NEUTRINOS - HOMEWORK PROBLEMS

1. The charged pion decays almost 100% of the time into a muon and a (muon-type) neutrino ($\pi^+ \to \mu^+\nu_\mu$). In the reference frame where the parent pion is at rest, compute the muon energy as a function of the muon-mass (m_μ), the charged pion mass (m_π), and the neutrino mass (m_ν). What is the absolute value of the muon momentum (tri)vector? Numerically, what is the relative change of the muon momentum between $m_\nu = 0$ and $m_\nu = 0.1$ MeV? It is remarkable that the muon momentum from pion decay at rest has been measured at the 3.4×10^{-6} level (Phys. Rev. D**53**, 6065 (1996)). This provides the most stringent current constraint on the "muon-neutrino mass."

2. At small enough energies ($\sqrt{s} \ll 100$ GeV), the neutrino scattering cross section with matter is approximately given by $\sigma_\nu \sim G_F^2 s/\pi$, where G_F is the Fermi constant and $s = (p+P)^2$ is the square of the center-of-mass energy of the neutrino (with four-momentum p_μ) plus target (with four-momentum P_μ) system. (a) Estimate the cross section, in cm^2, for neutrino–electron scattering and neutrino–neutron charged-current scattering (e.g. $\nu_e + n \to e^- + p$) when e and n are at rest and the neutrino energy $E_\nu = 10$ MeV. (b) Estimate the mean free path of a 10 MeV electron-type neutrino through lead, in AU (one Astronomical Unit. This is the average Earth–Sun distance, equal to 1.5×10^{11} m (or 500 light-seconds)). This is a good estimate of the average distance a high-energy solar neutrino will travel inside lead before converting into an electron. Assume that, at these energies, you can treat the nucleus as a collection of free nucleons and electrons (this is not completely correct but is not a bad approximation). Note that, at these energies, the charged-current process $\nu_e + p \to e^+ + n$ is forbidden (or very suppressed) due to the conservation of lepton number.

3. In February 1987, neutrinos from a Supernova that exploded in the Large Magellanic Cloud, located 50 kpc away from the Earth, reached the Kamiokande and the IMB experiments. This neutrino burst was reported in K. Hirata *et al.*, Phys. Rev. Lett. **58**, 1490 (1987) and R.M. Bionda *et al.*, Phys. Rev. Lett. **58**, 1494 (1987). Read the two papers and address the following questions: (a) Both experiments detect electron antineutrinos via $\bar{\nu}_e p \to e^+ n$. Calculate the incoming antineutrino energy, in the reference frame where the target protons are at rest, as a function of the recoil angle and energy of the positron, and the neutron and proton masses. You may set the positron mass to zero. Using the table of events in the Kamiokande paper, compute the energies of the antineutrinos that were deteced by the Kamiokande experiment. What is the highest (lowest) observed antineutrino energy? (b) Use the result from (a) to obtain an upper bound on the "electron antineutrino mass" (don't worry about mixing). The reasoning is the following: if neutrinos have mass, neutrinos with different energies propagate with slightly different velocities. Hence, the higher energy neutrinos should have arrived at the detectors before the lower energy ones. Compute the relative arrival time of two antineutrinos with two different energies (assume that the neutrino mass is a lot smaller than any neutrino energy in the problem). The time distribution of the events observed by Kamiokande is consistent with the expected antineutrino energy spread (a few seconds). From this fact, place an upper bound on the neutrino mass. [The last three events arrives more than nine seconds after the first event, and there is still a debate regarding whether these are really from the Supernova. Discard them, and consider only the first nine events.] Useful information: 1 parsec is about 3×10^{16} m.

4. To understand the effect of neutrino oscillations (consider two flavor $\nu_\mu \leftrightarrow \nu_\tau$ transitions) on the atmospheric muon-neutrino data, numerically calculate and draw histograms of the average muon neutrino survival probability in ten equal-size bins of $\cos\theta_z$, where θ_z is the angle between the neutrino direction and the vertical-axis at the detector's location ($\theta_z = 0$ for neutrinos coming straight from above, and $\theta_z = \pi$ for neutrinos coming from below). Make one histogram for $E_\nu = 0.2$ GeV, 2 GeV, and 20 GeV plus $\Delta m^2 = 2.5 \times 10^{-4}$ eV^2, 2.5×10^{-3} eV^2, and 2.5×10^{-2} eV^2, for a grand total of

nine plots. Assume throughout that the mixing is maximal, i.e., $\sin^2 2\theta = 1$, and that neutrinos are produced 20 km above the surface of the Earth.

5. Read the article "Super-Kamiokande Atmospheric Neutrino Results" by T. Toshito, hep-ex/0105023. It contains an old but not too outdated summary of the atmospheric neutrino data. A talk by T. Kajita, presented at the Neutrino 1998 Conference, may also prove helpful in understanding some of the Super-Kamiokande terminology: hep-ex/981001.

   (a) From Table 1, compute the value of the "ratio-of-ratios" R (the measured ν_μ to ν_e flux ratio divided by the theoretical calculation) for sub-GeV and multi-GeV single ring events, and compare them to the numbers quoted in the paper. How do these numbers compare to 1, to each other, and to the ratio of observed partially contained events to the Monte Carlo calculation (this are all muon-type events, and consist of events whose average energy is larger than that of the multi-GeV events)?

   (b) Look at Figure 1, and compare with the results you got in problem 1. Can you verify that $\Delta m^2 \sim 2.5 \times 10^{-3}$ eV^2 and $\sin^2 2\theta \sim 1$ is a good fit to the data (200 MeV is characteristic of sub-GeV events, 2 GeV is typical of multi-GeV events, and 20 GeV is typical of upward stopping muons. The fourth category, upward-through-going muons, has an average energy above 100 GeV)? In particular, explain why there is almost no depletion for $\cos\theta_z > 0.2$ in the multi-GeV data, but some depletion in the sub-GeV data.

   (c) Use the number of observed sub-GeV "e-like" events (as these seem to agree well with Monte Carlo predictions) to obtain an order of magnitude estimate of the electron neutrino flux (neutrinos per unit time and unit area). The cross section for detecting neutrinos at this energy range is roughly 5 fb.

6. **Understanding SNO data** – Read Phys. Rev. Lett. **89**, 011301 (2002), which describes the first results of the SNO (Sudbury Neutrino Observatory) experiment [nucl-ex/0204008]. In page 4, the collaboration quotes the measured values of the "solar neutrino flux," obtained by using different physical processes: $\phi_{\rm CC}$ is determined from the Charged Current reaction $\nu + d \to p + p + e^-$ (d is a deuteron nucleus), $\phi_{\rm NC}$ is determined from the Neutral Current reaction $\nu + d \to n + p + \nu$, while $\phi_{\rm ES}$ is determined from the Elastic Scattering reaction $\nu + e^- \to \nu + e^-$. These flux-measurements are obtained assuming that only electron-type neutrinos are coming from the Sun.

   The key point is that the Charged Current process is only sensitive to electron-type neutrinos, the Neutral Current reaction is flavor blind (i.e. does not care whether the neutrino is of the electron-, muon- or tau-type), while the Elastic Scattering reaction is sensitive to electron-type neutrinos and muon/tau-type neutrinos in a different way. At the energy range of interest to the SNO experiment, the ratio of the elastic scattering cross sections is $\sigma_{\rm ES}(\nu_a + e)/\sigma_{\rm ES}(\nu_e + e) = 0.154$, and very close to being energy independent. Here, $a = \mu$ and/or τ.

   (a) Rewrite $\phi_{\rm CC}$, $\phi_{\rm NC}$ and $\phi_{\rm ES}$ in terms of ϕ_e and ϕ_a, the flux of electro-type solar neutrinos and the flux of muon/tau-type solar neutrinos. Given the experimental results obtained by SNO, compute ϕ_e and ϕ_a.[1] Compare your results with those obtained by the collaboration, quoted in page 5. The fact that $\phi_a \neq 0$ is, currently, the most concrete evidence we have of neutrino flavor conversion, since there are no physical processes capable of producing non-electron-type neutrinos inside the Sun.

[1] This system is overconstrained – there are three equations and two unknowns. You can either make sure that the three measurements are consistent, or you can perform a quick fit to the three measurements. If you choose to do this, for simplicity, add the statistical and systematic errors in quadrature, and assume that this combined error is Gaussian. I recommend the second option.

(b) Assume that the survival probability of electron-type neutrinos P_{ee} is energy independent, so you can rewrite $\phi_e = P_{ee}\phi_\odot$, $\phi_a = (1 - P_{ee})\phi_\odot$, where $\phi_\odot$ is the total neutrino flux from the Sun. From the SNO data, calculate the values of P_{ee} and $\phi_\odot$. Note that $P_{ee} < 0.5$ is indicative of "strong" matter effects inside the Sun combined with the fact that the electron-type neutrino is predominantly light, i.e., $\sin^2\theta < 0.5$.

7. **Day-Night Effect** – Solar neutrino oscillations can also be modified by the fact that, during the night, the neutrinos have to cross some significant amount of the Earth in order to reach the detectors. Hence, the oscillation probability is different for neutrinos arriving during the day and the night (experiments with real-time event reconstruction capabilities search for a day-night asymmetry in the measured solar neutrino flux).

   To understand this effect, assume that solar neutrinos arrive at the surface of the Earth in the $|\nu_2\rangle$ state (a mass eigenstate). This is true of ^{8}B solar neutrinos as long as few$\times 10^{-9}$ eV$^2 < \Delta m^2 <$ few $\times 10^{-5}$ eV^2 and $\sin^2\theta$ is not too small ($\sin^2\theta > 0.1$ is safe). Even for the "real" value of $\Delta m^2_{12} \sim 8\times 10^{-5}$ eV^2 the approximation is pretty good (at the several percent level).

   (a) First, express $|\nu_2\rangle$ as a linear combination of $|\nu_e\rangle$ and $|\nu_\mu\rangle$ (define the mixing angle such that $|\nu_e\rangle = \cos\theta|\nu_1\rangle + \sin\theta|\nu_2\rangle$, as we have been doing in class), and compute the probability that $|\nu_2\rangle$ is detected as an electron-type neutrino.

   (b) Assume that the solar neutrino, originally in the $|\nu_2\rangle$ state, propagates a distance L through the Earth before reaching the detector. Assume that the electron number density in the neutrino's path is constant. Compute that probability that this neutrino is detected as an electron-type neutrino.

   (c) Assume $L = 3000$ km, $\sqrt{2}G_F N_e = 1.5\times 10^{-7}$ eV^2/MeV, $\Delta m^2 = 10^{-5}$ eV^2, $\sin^2\theta = 0.3$, and $E_\nu = 8$ MeV, and compute $P_{ee}^{\rm night} - P_{ee}^{\rm day}$.

   Useful Formulae: In matter of constant density, the oscillation frequency is

$$\Delta_M = \sqrt{\Delta^2\sin^2 2\theta + (\Delta\cos 2\theta - \sqrt{2}G_F N_e)^2}, \tag{1}$$

   where $\Delta \equiv \Delta m^2/(2E_\nu)$. The matter mixing angle obbeys

$$\Delta_M \sin 2\theta_M = \Delta\sin 2\theta \quad \text{and} \quad \Delta_M\cos 2\theta_M = \Delta\cos 2\theta - \sqrt{2}G_F N_e. \tag{2}$$

   The second expression is easy to derive (try it!) and may help you simplify your answer in (b).

8. It is easy to show that the $\nu_\mu \to \nu_e$ oscillation probability for three active flavors in vacuum can be written as

$$P(\nu_\mu \to \nu_e) = \sum_{i,j=1}^{3} U^*_{ei}U_{\mu i}U_{ej}U^*_{\mu j}\exp\left(-i\frac{(m_i^2 - m_j^2)L}{2E}\right). \tag{3}$$

   This form proves useful to address the following questions:

   (a) Show that time-reversal invariance is not necessarily conserved, *i.e.*, that $P(\nu_\mu \to \nu_e) \neq P(\nu_e \to \nu_\mu)$ unless U is a real matrix. Show that the different $P(\nu_\mu \to \nu_e) - P(\nu_e \to \nu_\mu)$ is proportional to the imaginary part of $U^*_{ei}U_{\mu i}U_{ej}U^*_{\mu j}$.

   (b) The mixing matrix for antineutrinos is the same as the one for neutrinos, except for $U \leftrightarrow U^*$. Show that CP-invariance is not necessarily conserved, *i.e.*, that $P(\nu_\mu \to \nu_e) \neq P(\bar{\nu}_\mu \to \bar{\nu}_e)$. How does $P(\nu_\mu \to \nu_e) - P(\bar{\nu}_\mu \to \bar{\nu}_e)$ relate to the expression for $P(\nu_\mu \to \nu_e) - P(\nu_e \to \nu_\mu)$ you worked out in (a)?

   (c) Show that CPT invariance is conserved, *i.e.*, $P(\nu_\mu \to \nu_e) = P(\bar{\nu}_e \to \bar{\nu}_\mu)$.