamics

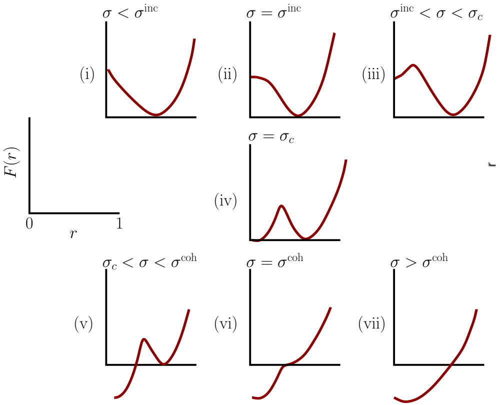
For $m = 20$, $T = 0.25$, $\sigma^{\text{inc}} \approx 0.10076...$
Let us choose $\sigma \lesssim \sigma^{\text{inc}}$.

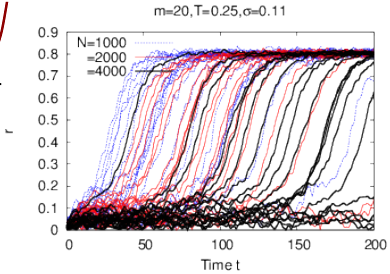




Relaxation dynamics

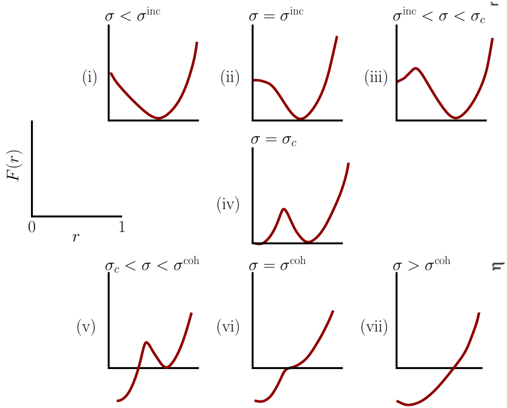
For $m = 20$, $T = 0.25$, $\sigma^{\text{inc}} \approx 0.10076\ldots$
Let us choose $\sigma \gtrsim \sigma^{\text{inc}}$.

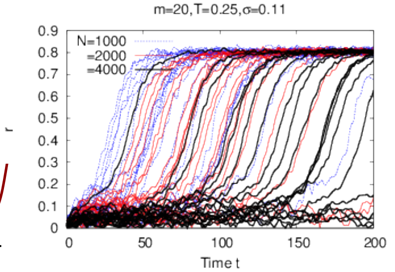


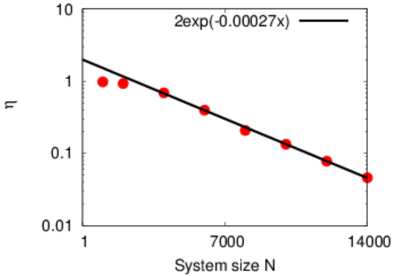


Relaxation dynamics

For $m = 20$, $T = 0.25$, $\sigma^{\rm inc} \approx 0.10076...$
Let us choose $\sigma \gtrsim \sigma^{\rm inc}$.
η: fraction of realizations
relaxing to sync. state in a fixed time.





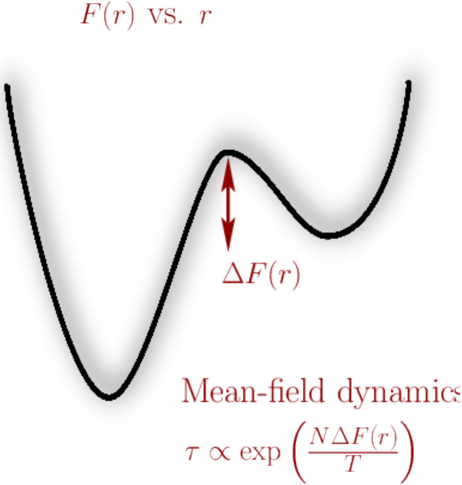


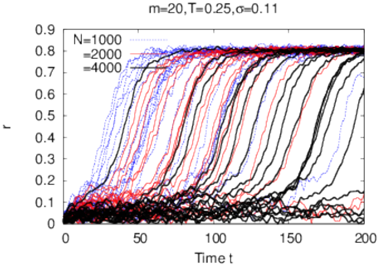
Relaxation dynamics

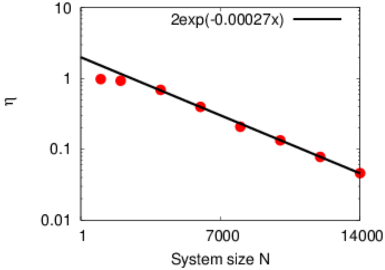
For $m = 20$, $T = 0.25$, $\sigma^{\text{inc}} \approx 0.10076...$

Let us choose $\sigma \gtrsim \sigma^{\text{inc}}$

η: fraction of realizations relaxing to sync. state in a fixed time.

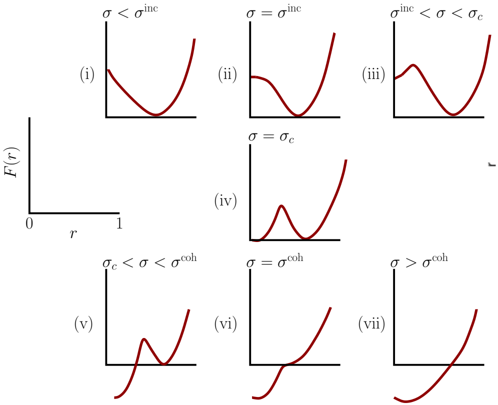


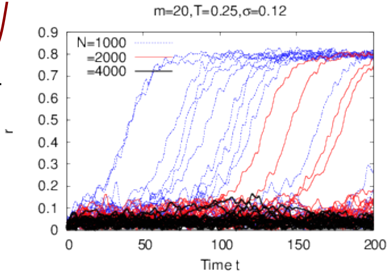




Relaxation dynamics

For $m = 20$, $T = 0.25$, $\sigma^{\text{inc}} \approx 0.10076\ldots$
Let us choose $\sigma > \sigma^{\text{inc}}$.





Summary

- Kuramoto model as an overdamped limit of a long-range interacting system.

Summary

- Kuramoto model as an overdamped limit of a long-range interacting system.
- General dynamics: (1) External drive, (2) Quenched vs. annealed randomness.

Summary

- Kuramoto model as an overdamped limit of a long-range interacting system.
- General dynamics: (1) External drive, (2) Quenched vs. annealed randomness.
  - No quenched randomness $\Rightarrow$ Equilibrium.

Summary

- Kuramoto model as an overdamped limit of a long-range interacting system.
- General dynamics: (1) External drive, (2) Quenched vs. annealed randomness.
  - No quenched randomness $\Rightarrow$ Equilibrium.
    - Prob. distr. $\sim \exp\Big[-\beta(K.E. + P.E.)\Big]$; Product measure.

Summary

- Kuramoto model as an overdamped limit of a long-range interacting system.
- General dynamics: (1) External drive, (2) Quenched vs. annealed randomness.
  - No quenched randomness $\Rightarrow$ Equilibrium.
    - Prob. distr. $\sim \exp\Big[-\beta(K.E. + P.E.)\Big]$; Product measure.
    - Phase transition given by P. E., same for underdamped and overdamped dynamics.

Summary

- Kuramoto model as an overdamped limit of a long-range interacting system.
- General dynamics: (1) External drive, (2) Quenched vs. annealed randomness.
  - No quenched randomness $\Rightarrow$ Equilibrium.
    - Prob. distr. $\sim \exp\Big[-\beta(K.E. + P.E.)\Big]$; Product measure.
    - Phase transition given by P. E., same for underdamped and overdamped dynamics.
  - Quenched randomness $\Rightarrow$ Noneqlbm. stat. state.

Summary

- Kuramoto model as an overdamped limit of a long-range interacting system.
- General dynamics: (1) External drive, (2) Quenched vs. annealed randomness.
  - No quenched randomness $\Rightarrow$ Equilibrium.
    - Prob. distr. $\sim \exp\Big[-\beta(K.E. + P.E.)\Big]$; Product measure.
    - Phase transition given by P. E., same for underdamped and overdamped dynamics.
  - Quenched randomness $\Rightarrow$ Noneqlbm. stat. state.
    - Prob. distr. $\neq \exp\Big[-\beta(K.E. + P.E.)\Big]$; In general, Not product measure.

Summary

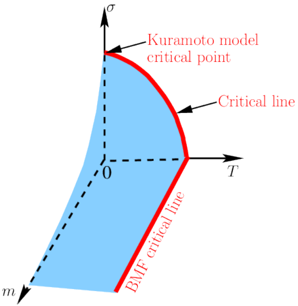
- Kuramoto model as an overdamped limit of a long-range interacting system.
- General dynamics: (1) External drive, (2) Quenched vs. annealed randomness.
  - No quenched randomness ⇒ Equilibrium.
    - Prob. distr. $\sim \exp\Big[-\beta(K.E. + P.E.)\Big]$; Product measure.
    - Phase transition given by P. E., same for underdamped and overdamped dynamics.
  - Quenched randomness ⇒ Noneqlbm. stat. state.
    - Prob. distr. $\neq \exp\Big[-\beta(K.E. + P.E.)\Big]$; In general, Not product measure.
    - Dynamics matters: Phase transitions different for underdamped and overdamped dynamics.

Summary

- Kuramoto model as an overdamped limit of a long-range interacting system.
- General dynamics: (1) External drive, (2) Quenched vs. annealed randomness.
  - No quenched randomness $\Rightarrow$ Equilibrium.
    - Prob. distr. $\sim \exp\Big[-\beta(K.E. + P.E.)\Big]$; Product measure.
    - Phase transition given by P. E., same for underdamped and overdamped dynamics.
  - Quenched randomness $\Rightarrow$ Noneqlbm. stat. state.
    - Prob. distr. $\neq \exp\Big[-\beta(K.E. + P.E.)\Big]$; In general, Not product measure.
    - Dynamics matters: Phase transitions different for underdamped and overdamped dynamics.

Summary

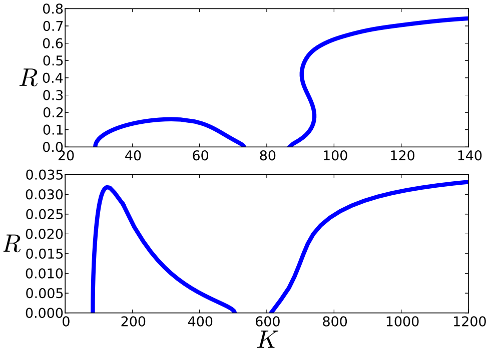
- Kuramoto model as an overdamped limit of a long-range interacting system.
- General dynamics: (1) External drive, (2) Quenched vs. annealed randomness.
  - No quenched randomness $\Rightarrow$ Equilibrium.
    - Prob. distr. $\sim \exp\Big[-\beta(K.E. + P.E.)\Big]$; Product measure.
    - Phase transition given by P. E., same for underdamped and overdamped dynamics.
  - Quenched randomness $\Rightarrow$ Noneqlbm. stat. state.
    - Prob. distr. $\neq \exp\Big[-\beta(K.E. + P.E.)\Big]$; In general, Not product measure.
    - Dynamics matters: Phase transitions different for underdamped and overdamped dynamics.



More general situations

1. Distr. of moment of inertia, $G(m)$.
2. More general $g(\omega)$.

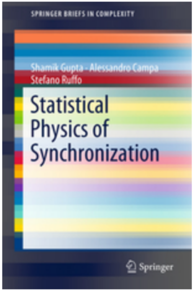
An analytically exact self-consistent approach predicts complex and non-trivial phase diagrams with reentrant transitions.



(Komarov, Gupta, Pikovsky (2014))

Crass Commercialism

SpringerBriefs in Complexity



© 2018

Statistical Physics of Synchronization

Authors: **Gupta**, Shamik, **Campa**, Alessandro, **Ruffo**, Stefano