Chimera patterns and solitary synchronization waves in distributed oscillator populations

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Contents of this talk:

Chimera patterns in the Kuramoto-Battogtokh model

[based on papers L.A. Smirnov, G.V. Osipov, A. Pikovsky J. Phys. A: Math. Theor. 50 , 08LT01 (2017);
In: Abcha N., Pelinovsky E., Mutabazi I. (eds) Nonlinear Waves and Pattern Dynamics. Springer, Cham. p 159-180. (2018)]

Solitary synchronization waves

[based on paper L.A. Smirnov, G.V. Osipov, A. Pikovsky Phys. Rev. E 98 , 062222 (2018)]

Kuramoto-Battogtokh model as a set of partial differential equations

Original KB model: Integral equation

$$\frac{\partial \varphi}{\partial t} = \omega - \int_0^L G(x - \tilde{x}) \sin(\varphi(x,t) - \varphi(\tilde{x},t) - \alpha) d\tilde{x}$$

Step 1: introduce a coarse-grained complex order parameter

$$Z(x,t) = \langle e^{i\varphi(x,t)} \rangle \big|_{x-\Delta<x<x+\Delta}$$

Step 2: Apply the Ott-Antonsen ansatz for the dynamics of the order parameter

$$\frac{\partial Z}{\partial t} = i\omega Z + \frac{1}{2}\left(He^{-i\alpha} - H^* Z^2 e^{i\alpha}\right) \qquad H = \int_0^L G(x - \tilde{x}) Z(\tilde{x},t) d\tilde{x}$$

Step 3: Exponential kernel corresponds to a differential operator (cf. Lecture of C. Laing)

$$H = \int_{-\infty}^{\infty} \exp[-\kappa|x - \tilde{x}|] Z(\tilde{x}, t) d\tilde{x} \quad \Leftrightarrow \quad \frac{\partial^2 H}{\partial x^2} - \kappa^2 H = -\kappa^2 Z$$

Step 4: Apply this to a periodic domain (this slightly modifies the kernel)

$$H(0) = H(L),\ \partial_x H(0) = \partial_x H(L) \quad \Leftrightarrow \quad G(x) = \frac{\kappa \cosh \kappa(|x| - L/2))}{2 \sinh(\kappa L/2)}$$

Result: a system of PDEs with periodic boundary conditions

$$\frac{\partial Z}{\partial t} = i\omega Z + \frac{1}{2}\left(He^{-i\alpha} - H^* Z^2 e^{i\alpha}\right) \qquad \frac{\partial^2 H}{\partial x^2} - \kappa^2 H = -\kappa^2 Z$$

We do not solve the consistency equation (nonlinear eigenvalue problem) for fixed length of the domain *L*, but find periodic in space and time solutions (standing waves) of the system of PDEs

ODE for the chimera patterns

$$\frac{\partial Z}{\partial t} = i\omega Z + \frac{1}{2}\left(He^{-i\alpha} - H^{*}Z^{2}e^{i\alpha}\right) \qquad \frac{\partial^2 H}{\partial x^2} - \kappa^2 H = -\kappa^2 Z$$

Rotating wave ansatz: $Z(x,t) = z(x)e^{i(\omega+\Omega)t}, \quad H(x,t) = h(x)e^{i(\omega+\Omega)t}$

Quadratic equation for *z*: $e^{i\alpha}h^{*}z^{2} + 2i\Omega z - e^{-i\alpha}h = 0$

Second-order ordinary differential equation for complex field *h*

$$\frac{d^2}{dx^2}h - h = \frac{\Omega + \sqrt{\Omega^2 - |h|^2}}{h^{*}\exp[-i\beta]} \qquad \beta = \pi/2 - \alpha$$

Fourth-order complex ODE can be reduced, due to phase-rotation invariance, to a three-dimensional real system

$$h(x) = r(x)e^{i\theta(x)}, \quad q(x) = r^2(x)\theta'(x)$$

$$r'' = r + \frac{q^2}{r^3} + \frac{\Omega}{r}\cos\beta - \frac{\sqrt{r^2-\Omega^2}}{r}\sin\beta$$

<-synchronous domain

$$q' = \Omega\sin\beta + \sqrt{r^2-\Omega^2}\cos\beta$$

if $|r| > |\Omega|$

$$r'' = r + \frac{q^2}{r^3} + \frac{\Omega + \sqrt{\Omega^2 - r^2}}{r}\cos\beta$$

asynchronous domain ->

$$q' = \left(\Omega + \sqrt{\Omega^2 - r^2}\right)\sin\beta$$

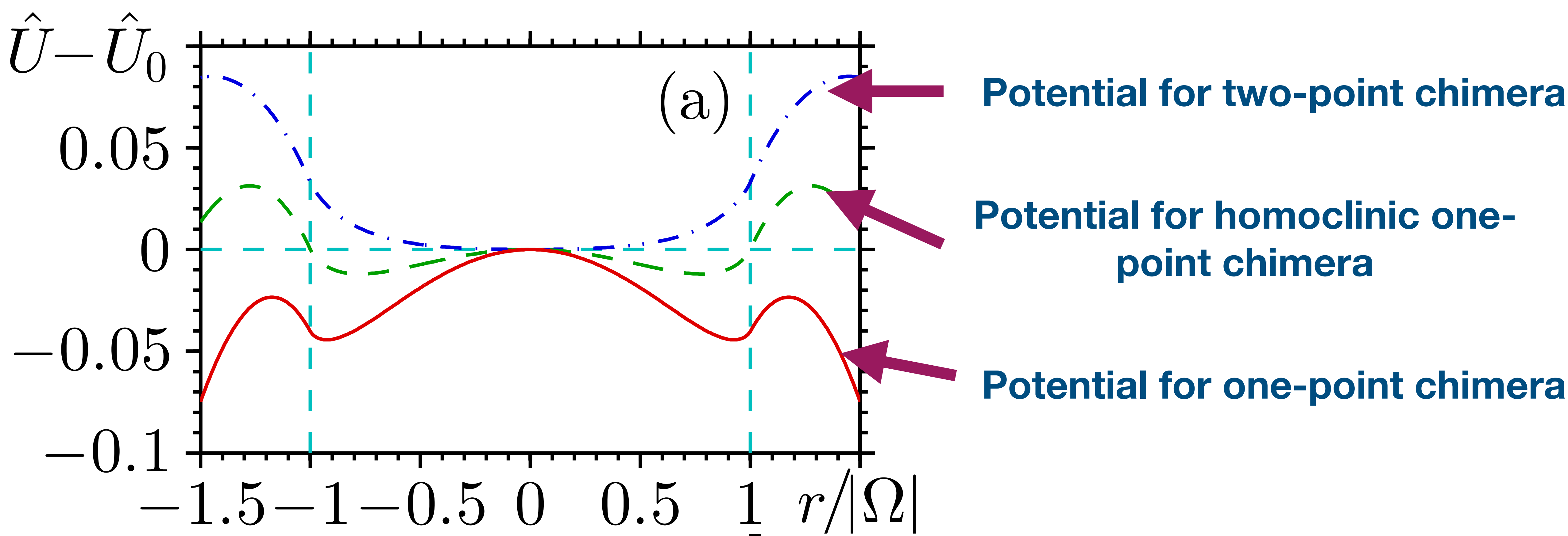
if $|r| < |\Omega|$

Analytic solutions: one- and two-point chimeras, chimera soliton

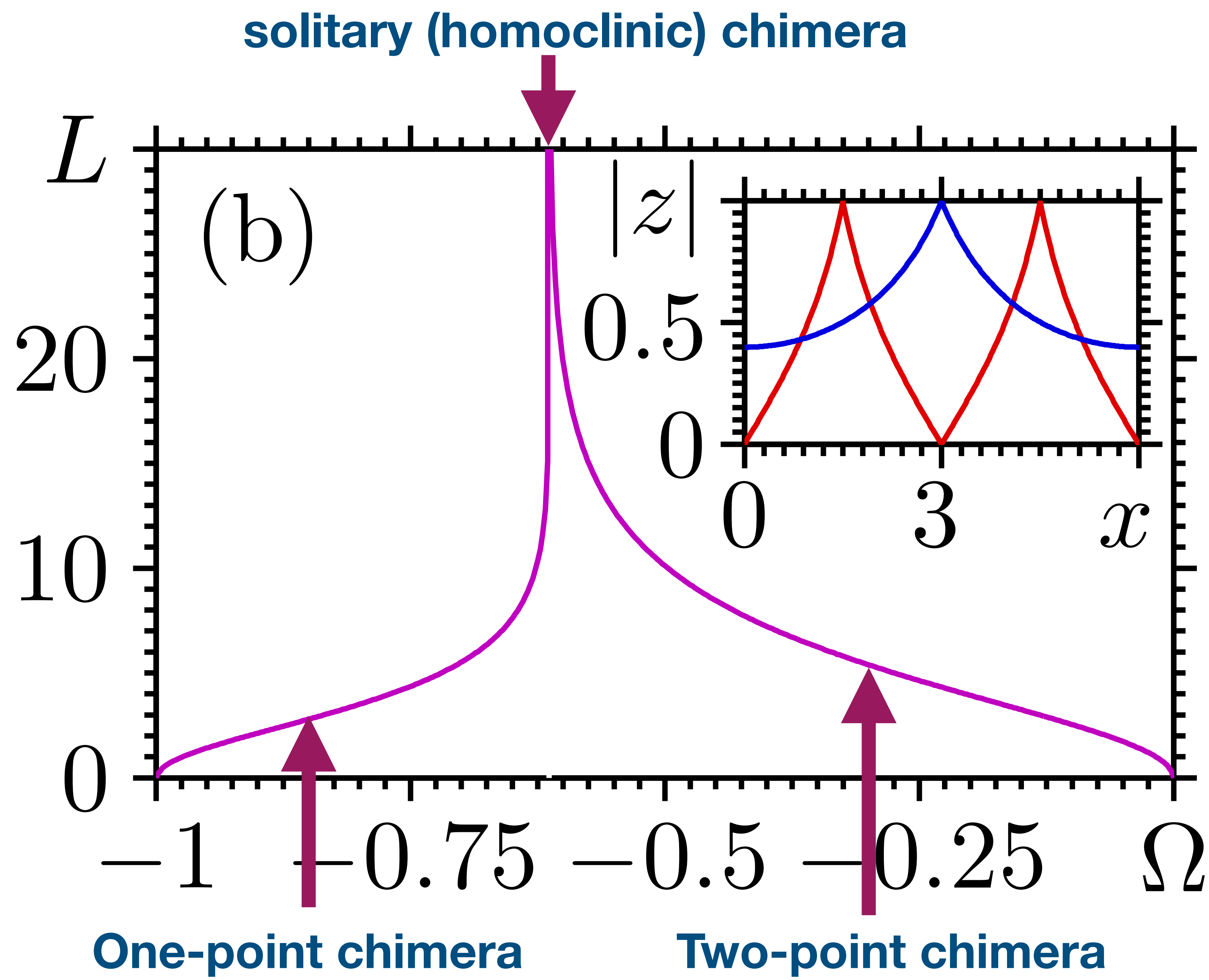
Case $\alpha = \pi/2, \beta = 0$ **is integrable!**

Dynamics only in the asynchronous domain, but synchrony can be achieved at one or two points

$$\frac{d^2 r}{dx^2} = -\frac{dU(r)}{dr}, \quad U(r) = -\frac{r^2}{2} - \sqrt{\Omega^2 - r^2} - \Omega \ln\left(\sqrt{\Omega^2 - r^2} - \Omega\right)$$



Period-frequency dependencies of singular one- and two-point chimeras

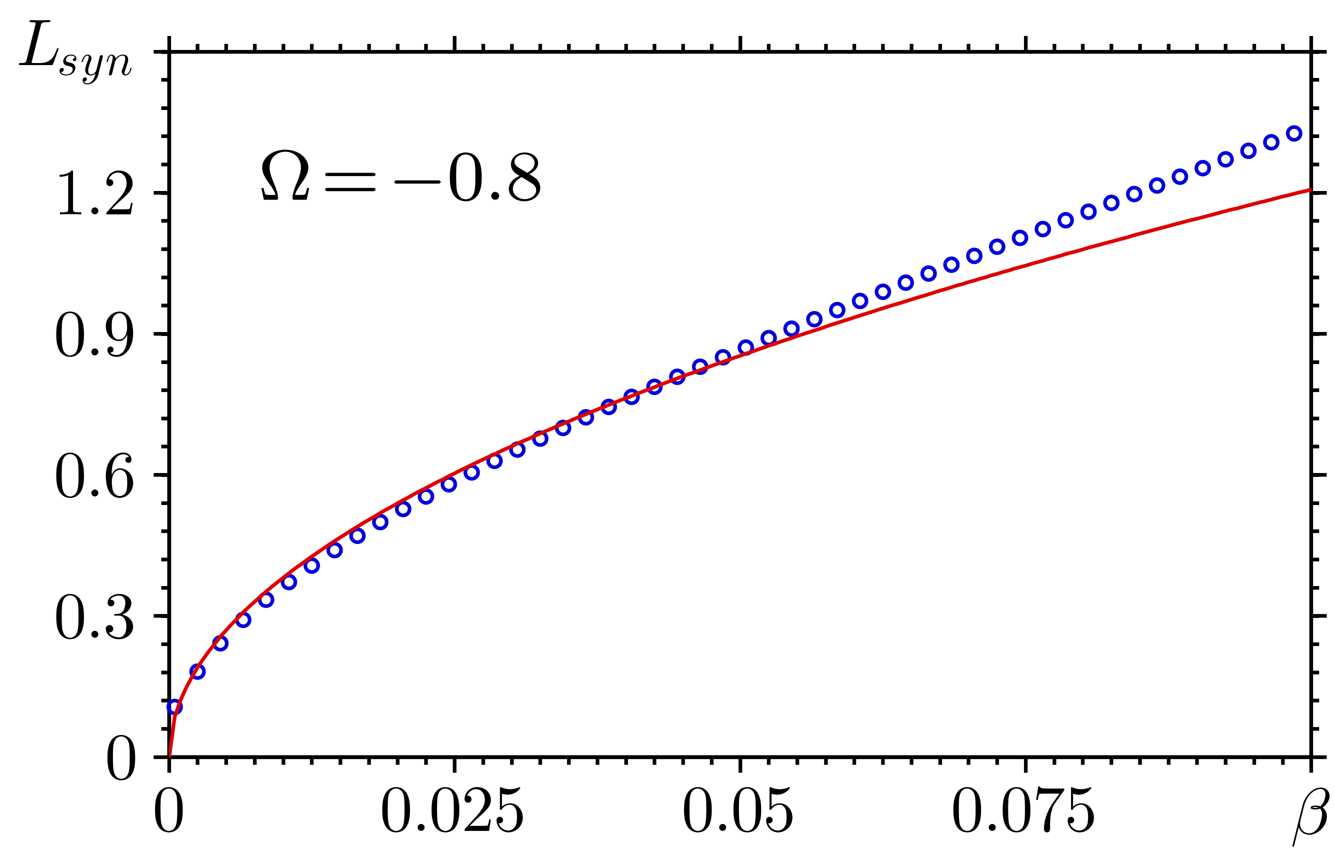


Perturbation theory close to the integrable case: $\frac{\pi}{2} - \alpha = \beta \ll 1$

Synchronous domain now is not a point but has finite length:

$$L_{syn} \approx \sqrt{\frac{8\beta}{\pi N_{SR}\sqrt{|\Omega|(1-|\Omega|)}} \oint (R'^2 + R^2)dx}$$

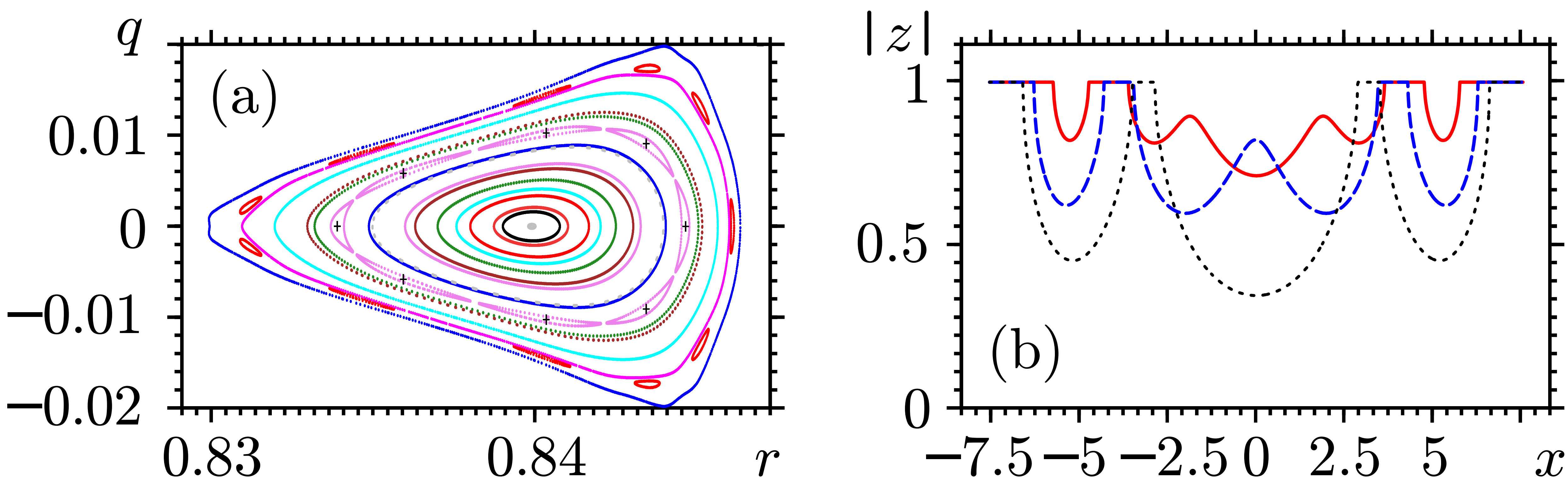
Here R is the solution at $\beta = 0$ and N_{SR} is the number of synchronous regions (1 or 2)

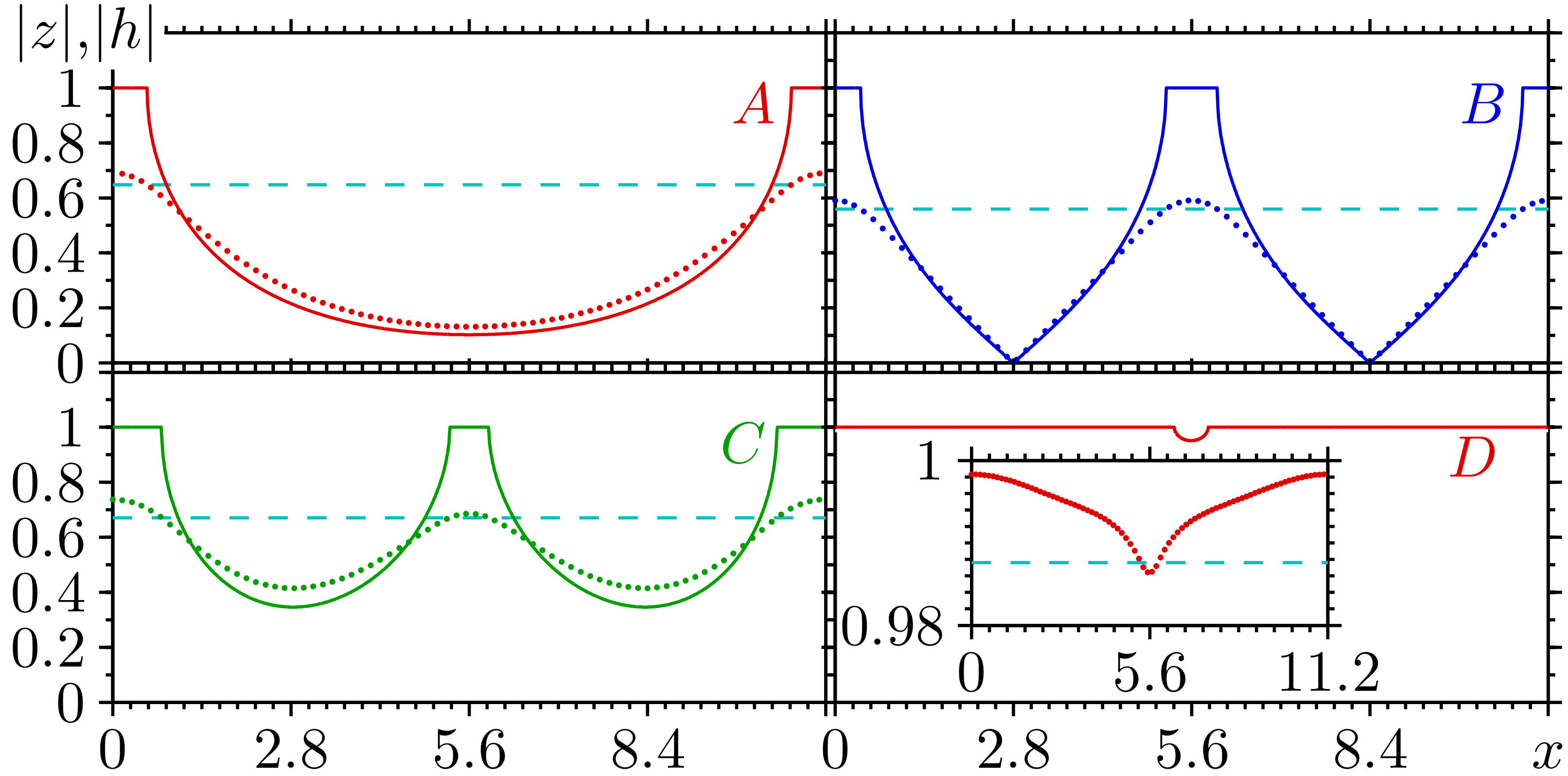


Chimera patterns as periodic orbits of ODE

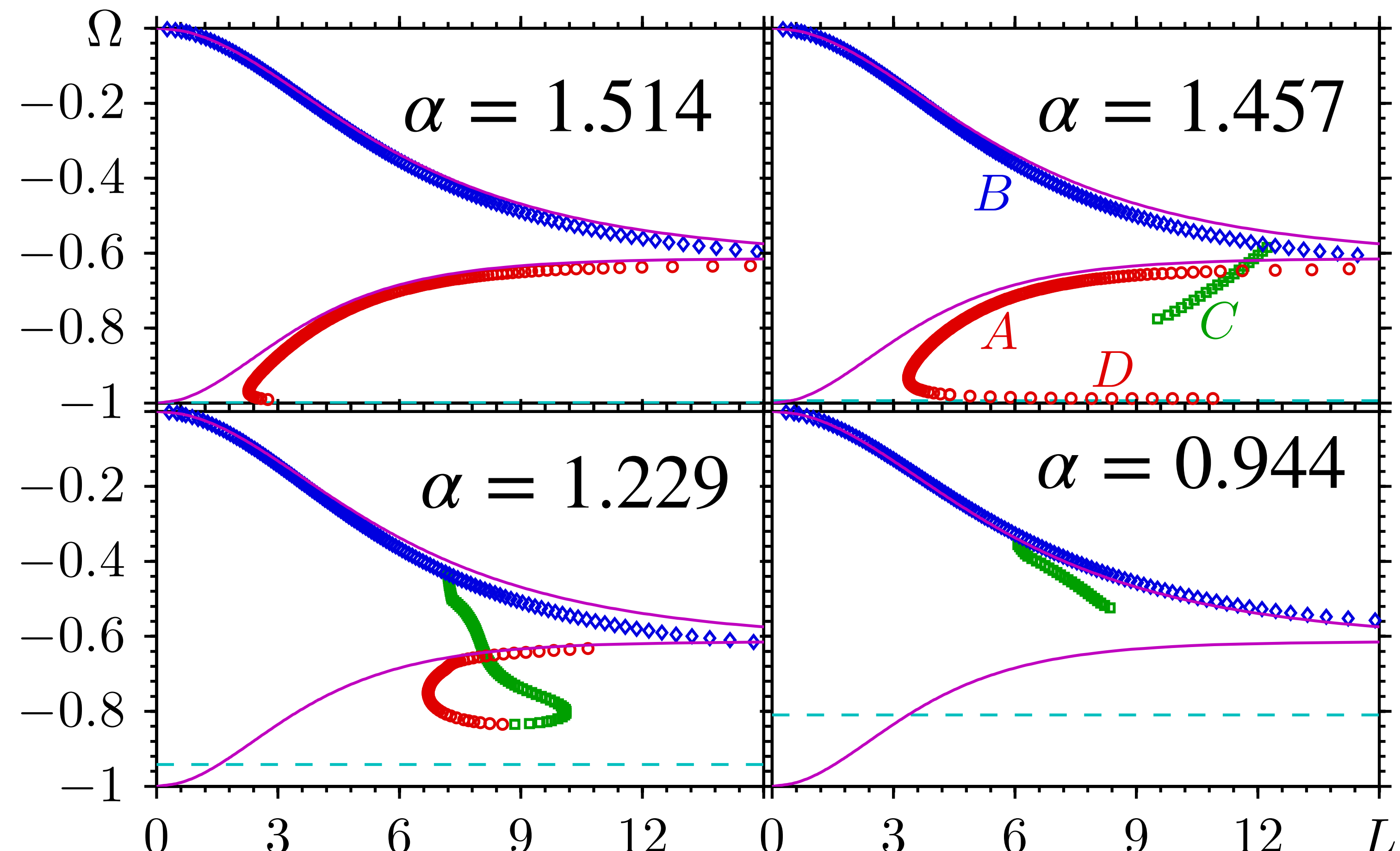
The system of ODEs for r(x) and q(x) is a reversible third-order system of ODEs with a plethora of solutions, including chaotic ones.

Poincare map | **Examples of chimera patterns**



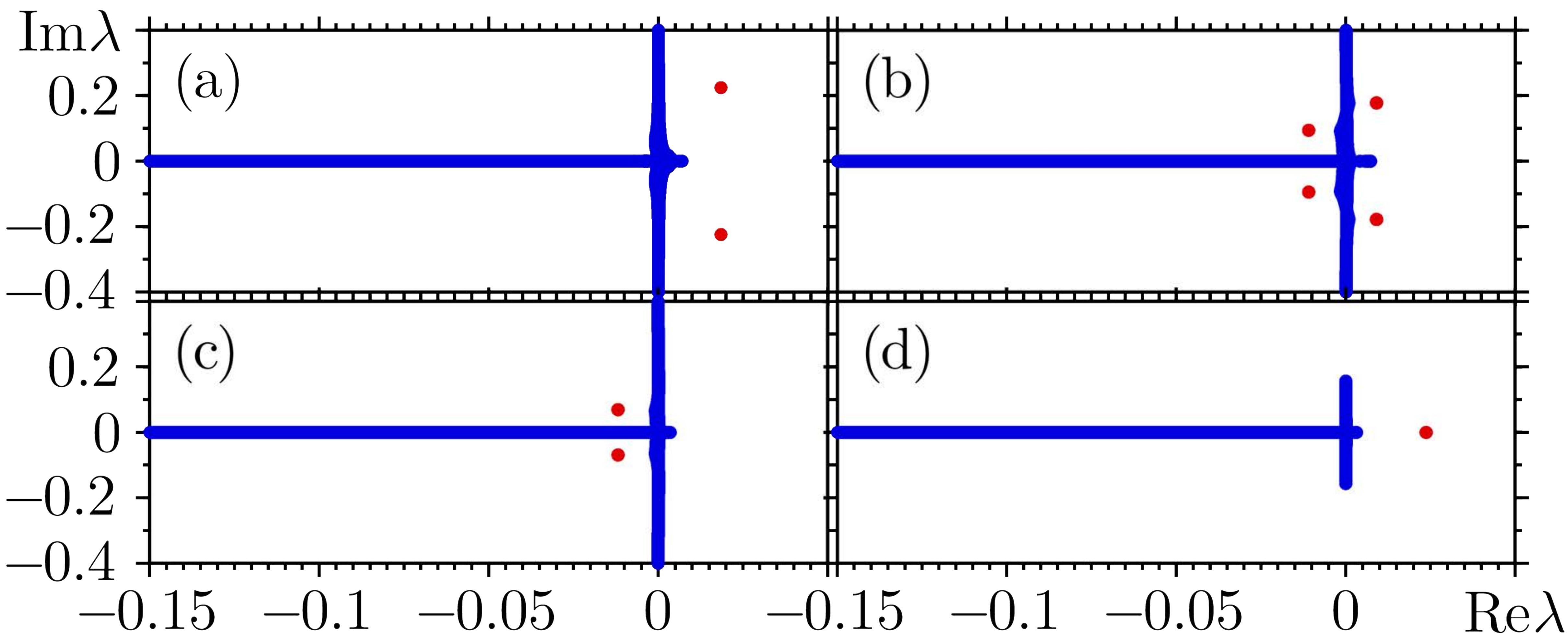


Four simple chimera patterns coexist for a particular domain length



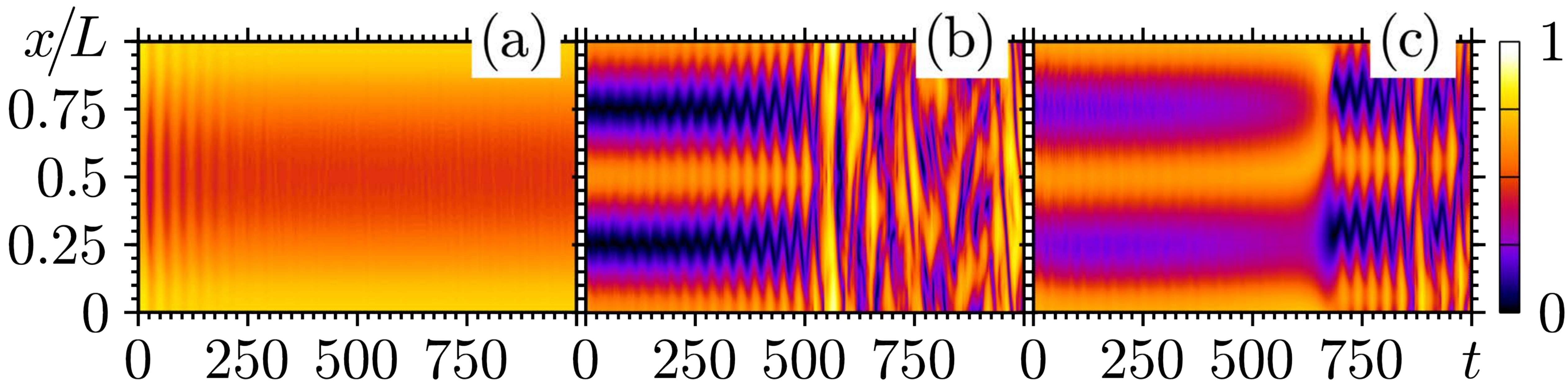
Stability properties

Essential and discrete spectra [according to O.E. Omel'chenko Nonlinearity 26, 2469 (2013); J. Xie et al. PRE 90, 022919 (2014)]



Only the “standard” Kuramoto-Battogtokh chimera is stable

Direct numerical simulations



Conclusions to this part

- Many chimera patterns can be found as periodic orbits of an ODE (potentially easier than solving a self-consistency problem)
- For neutral coupling, one-point and two-point chimera can be found analytically (represented as integrals), for nearly neutral coupling a perturbation theory on top of these solutions is developed
- No stable complex chimera patterns found, the only stable one is the KB chimera

Part 2: Solitary synchronyzation waves

Oscillatory medium with Laplacian coupling

Start with the KB-type model (1-d medium with non-local coupling)

$$\frac{\partial \varphi}{\partial t} = \mathbf{Im}\left(He^{-i\varphi}\right), \quad H(x,t) = e^{-\alpha}\int G(x-\tilde{x})e^{i\varphi(\tilde{x},t)}d\tilde{x}$$

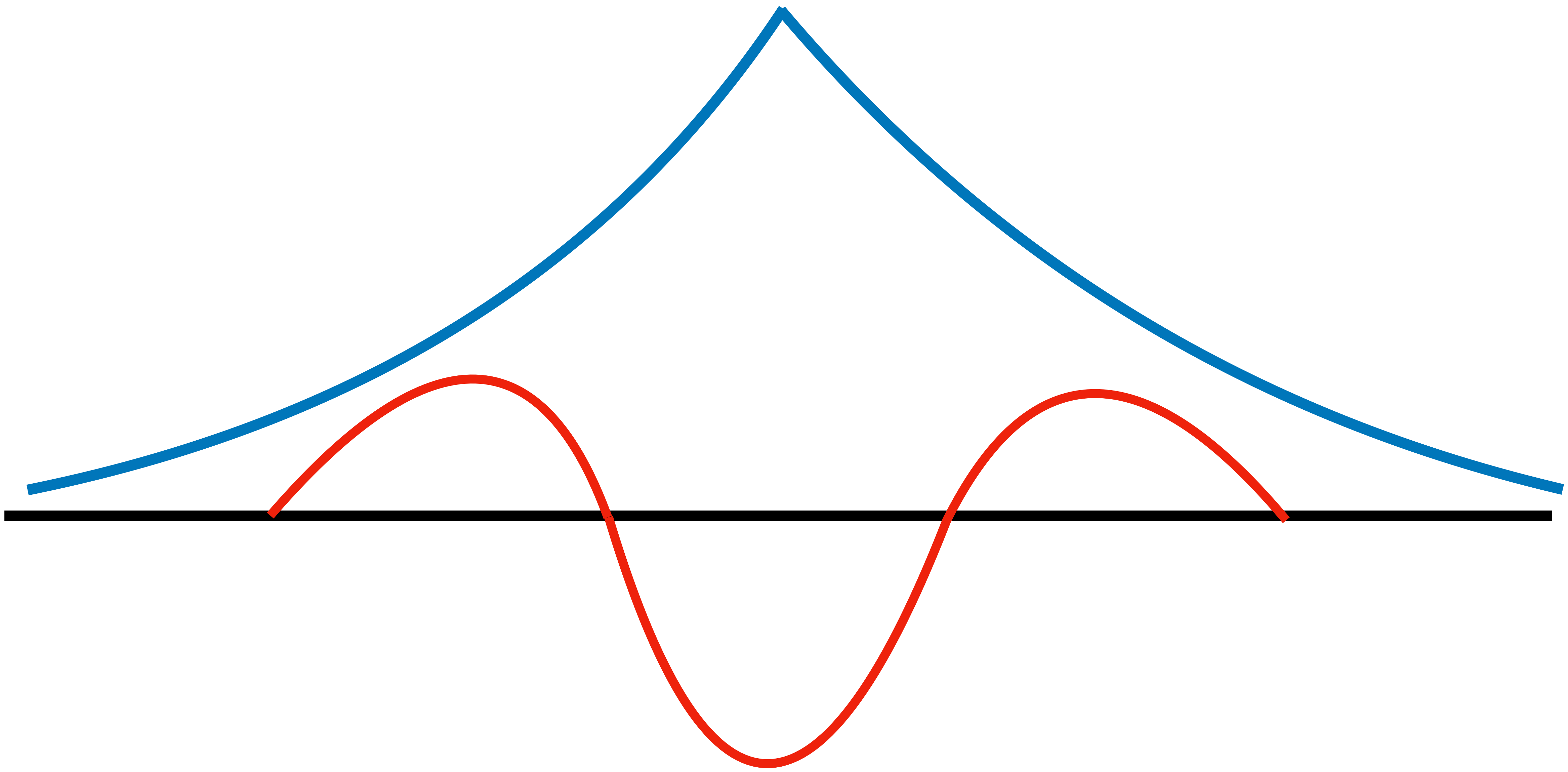
With Ott-Antonsen ansatz and coarse-grained order parameter Z

$$\frac{\partial Z}{\partial t} = \frac{1}{2}\left(e^{-i\alpha}H - H^{*}Z^{2}e^{i\alpha}\right), \quad H(x,t) = \int G(x-\tilde{x})Z(\tilde{x},t)d\tilde{x}$$

We assume a kernel with vanishing mean value (Laplacian coupling)

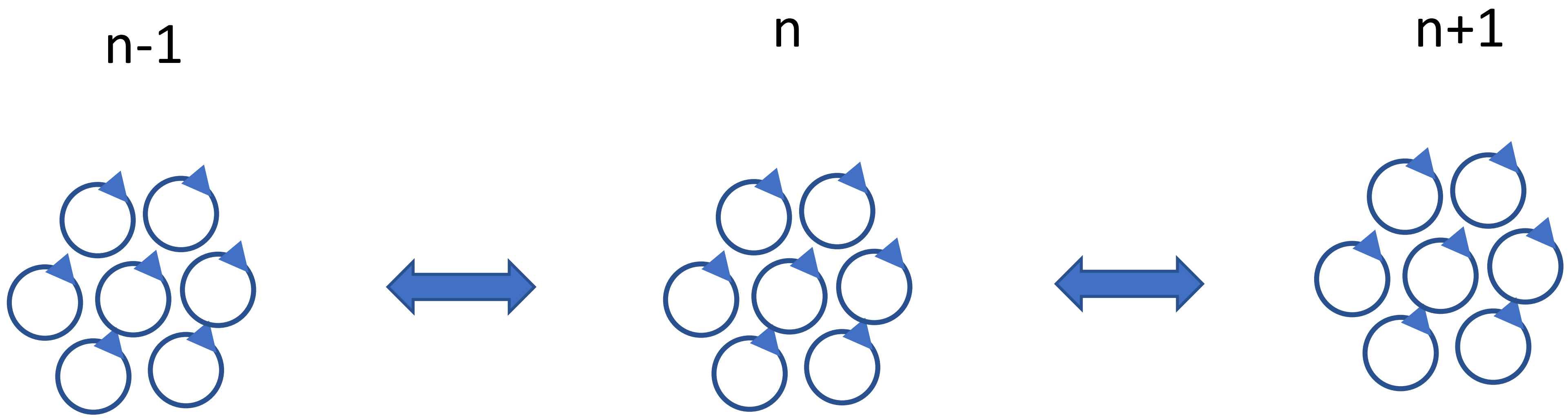
$$\int G(x)dx = 0 \quad \textbf{for example} \quad G(x) \sim (x^2-\sigma^2)e^{-\frac{x^2}{2\sigma^2}}$$

Exponential kernel - like mean field coupling, enables synchrony



Laplacian coupling - allows for any constant level of synchrony

Lattice with Laplacian coupling



Local dynamics at site *n* is described by the local order parameter Z_n

Coupling with nearest neighbours: $H_n = e^{-i\alpha}\left(Z_{n-1} + Z_{n+1} - 2Z_n\right)$

$$\boxed{2\frac{dZ_n}{dt} = e^{-i\alpha}(Z_{n-1} + Z_{n+1} - 2Z_n) - e^{i\alpha}(Z^*_{n-1} + Z^*_{n+1} - 2Z^*_n)Z_n^2}$$

A lattice with linear and nonlinear coupling of "complex Ginzburg-Landau" or of "nonlinear Schroedinger" type

Conservative coupling

We choose $\alpha = -\pi/2$ and obtain a conservative lattice

$$2\frac{dZ_n}{dt} = i(Z_{n-1} + Z_{n+1} - 2Z_n) + i(Z^*_{n-1} + Z^*_{n+1} - 2Z^*_n)Z_n^2$$

Spatially uniform solutions: $Z_n = \varrho e^{i\theta}$ **with any** $0 \le \varrho \le 1$

Linear waves on top of this background have dispersion

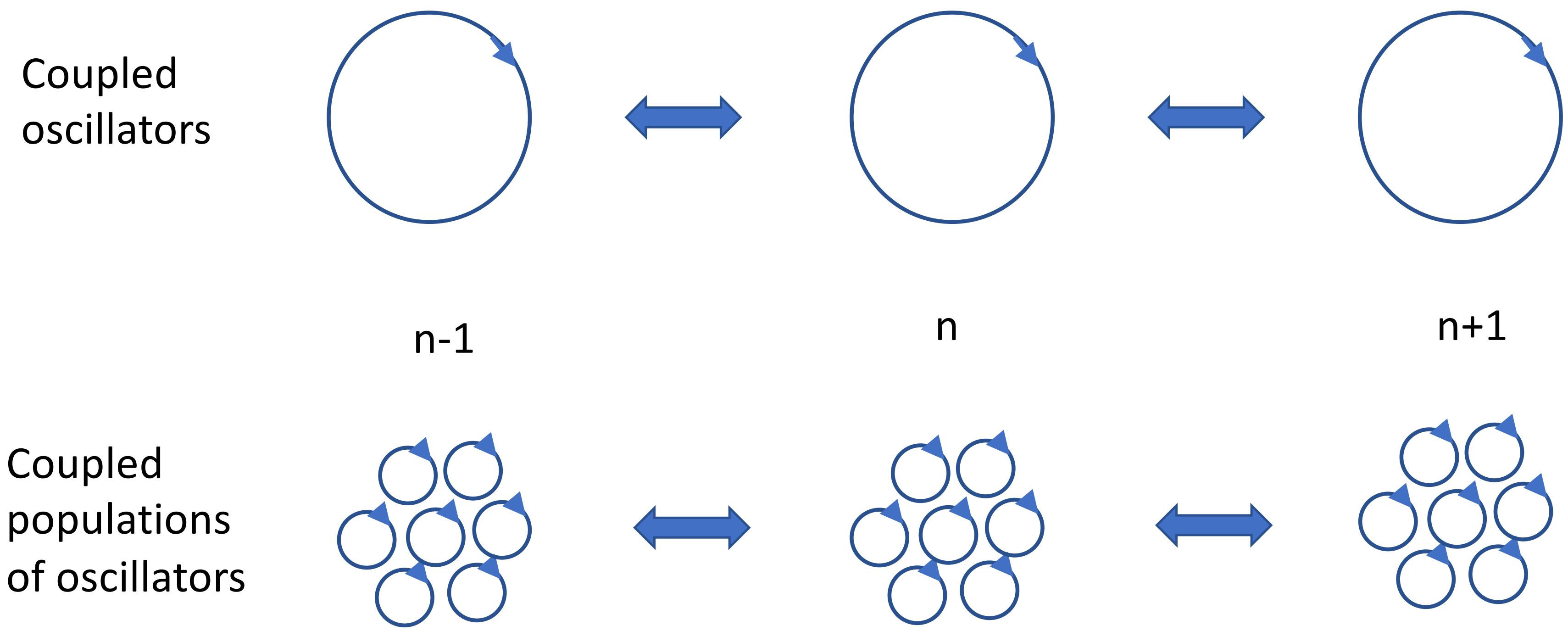
$$\omega(k) = \sqrt{1 - \varrho^4}(1 - \cos k)$$

Phase and group velocities:

$$\lambda_{ph} = \sqrt{1 - \varrho^4}\frac{1 - \cos k}{k}, \qquad \lambda_{gr} = \sqrt{1 - \varrho^4}\sin k$$

Solitary waves in the limit of full synchrony

If all oscillators on a site a synchronised, the problem reduces to a lattice of phase oscillators



The order parameter can be represented as $Z_n = e^{i\Theta_n}$

For the phase difference $V_n = \Theta_n - \Theta_{n-1}$ **we obtain**

$$\frac{dV_n}{dt} = \cos V_{n+1} - \cos V_{n-1}$$

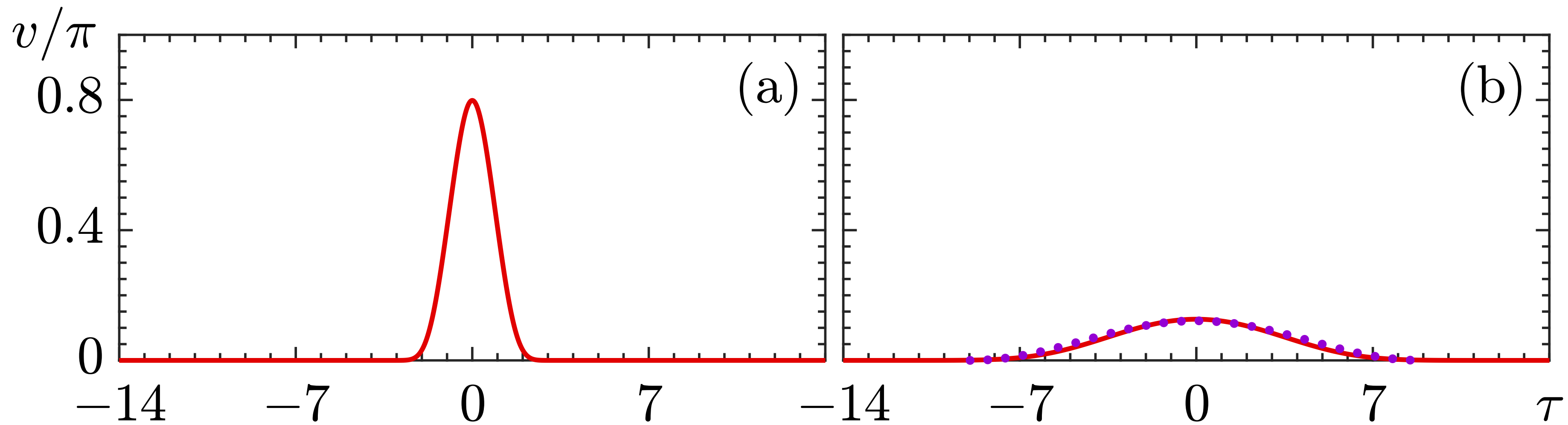
Compactons and kovatons

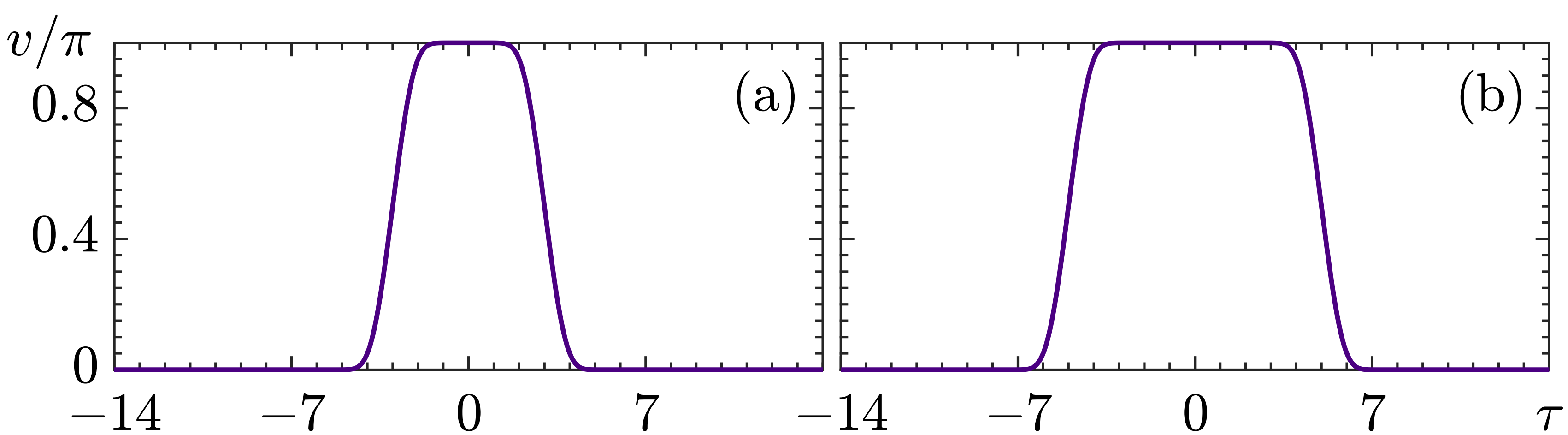
Equation $$\frac{dV_n}{dt} = \cos V_{n+1} - \cos V_{n-1}$$ has been studied in
P. Rosenau, A.P., PRL (2005); Physica D (2006)

Traveling wave ansatz $$V_n(t) = V\left(t - \frac{n}{\lambda}\right)$$

<u>Compactons</u>
are solitary
waves with
compact support

$$0 < \lambda < \lambda_c = \frac{4}{\pi}$$





<u>Kovatons</u>
are glued $0 \leftrightarrow \pi$
kinks with
arbitrary length

$$\lambda = \lambda_c$$

Solitary waves close to compactors

Full equations for the lattice: $Z_n = \rho_n e^{i\theta_n} \quad v_n = \theta_n - \theta_{n-1}$

$$\frac{d\rho_n}{dt} = \frac{1-\rho_n^2}{2}\left(\rho_{n-1}\sin v_n - \rho_{n+1}\sin v_{n+1}\right)$$

$$\frac{d\theta_n}{dt} = \frac{1+\rho_n^2}{2\rho_n}\left(\rho_{n-1}\cos v_n + \rho_{n+1}\cos v_{n+1} - 2\rho_n\right)$$

Traveling wave ansatz:

$$\rho_n(t) = \rho(\tau),\ \theta_n(t) = \theta(\tau)$$

$$\tau = t - \frac{n}{\lambda}$$

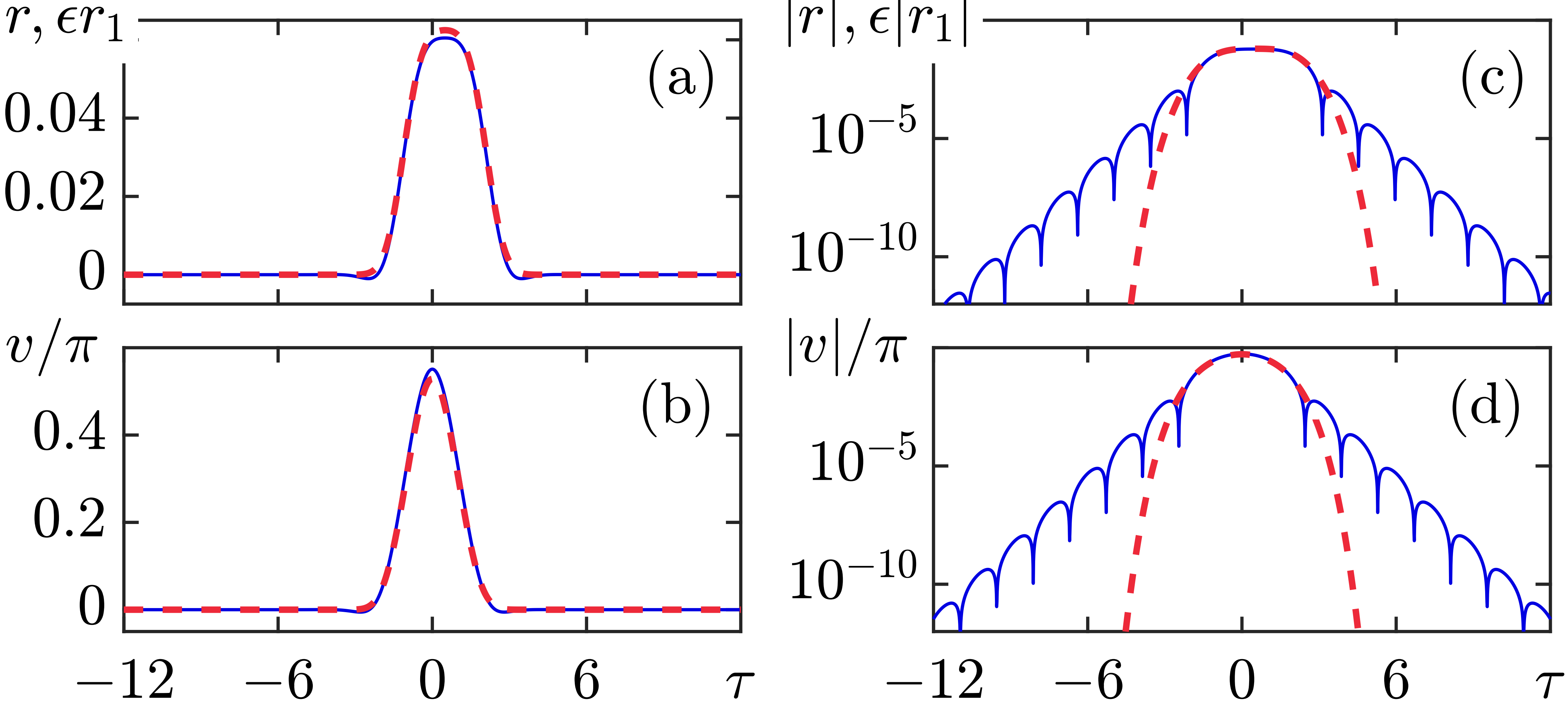
Perturbation approach close to full synchrony (close to true compactons)

$$\epsilon = 1 - \varrho \ll 1 \qquad \rho(\tau) = \varrho + \epsilon r_1(\tau) + \dots \quad v(\tau) = V(\tau) + \epsilon v_1(\tau) + \dots$$

Analytic expression for the 1st correction:

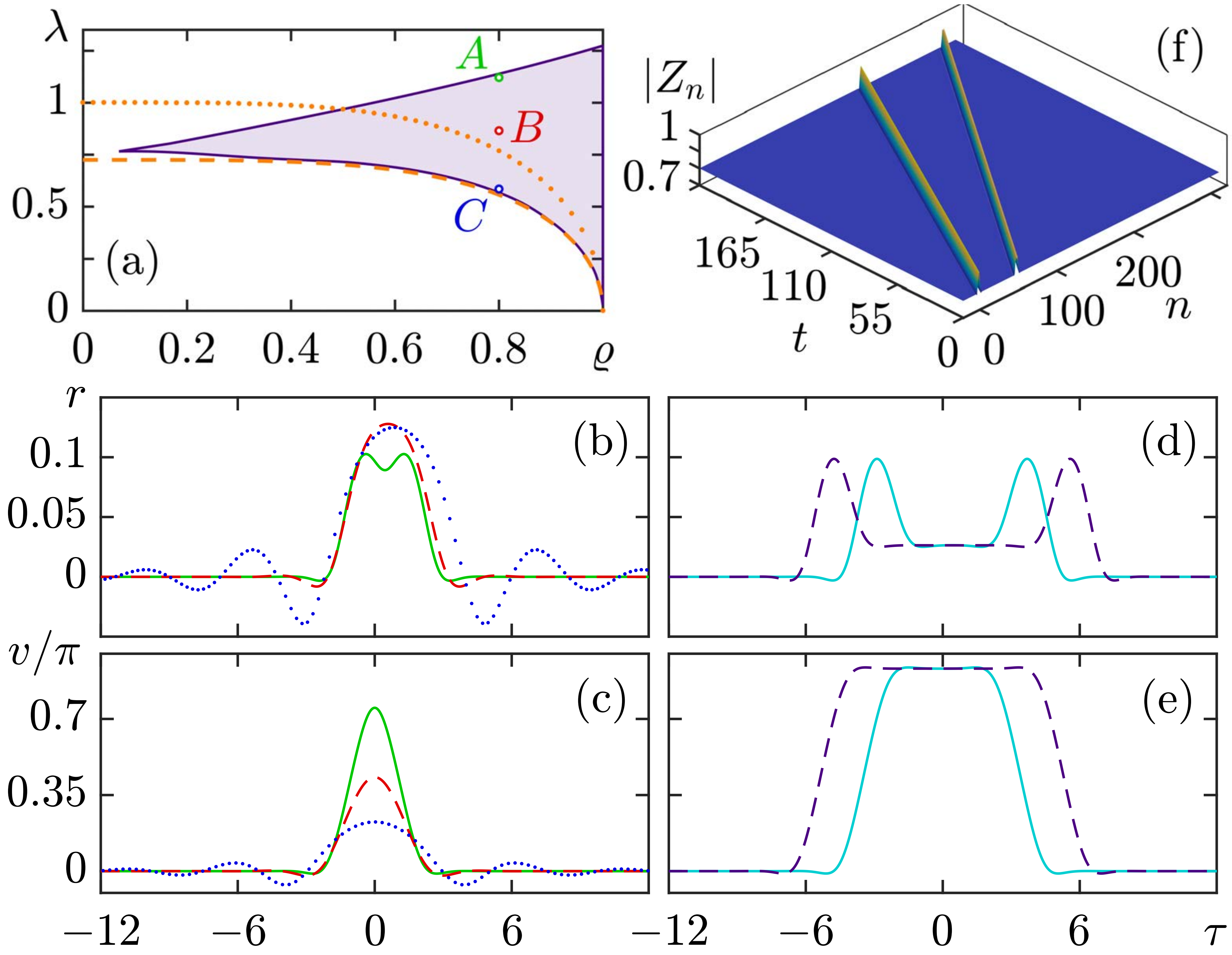
$$r_1(\tau) = 1 - \exp\left[\int_{-\infty}^{\tau}\left(\sin V(\tilde\tau - 1/\lambda) - \sin V(\tilde\tau)\right)d\tilde\tau\right]$$

Comparison of approximate and exact solitary waves for $\varrho = 0.9$



Dashed red curves: approximate solution
Blue curves: exact solution

Exact solitary wave is not compact, but has exponentially decaying, oscillating tails



Examples of compactor-like and kovaton-like solitary waves and the domain on their existence on (ϱ, λ) plane

Illustration of solitons in lattice equations

$$2\frac{dZ_n}{dt} = i(Z_{n-1} + Z_{n+1} - 2Z_n) + i(Z^*_{n-1} + Z^*_{n+1} - 2Z^*_n)Z_n^2$$

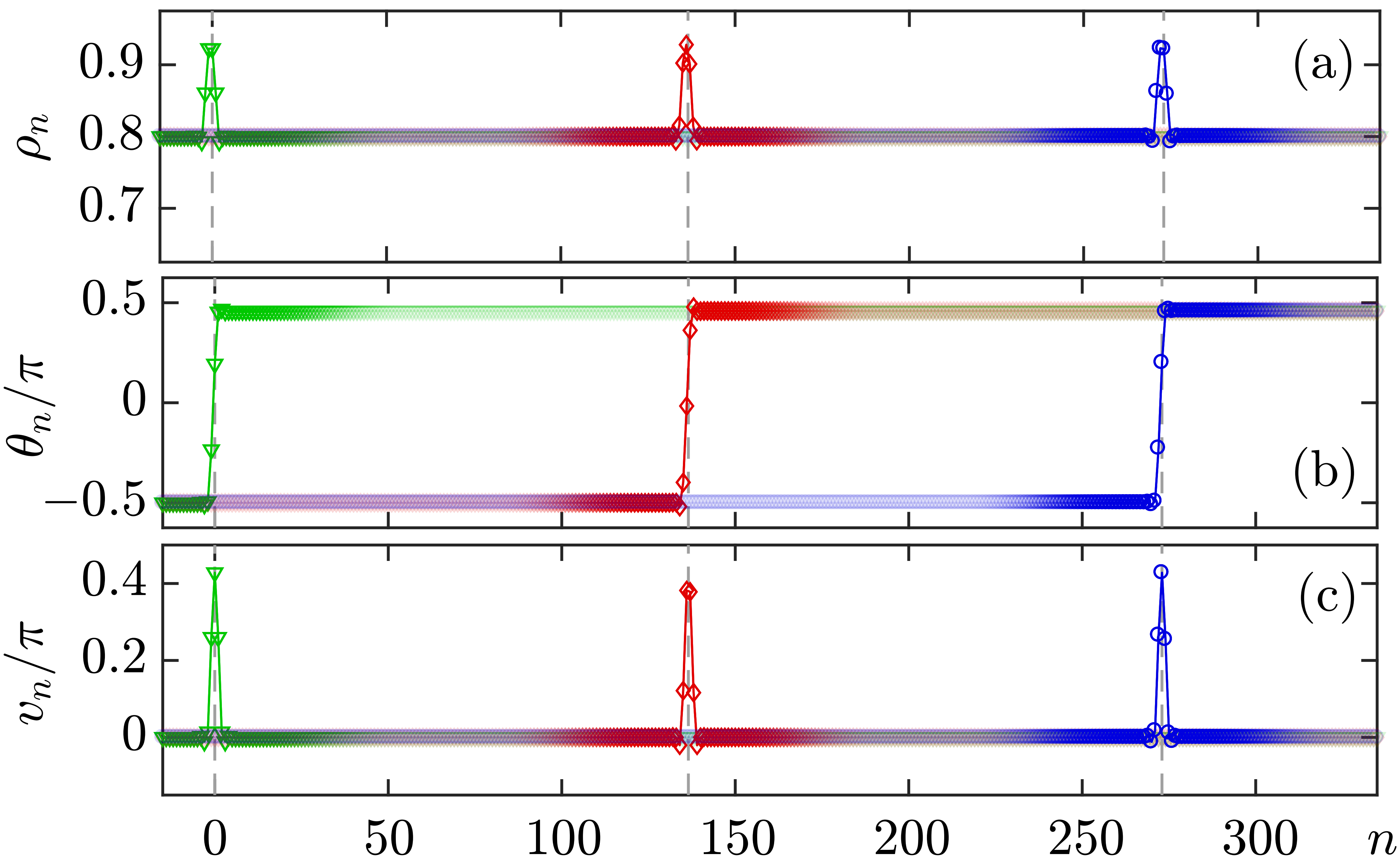
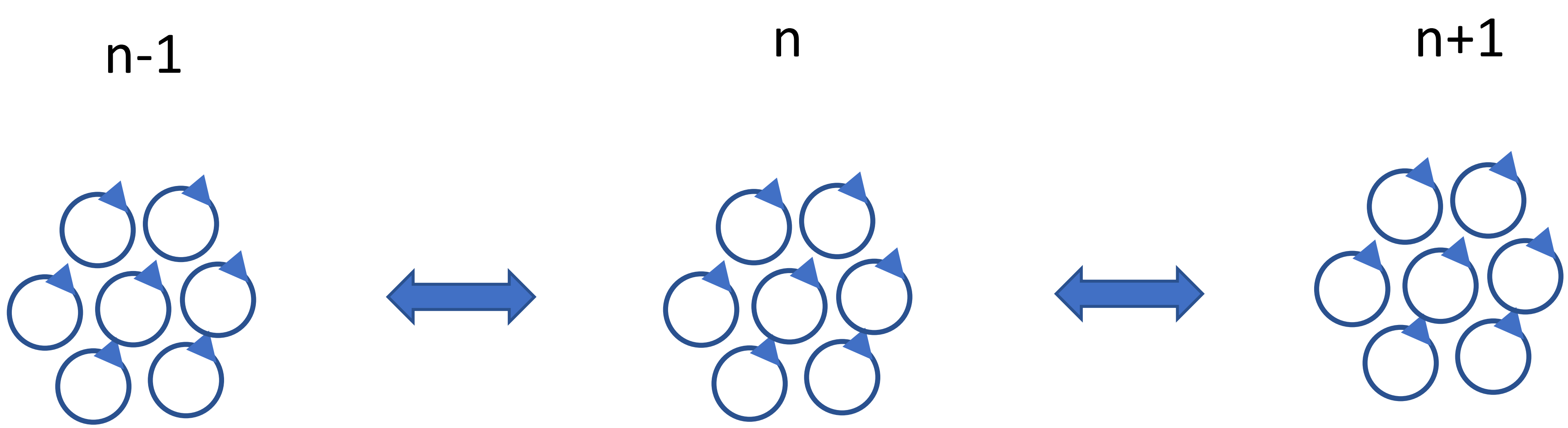
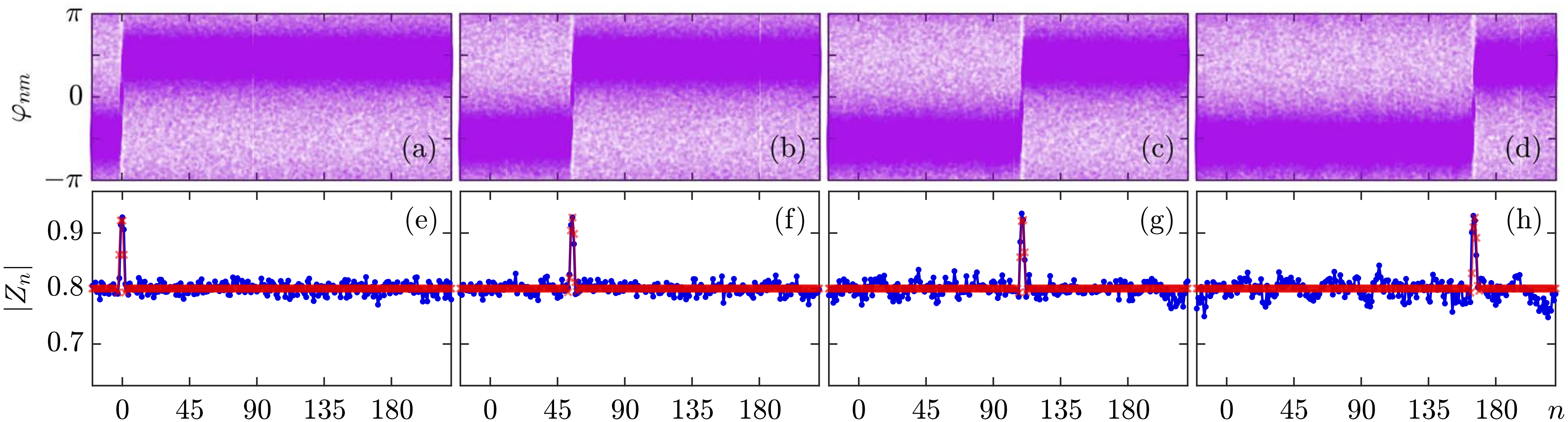


Illustration of solitons in phase equations



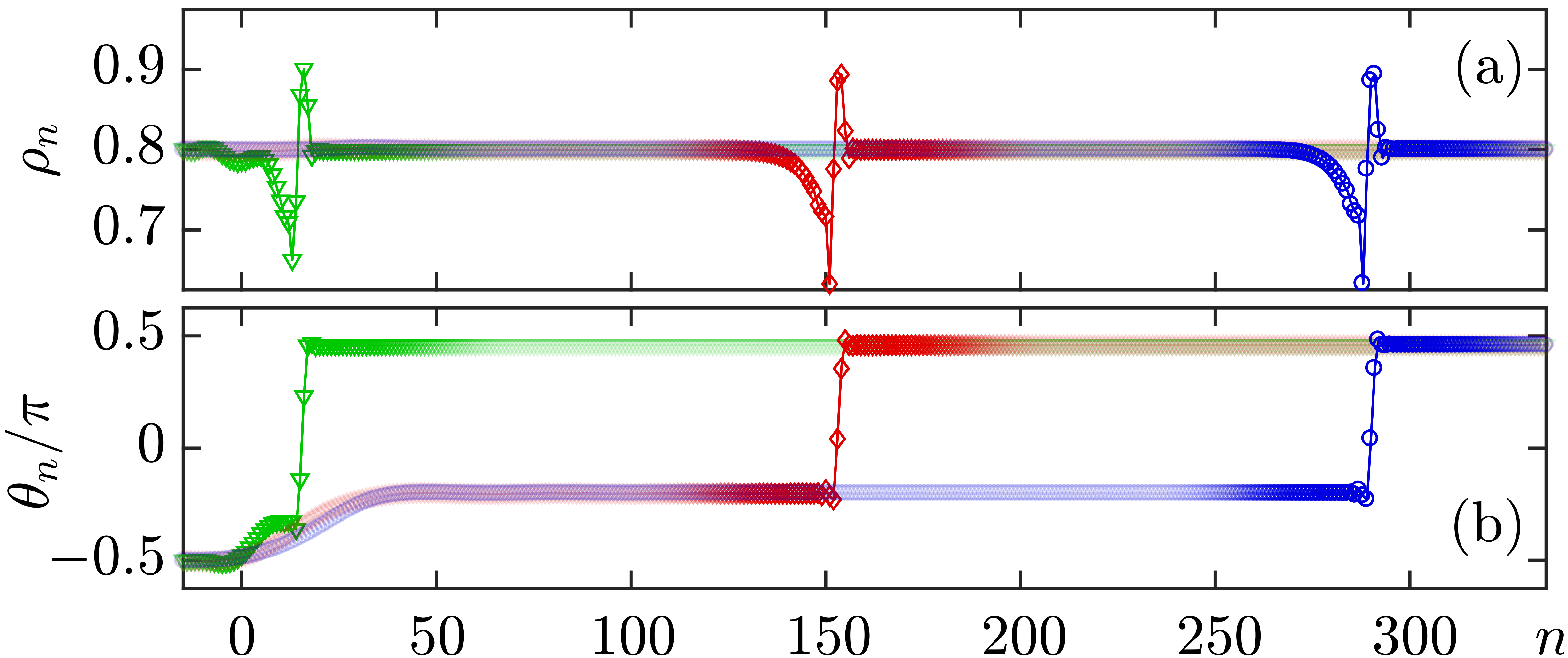


Nonconservative system: dissipative solitons

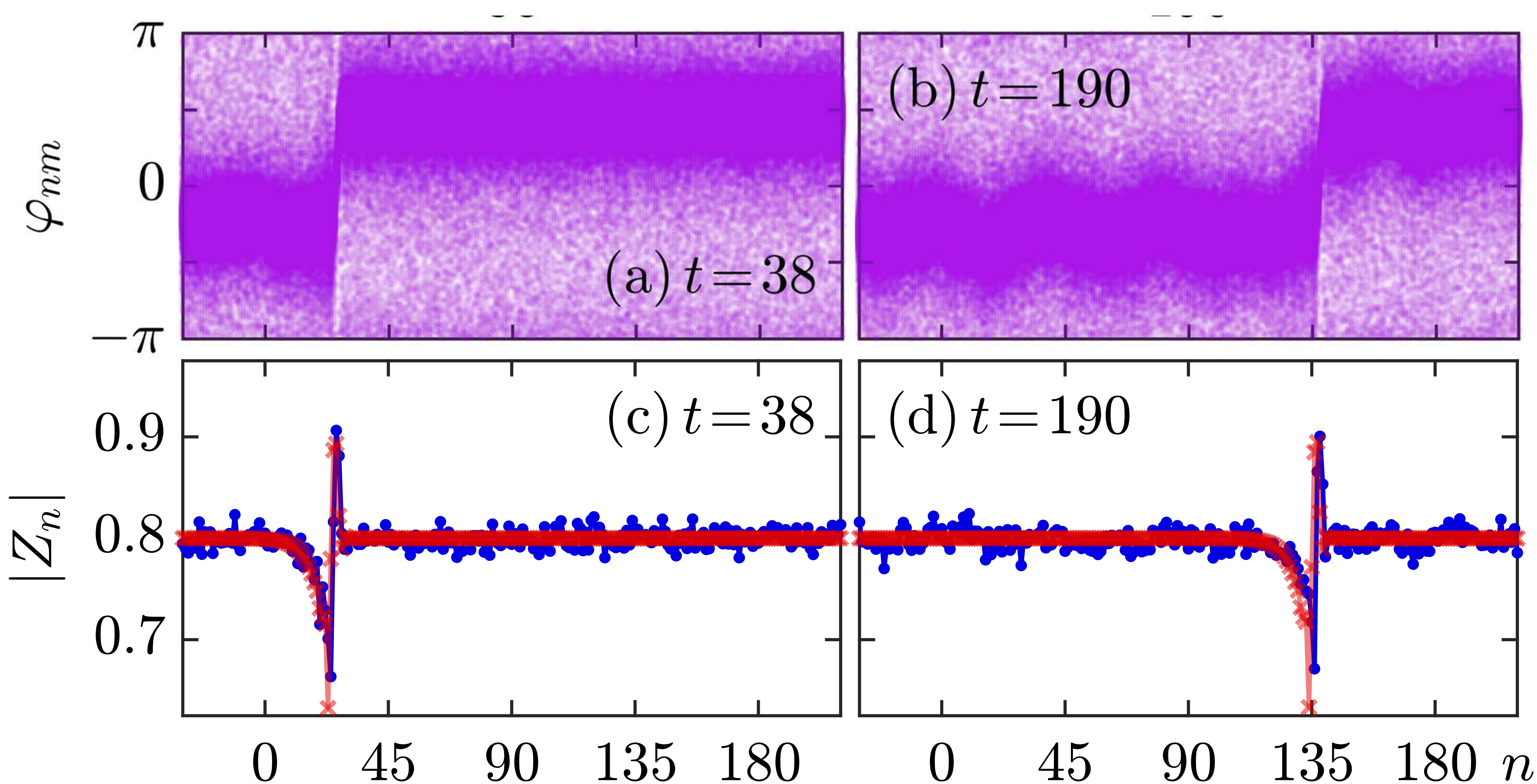
$$\frac{dZ_n}{dt} = \boxed{-\gamma Z_n} + \frac{1}{2}(H_n - H_n^* Z_n^2) \qquad H_n = i(Z_{n-1} + Z_{n+1} - 2Z_n) + \boxed{\mu Z_n}$$

Heterogeneity of oscillators **Local attractive coupling**

Spatially homogeneous stable level of synchrony $\varrho_* = \sqrt{(\mu_r - 2\gamma)/\mu_r}$



Dissipative soliton in a chain of phase populations



Continuous medium with Laplacian coupling

Medium of phase oscillators:

$$\frac{\partial \varphi}{\partial t} = \mathbf{Im}\left(He^{-i\varphi}\right), \quad H(x,t) = e^{-i\alpha}\int G(x-\tilde{x})e^{i\varphi(\tilde{x},t)}d\tilde{x}$$

Order parameter field (coarse-grained):

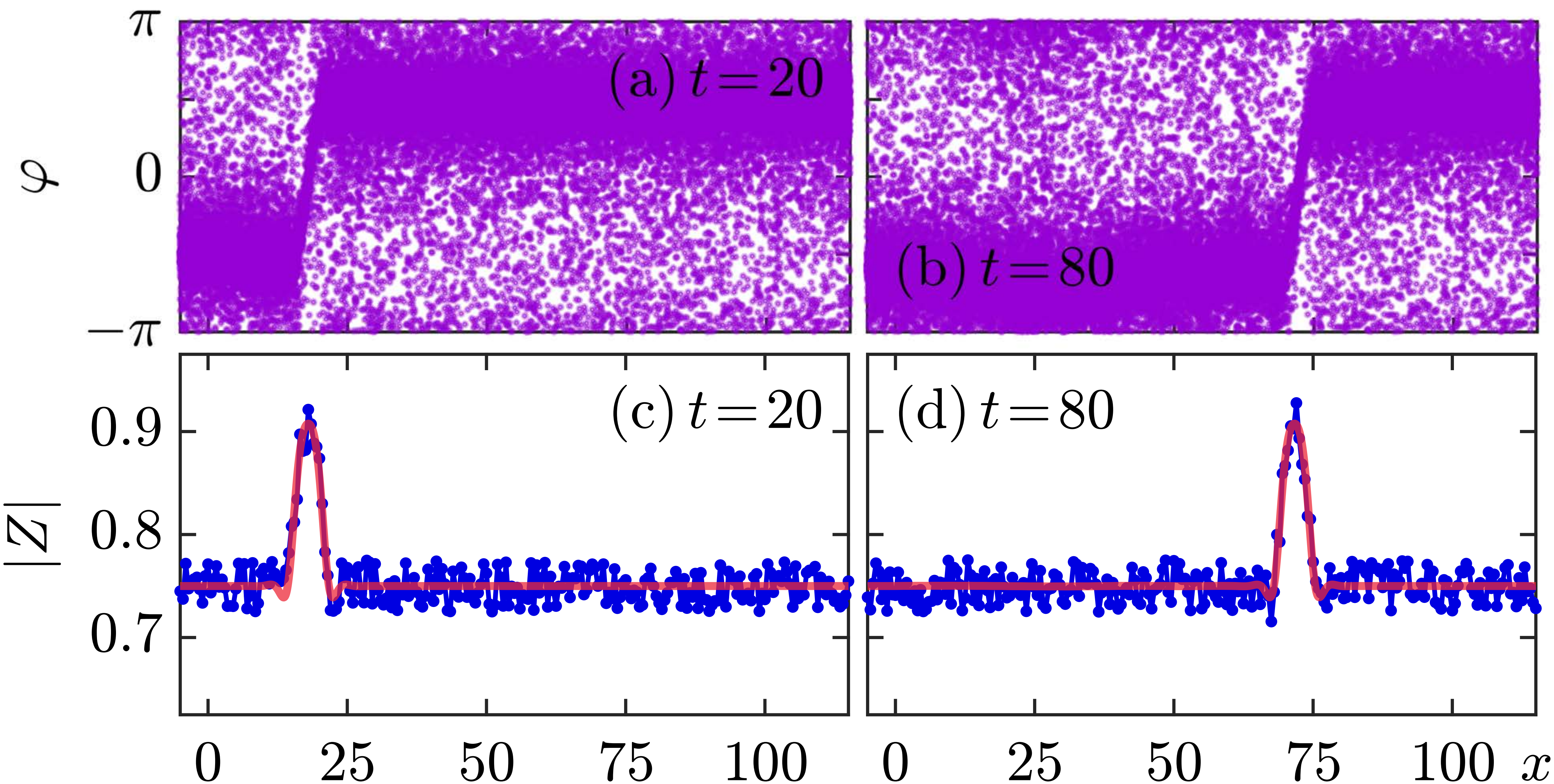
$$\frac{\partial Z}{\partial t} = \frac{1}{2}\left(H - H^{*}Z^{2}\right), \quad H(x,t) = e^{-i\alpha}\int G(x-\tilde{x})Z(\tilde{x},t)d\tilde{x}$$

Laplacian kernel coupling:

$$G(x) = A(x^2 - \sigma^2)e^{-\frac{x^2}{2\sigma^2}}$$

Solitary waves can be found numerically (Newton's method)

Conservative soliton in a medium with Laplacian coupling



Conclusions to this part

- Laplacian (local) coupling can be formulated for a medium or for a lattice
- Lattice equations for the complex order parameter resemble nonlinear Schroedinger lattice (for conservative case) or complex Ginzburg-Landau lattice
- Solitary waves can be traced from compactons and kovatons, existing in full synchrony limit
- No theory yet for dissipative solitons