Taming Non-Equilibrium Systems with QuTiP, the Quantum Toolbox in Python

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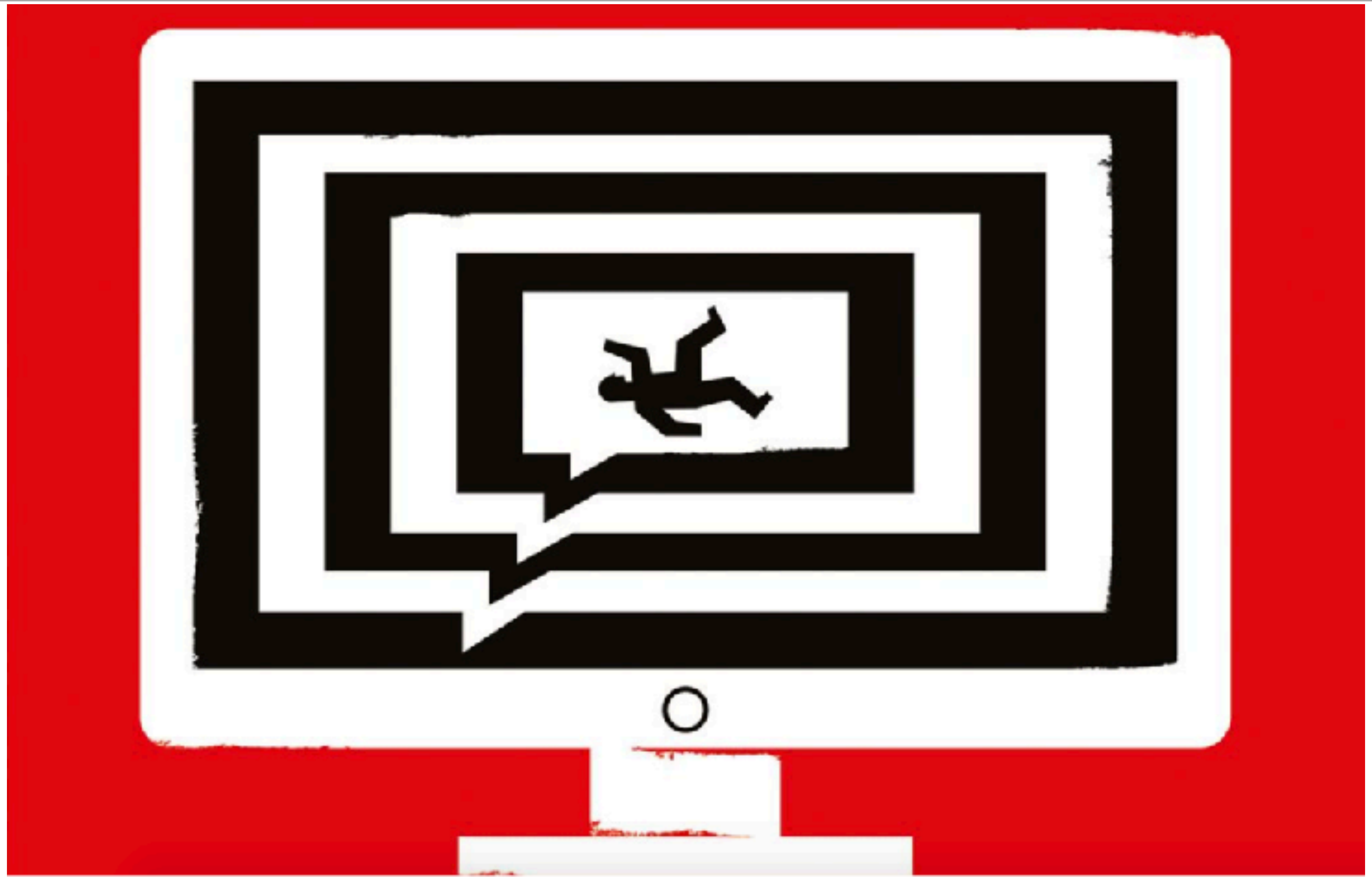
"Taming Non-Equilibrium Systems: from Quantum Fluctuation to Decoherence"
July 31st 2019
ICTP Trieste, Italy

TOOLBOX · 01 JULY 2019

How to support open-source software and stay sane

Releasing lab-built open-source software often involves a mountain of unforeseen work for the developers.

Anna Nowogrodzki



https://www.nature.com/articles/d41586-019-02046-0

QuTiP: The Quantum *Physics* Simulator

The **Qu**antum **T**oolbox **i**n **P**ython

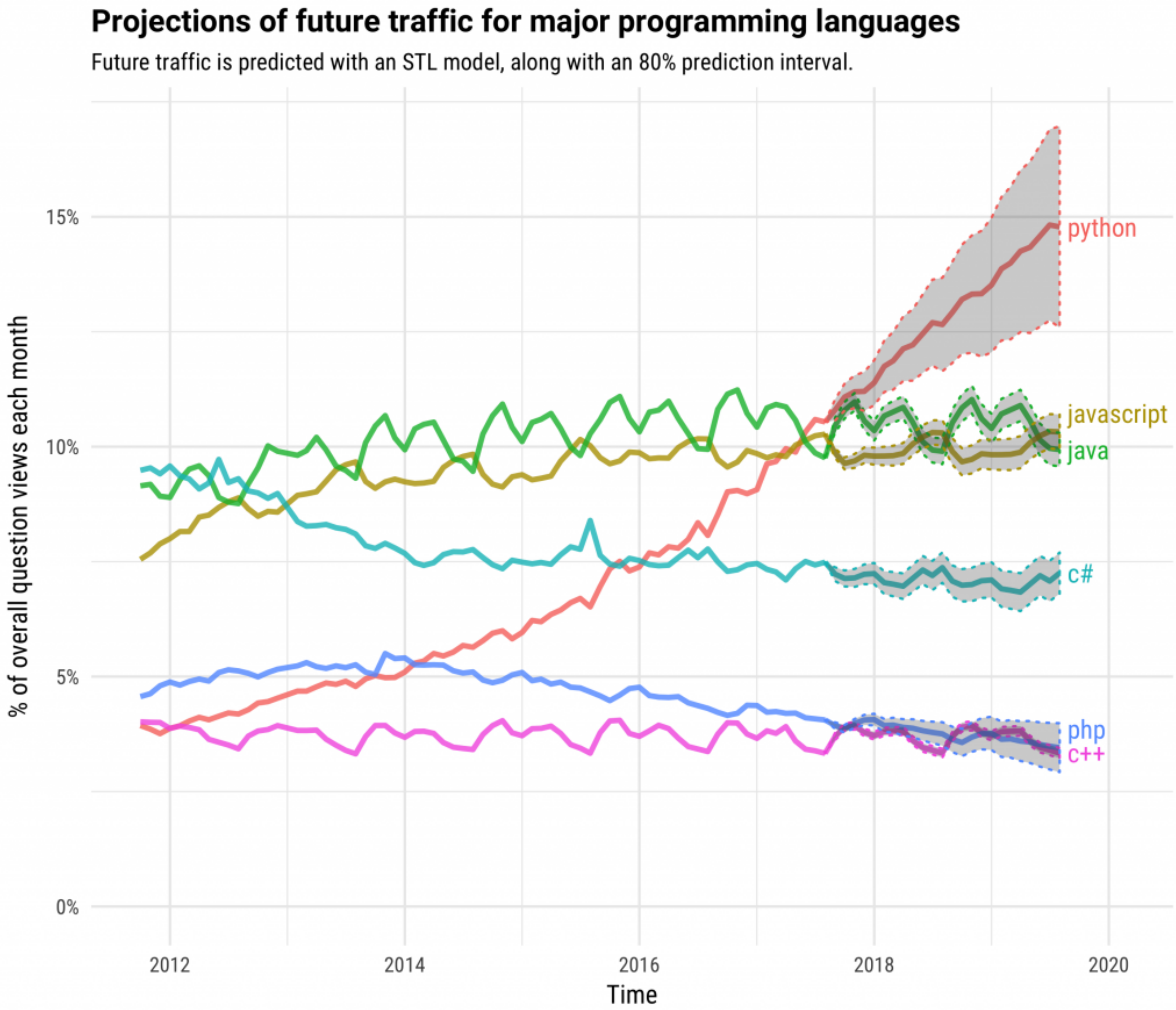
July 3rd, 2019: QuTiP 4.4.0 Released

`conda install qutip`

More info: qutip.org

The steady growth of Python

Empowered by a large open-source ecosystem



Source: David Robinson

Python’s strengths

A community-based programming language

Community

PyCons
Workshops
Sprints
EuroSciPy

Libraries

Tools

Notebooks
LaTeX comments
Interactive code
Jupyter

Open Science through Open Source

The tools of open source make your code count

Code & Testing

GitHub

Travis CI

Documentation

SPHINX
Python Documentation Generator

Read the Docs

Publication

CONDA

python™
Package
Index

zenodo

Open source and open science

Aligned vision

Open Source Coding | **Open Science Research**

- Allow **access** to the research/project results, sharing knowledge.
- **Collaboratively** advance the field, building upon others' results.
- Coordinate large and delocalized **teams** working remotely.
- Make supporting data and code available for fast **reproducibility**.

Beyond Reproducibility

From reproducible data to reusable code.

		DATA	
		Same	**Different**
ANALYSIS	**Same**	Reproducible	Replicable
	Different	Robust	Generalizable

Take a snapshot

Cloud-based Notebooks on Open Quantum Systems

The **Qu**antum **T**oolbox **in** **P**ython

Take a snapshot

You can find an interactive notebook at

https://github.com/nathanshammah/

Repository: interactive-notebooks

You can run the notebooks live

launch binder

Create an open-source scientific project

Use the tools of open source: https://github.com/nathanshammah/opensource

GitHub, Inc. [US] | https://github.com/nathanshammah/opensource/blob/master/README.md

A Guide to Building Your Open-Source Science Library

A cheatsheet to develop a scientific open-source library from scratch.

Zero to Library

Here you will find information to design, build, and release an open-source library to perform scientific research in Python from scratch to finish.

0 - Open Source for Open Science: Some information about the Python and open source ecosystem, and how they relate to open science are also given.

- StackOverflow

1 - Before Starting Coding: Setting up the working environment on your machine, including the tools you will need to write code efficiently.

- Sublime, Git, GitHub

2 - Developing your Project: A step-by-step guide with best practices for coding, and tips for making code development as effortless as possible.

- PEP 8, PEP 257
- jupyter notebook
- nbconvert

3 - Testing: Especially in software related to scientific research, at start, the destination is not always crystal clear. Code is written, optimized, reorganized. Unit testing is a crucial task to avoid getting lost in the process.

- nose2, pytest

Quantum Tech: Open Source Libraries

More open-source is empowering broad research in the field

Library	Year	Creators	Institution	Language	Description
QuTiP Quantum Toolbox in Python	2012	Rob Johannson Paul Nation Franco Nori	RIKEN	**Python**	Simulation of open quantum systems; quantum optics, cavity QED.
QNet	2012	Nikolas Tezak, Michael Goerz Hideo Mabuchi	Stanford	**Python**	Computer algebra package for quantum mechanics and photonic quantum networks
QuantumOptics.jl	2017	Sebastian Krämer *et al.*	U Innsbruck IQOQI	Julia	Quantum optics and open quantum systems framework inspired by the QO toolbox in Matlab and QuTiP
ProjectQ	2016	Damian S. Steiger Thomas Häner Matthias Troyer	ETH Zurich	**Python**	Hardware-agnostic framework with compiler and simulator with emulation capabilities.
OpenFermion	2017	Ryan Babbush *et al.*	Google (unofficial)	**Python**	Fermionic potential calculations for quantum chemistry
NetKet	2018	Giuseppe Carleo	The Simons Foundation	C++ **Python interface**	Studying many-body quantum systems with artificial neural networks and ML techniques.
STRAWBERRY FIELDS	2018	Nathan Killoran *et al.*	Xanadu Inc	**Python**	Photonic quantum computing with continuous-variable optical circuits

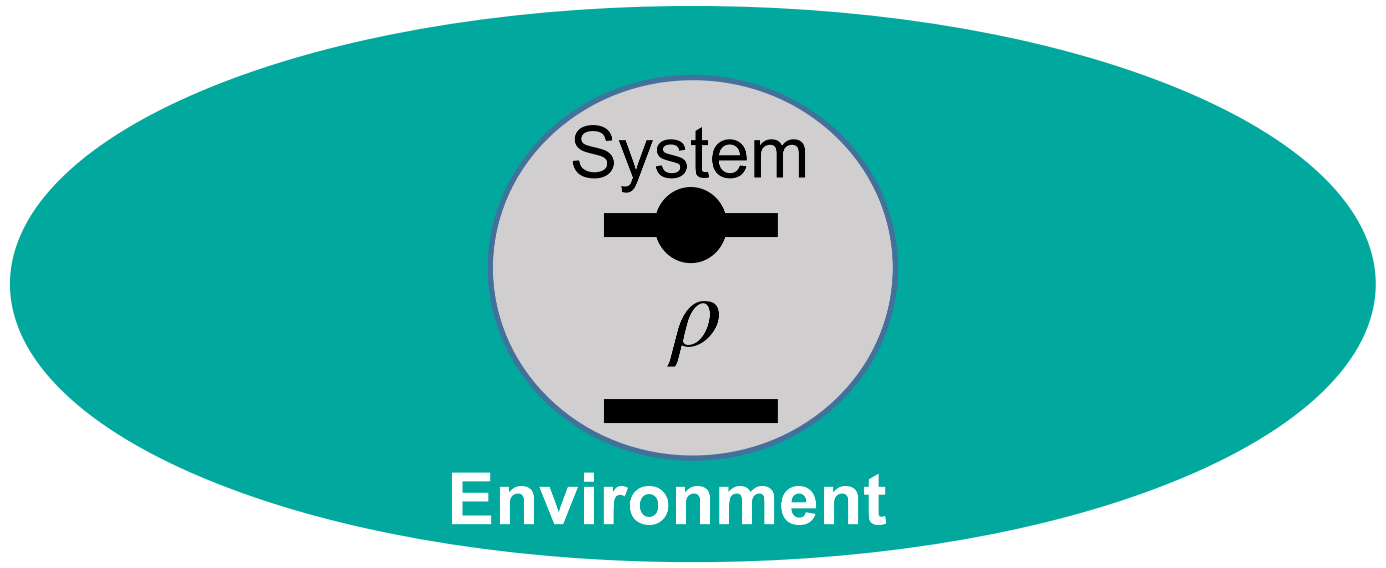
Checkout more open-source projects at https://qosf.github.io

QuTiP: The Quantum *Physics* Simulator

The **Qu**antum **T**oolbox **i**n **P**ython

QuTiP

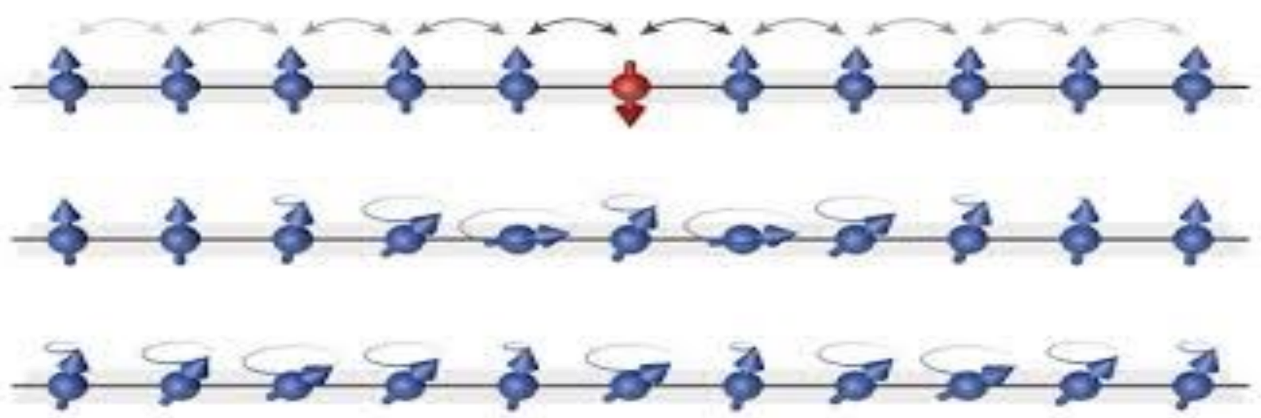
A toolbox to study the **open** quantum dynamics of realistic systems.



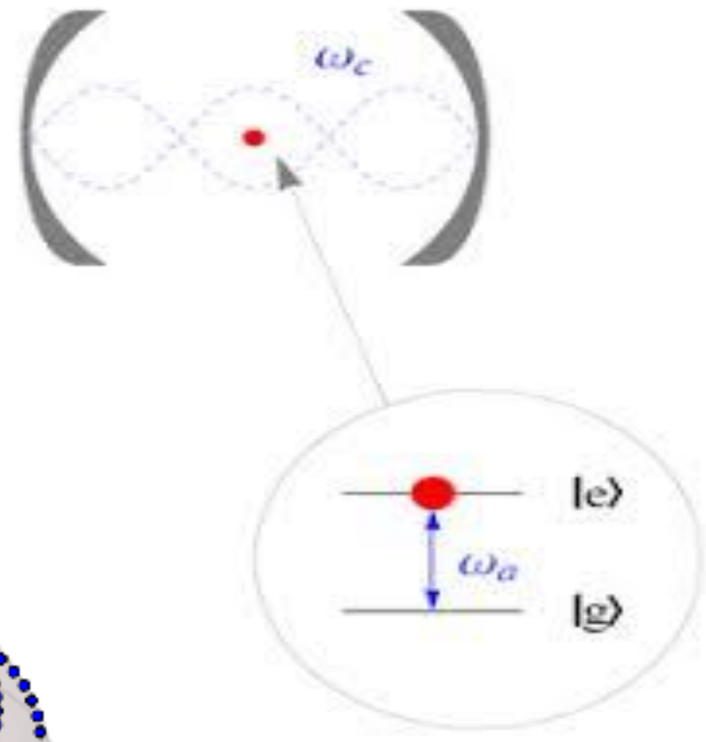
QuTiP: The Quantum *Physics* Simulator

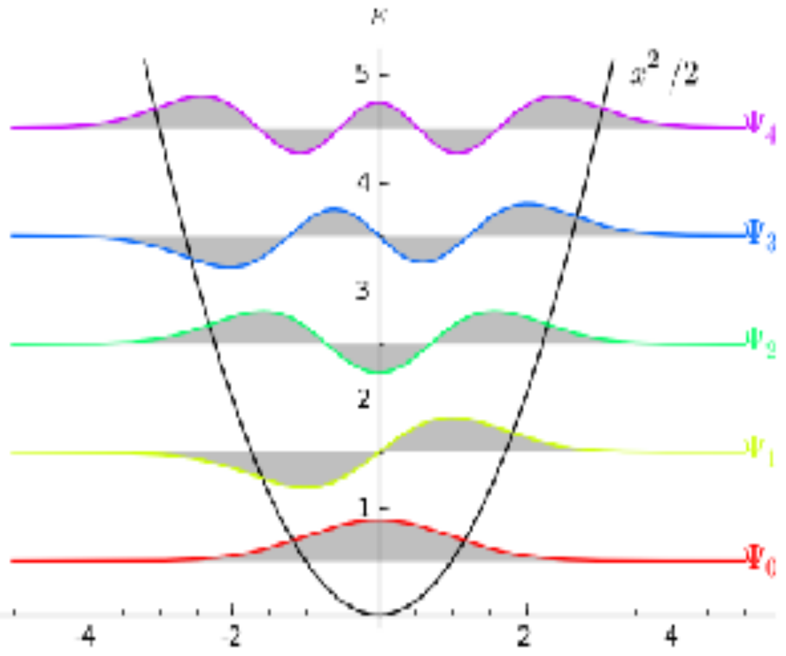
The **Qu**antum **T**oolbox **i**n **P**ython

Spin Lattices

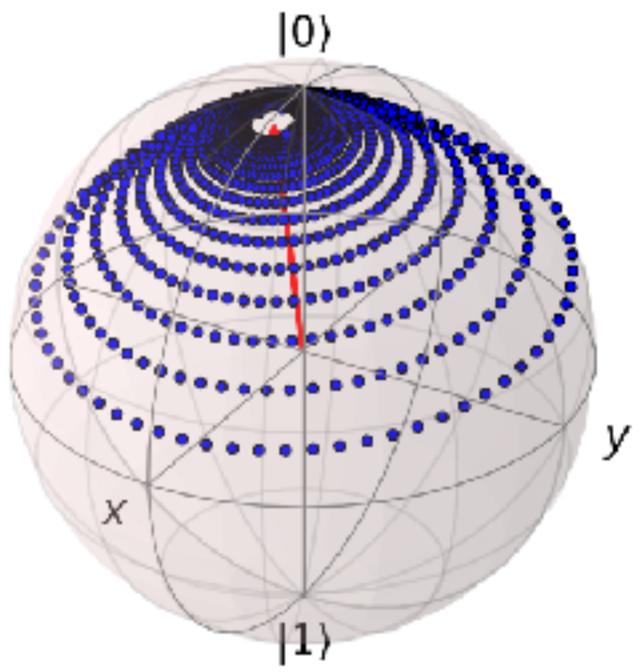


Cavity QED





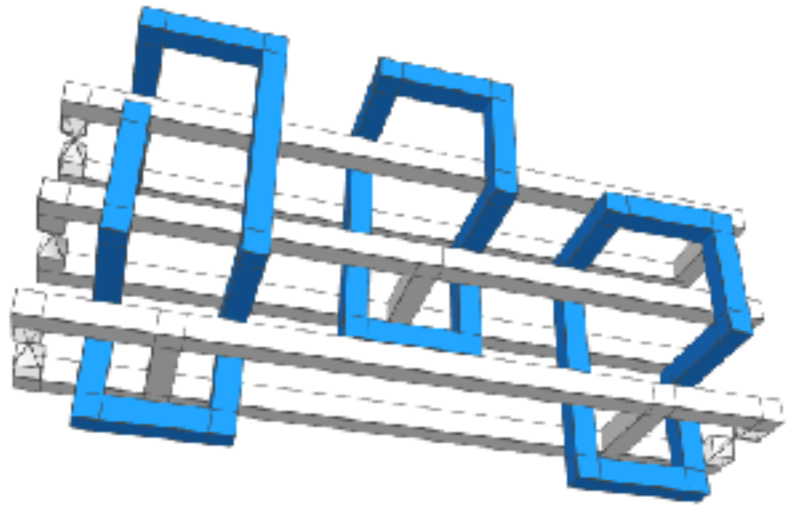
Condensed Matter



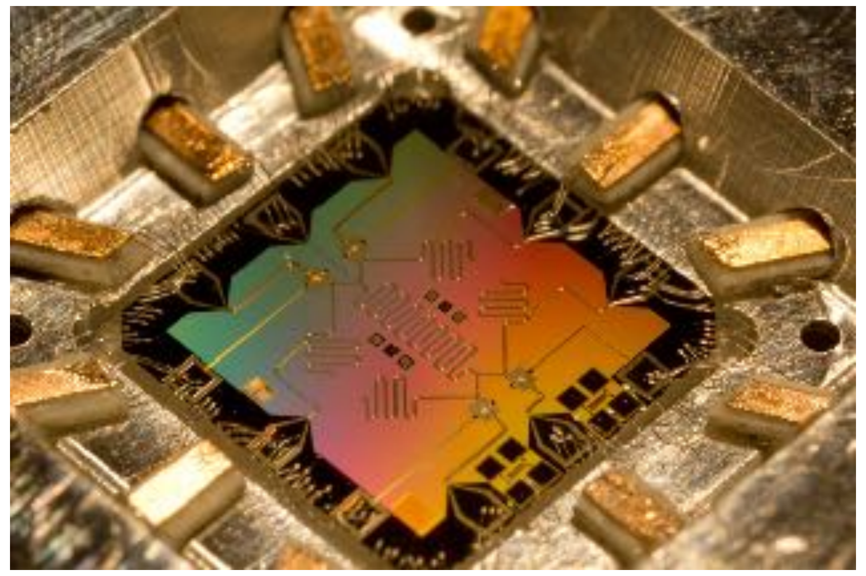
Quantum Optics

QuTiP

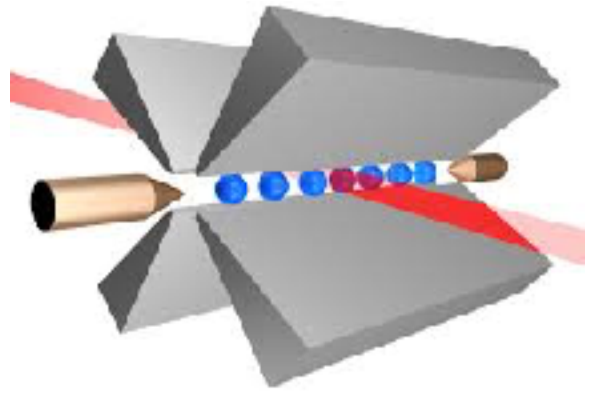
qutip.org



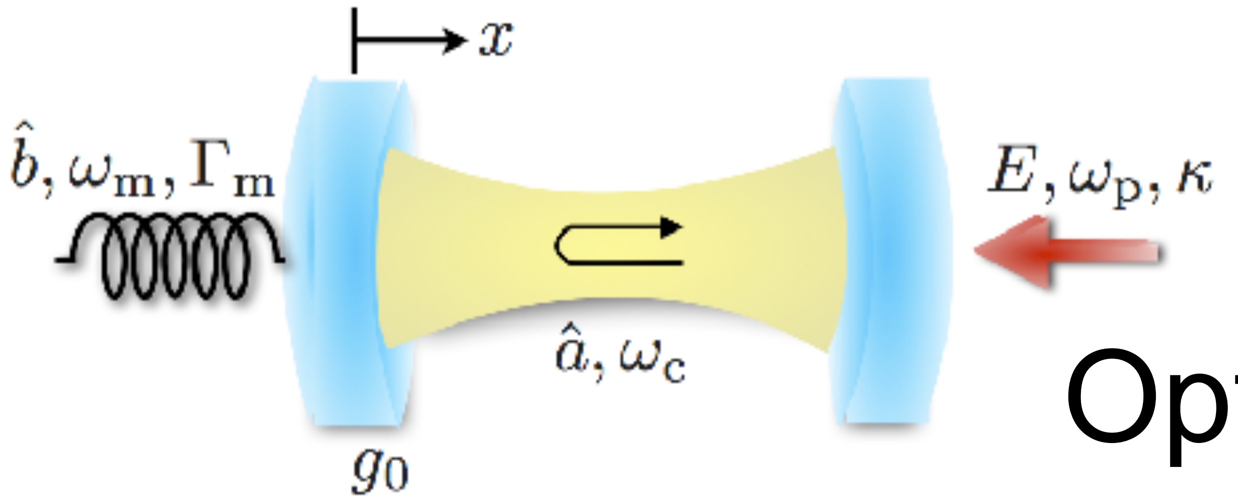
Quantum Error Correction



Superconducting Circuits



Ion Traps



Optomechanics

QuTiP in the lab: Rydberg atoms

The **Qu**antum **T**oolbox **i**n **P**ython

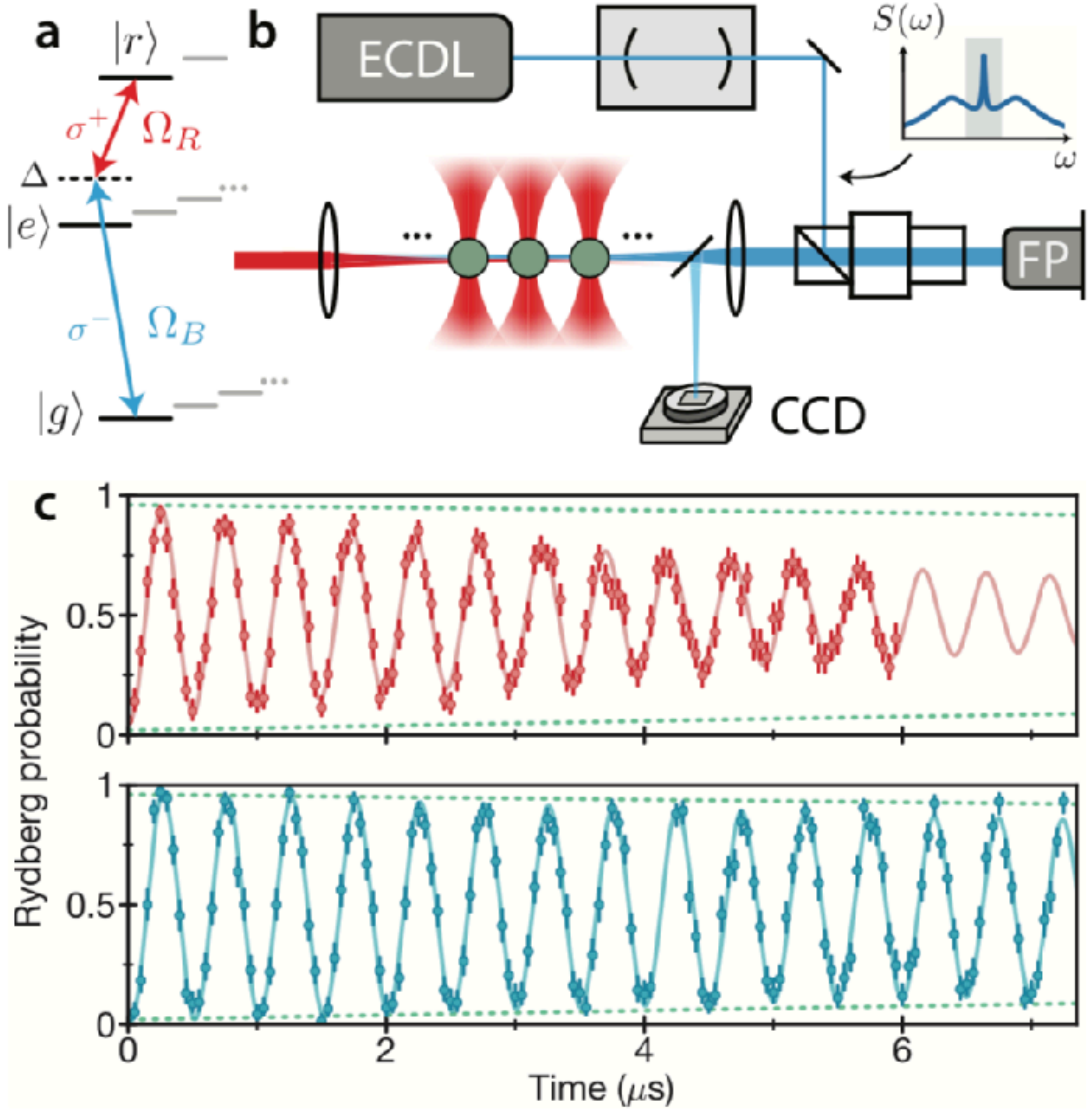
Experimental Research

arXiv:1806.04682v1 [quant-ph] 12 Jun 2018

High-fidelity control and entanglement of Rydberg atom qubits

H. Levine, [...] and M.D. Lukin

Phys. Rev. Lett. **121**, 123603 (2018)



Numerical model for single atoms

The numerical model is implemented using the Python package QuTiP [2].

It includes the following three effects:

1. A static but random Doppler shift in each iteration of the experiment [...].

2. Off-resonant scattering from the intermediate state |r⟩. [...].
   This process is modeled by Lindblad operators.

3. Finite lifetime of the Rydberg state |r⟩.

QuTiP in the lab: Superconducting qubits

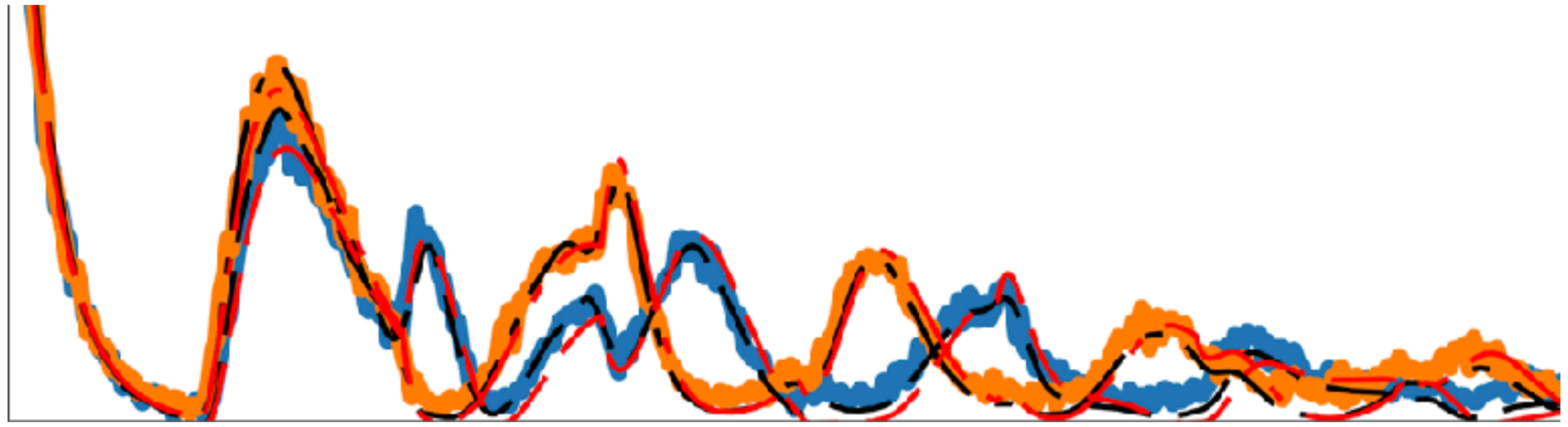
The **Qu**antum **T**oolbox **i**n **P**ython

Experimental Research

arXiv:1903.05672 [quant-ph] 13 Mar 2019

Phonon-mediated quantum state transfer and remote qubit entanglement

A. Bienfait, [...] and A.N. Cleland, Science **10**, 1126 (2019)



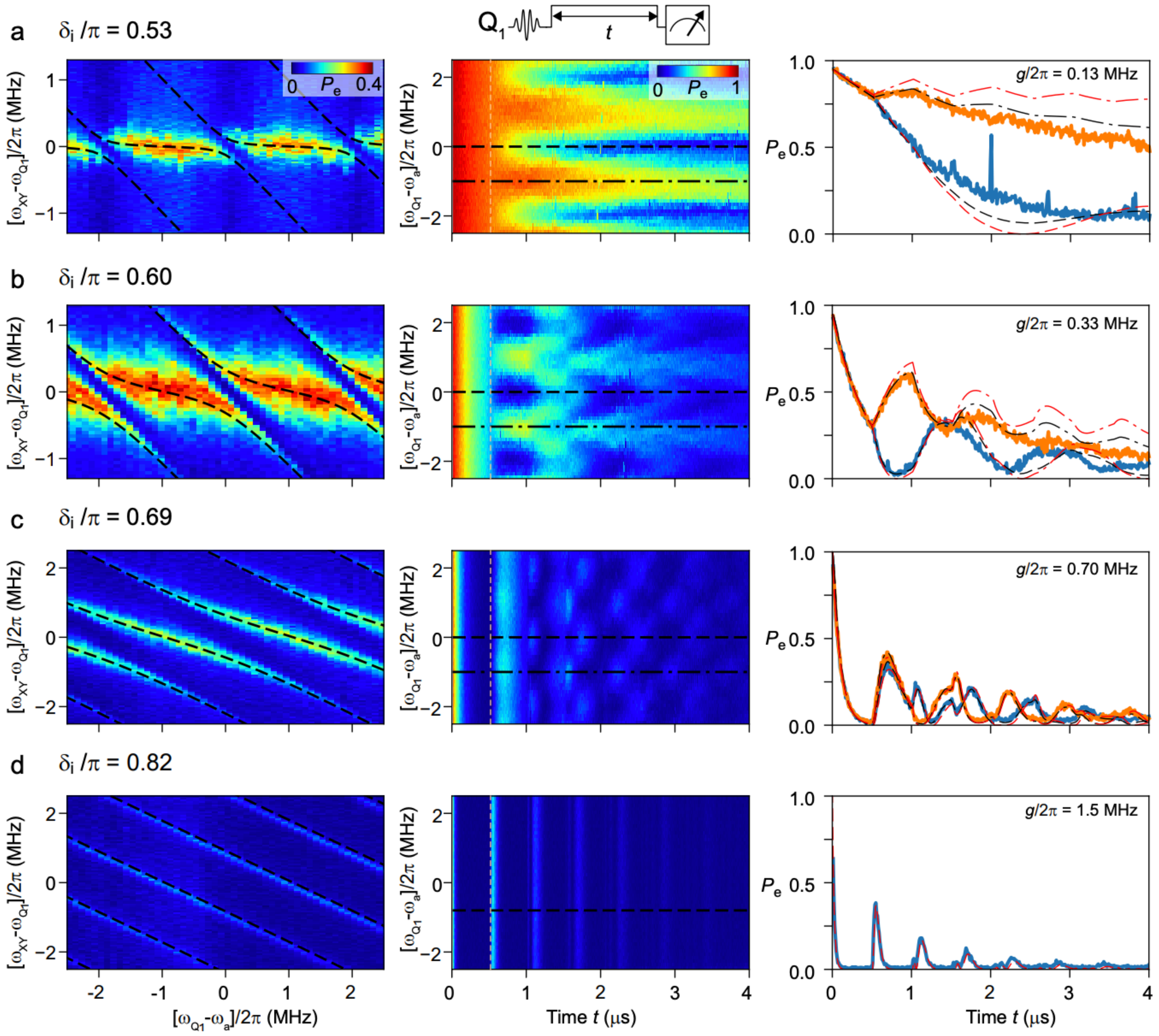
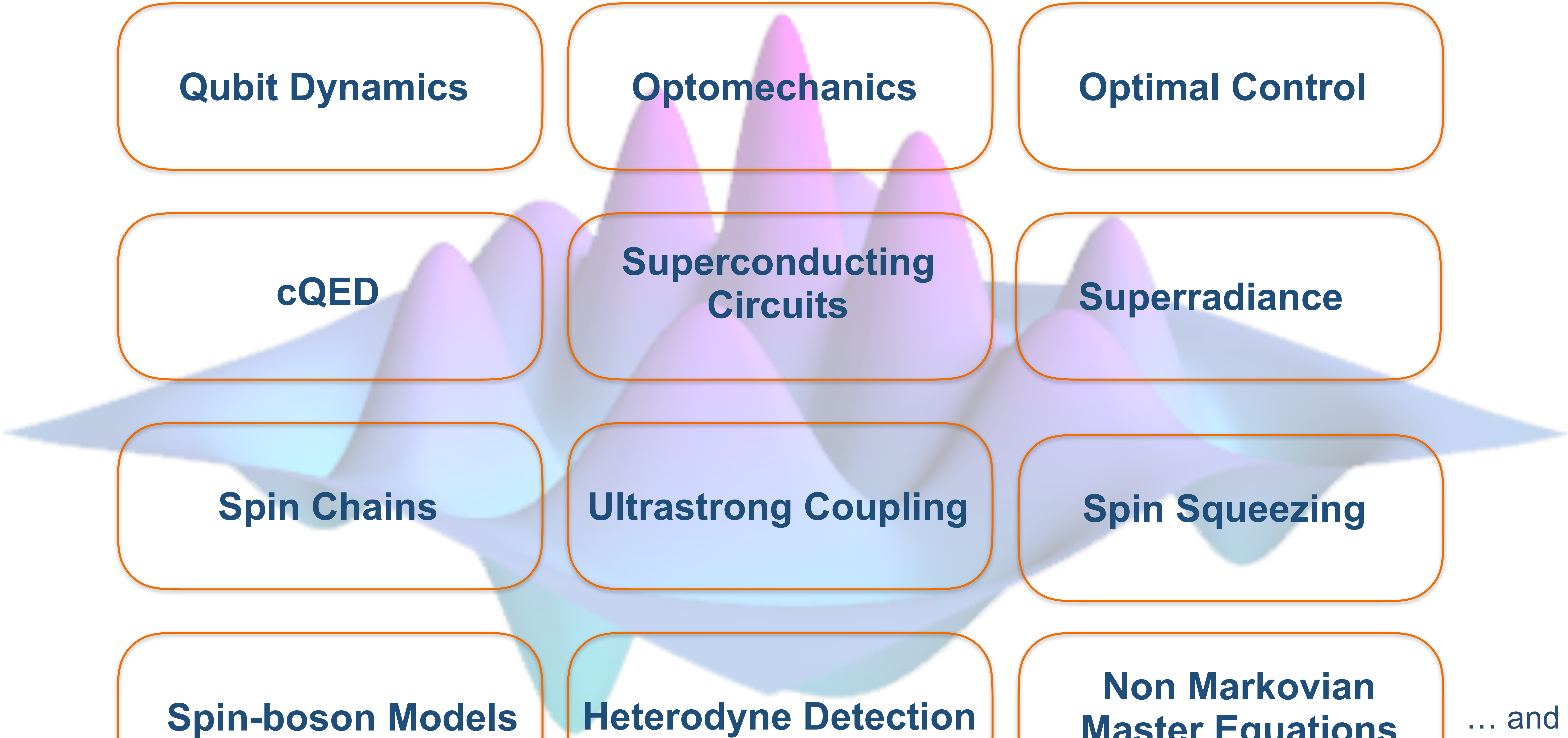


Figure S3. (a-d) *Left*: Qubit 1 spectroscopy near the SAW Fabry-Pérot mode $\omega_a/2\pi = 3.9542$ GHz for different coupling biases corresponding to the coupling strengths indicated on each of the right panels. *Center*: Corresponding energy decay rate measurements in the same frequency range. *Right*: Line cuts through the data in the central panels at $\omega_q = \omega_a$ (blue curves) and $\omega_q = \omega_a - \omega_{\mathrm{FSR}}/2$ (orange curves). Black dashed lines are QuTiP simulations as described in the text, red dashed lines are from Eq. S2. We did not perform a QuTiP simulation for d, as it requires the inclusion of more than 20 oscillators, which is very computationally intensive.

QuTiP: What research enables

The **Qu**antum **T**oolbox **i**n **P**ython



- Qubit Dynamics
- Optomechanics
- Optimal Control
- cQED
- Superconducting Circuits
- Superradiance
- Spin Chains
- Ultrastrong Coupling
- Spin Squeezing
- Spin-boson Models
- Heterodyne Detection
- Non Markovian Master Equations

… and more

QuTiP: The Quantum Toolbox in Python

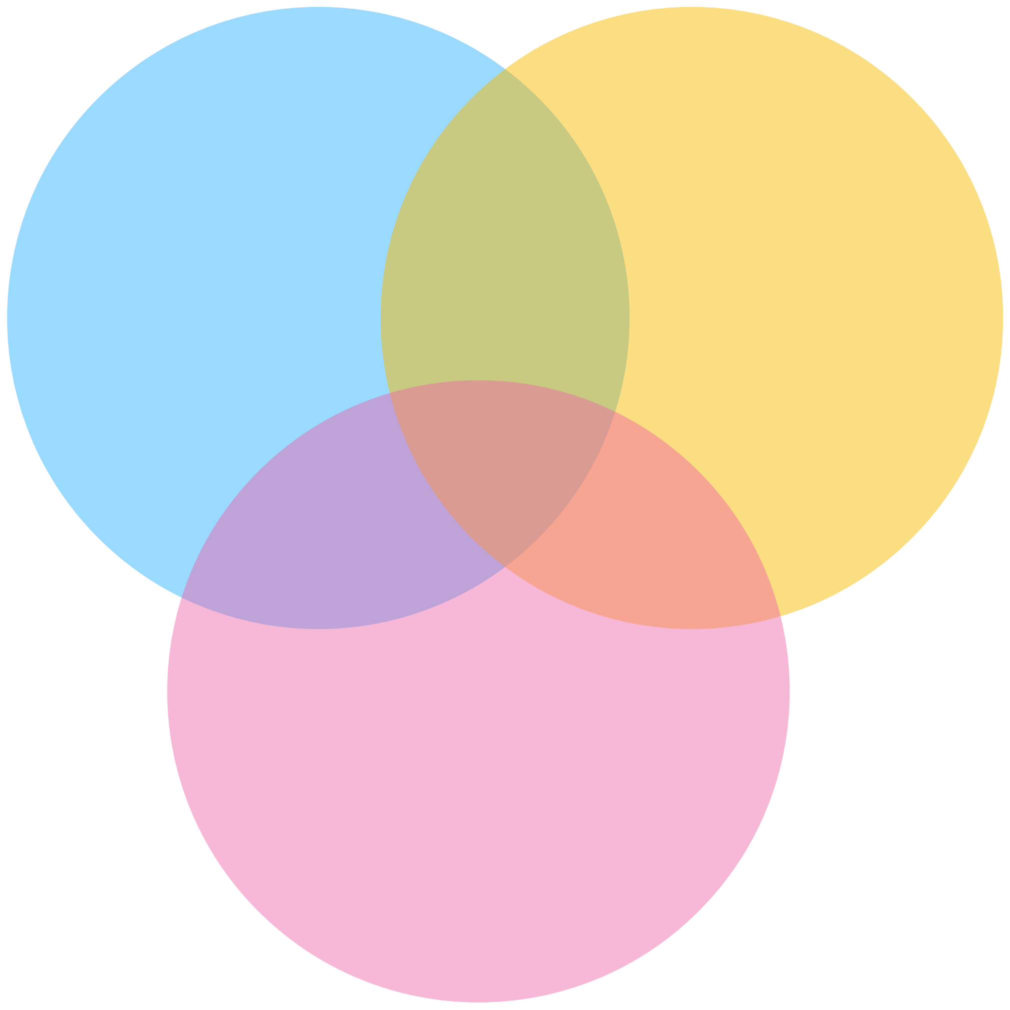
About QuTiP

QuTiP is a free and open source library for efficient simulation of a wide variety of quantum systems.

<u>Field-specific</u> intuitive framework

The mathematics of quantum mechanics

- Complex number matrices
- Non-commutative algebras
- Operators and superoperators
- Intuitive Python classes: `QObj`
  `Results`



<u>Advanced</u> mathematical techniques

- **Noisy Dynamics**: Master equation solvers
- Correlation functions (quantum regression formula)
- **Modularity**: Permutational invariant quantum solver
  Hierarchical equations of motion
  Waveguide photon scattering
  Topological quantum circuits

<u>Efficient</u> numerical calculations

- *Fast complex-complex matrix-vector multiplication* with SSE3 intrinsics.
- **Multiprocessing**: Enhanced parallel performance with OPENMP

- **Sparse Matrices**: Fast CSR (Compressed Sparse Row) matrix class.
- *Up to 100x improvement:* in CSR adjoint & transpose
  in Hermitian verification, Krönecker product
  in partial trace calculation

More info at <u>http://qutip.org/</u>

QuTiP: The building blocks: states, operators and gates

The **Qu**antum **T**oolbox **i**n **P**ython

$$H = \frac{\sigma_z}{2}$$

```
>> from qutip import *
>> H = sigmaz()/2
```

$$a, a^\dagger$$

```
>> a = destroy(2)
>> L = liouvillian(a.dag()*a,[a])
```

$$|\psi\rangle = \frac{1}{\sqrt{(2)}}\left(|0\rangle + |1\rangle\right)$$

```
>> psi1 = basis(2, 0)
>> psi2 = basis(2, 1)
>> psi = (psi1 + psi2)/1.414
```

$$\sigma_z \otimes \sigma_x$$

```
>> tensor(rho1, rho2)
>> tensor(sigmaz(), sigmax())
```

$$\rho$$

```
>> rho = ket2dm(psi)
>> op = vector_to_operator(rho)
```

QuTiP: History of the project at a glance

The **Qu**antum **T**oolbox **i**n **P**ython

Project Impact

QuTiP
Quantum Toolbox in Python

>**600** citations (Google Scholar)

downloads 130k total (conda forge)

More info at http://qutip.org/

Authors

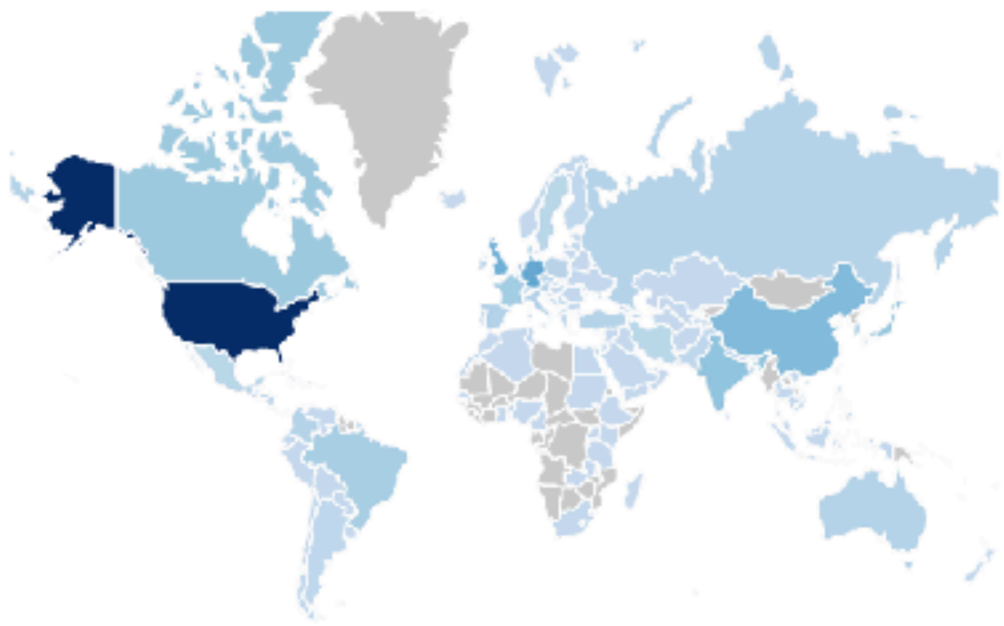
Comp. Phys. Comm. 183, 1760–1772 (2012); ibid. 184, 1234 (2013).

Code

Rakuten Inc.　IBM Q　RIKEN / U. Michigan

Users

Distribution of 25k website visitors (2016)



Timeline:

Inspired by the Quantum Toolbox in MatLAB.

- 2011-2012: **QuTiP** 1.0
- Aug 2015: 100 citations
- Aug 2016: 200 citations
- Jan 2017: QuTiP 4.0
- July 2018: QuTiP 4.3
- July 2019: QuTiP 4.4

Previous Lead Developers

Alex Pitchford
Aberystwyth University

Éric Giguère
U. Sherbrooke

Arne Grimsmo
Universite de Sherbrooke

Chris Granade
University of Sydney

Contributing Developers

- Neill Lambert (RIKEN)
- Denis Vasilyev (Leibniz)
- Kevin Fischer (Stanford)
- Jonathan Zoller (Ulm University)
- Ben Criger (RWTH Aachen)
- …
- Louis Tessler (RIKEN)
- Shahnawaz Ahmed (Chalmers) (**Lead**)
- Nathan Shammah (RIKEN) (**Lead**)

- **GitHub**: 44 contributors, 4k commits

License: BSD
(Berkeley Software Distribution)
Style: PEP8 compliant

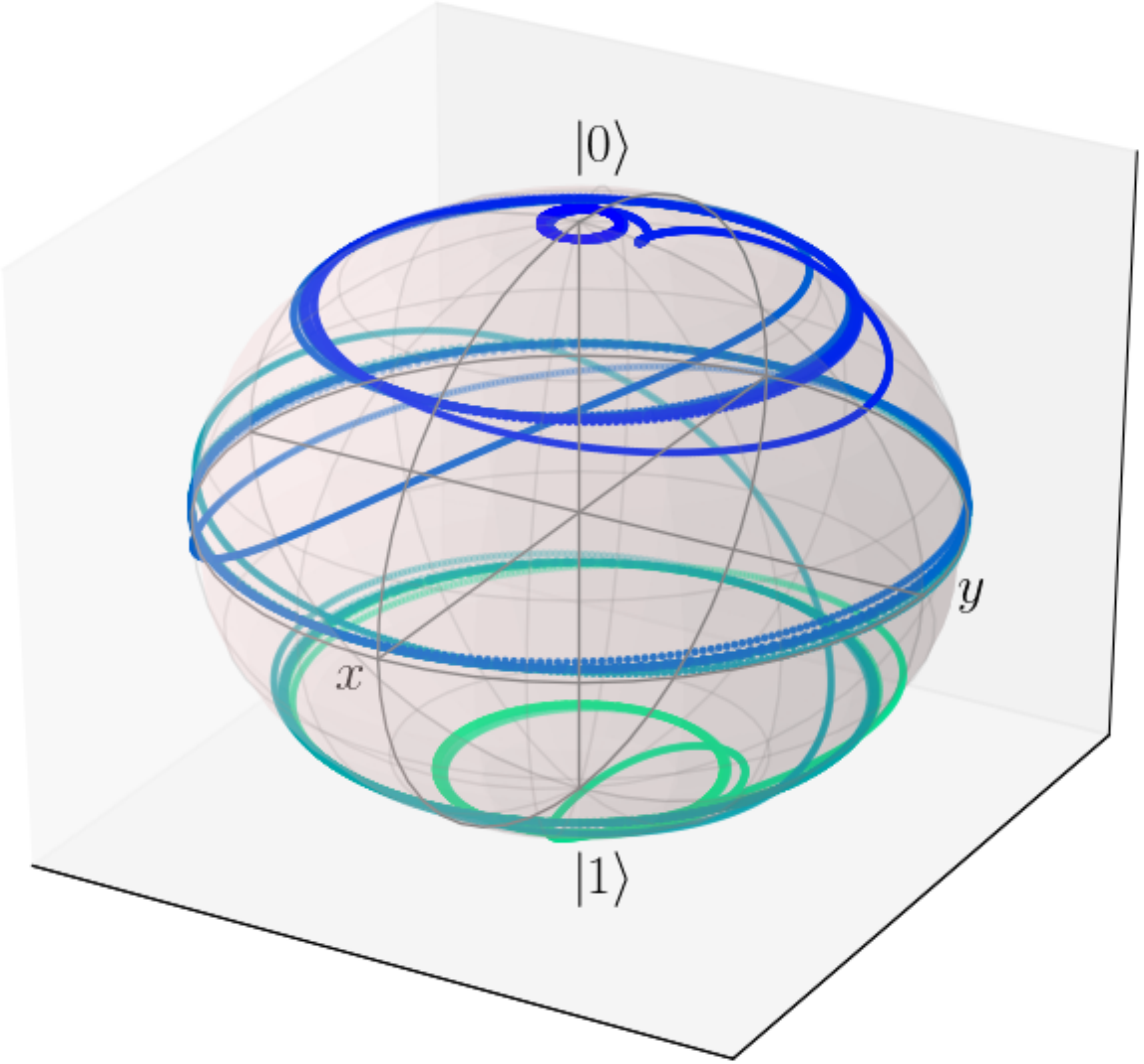
Libraries used:

- Scipy
- NumPy
- Cython
- Matplotlib
- SymPy

- Jupyter notebooks
- Online documentation
- Independent testing

QuTiP: Ad-hoc visualization tools

The **Qu**antum **T**oolbox **i**n **P**ython



QuTiP: Ad-hoc visualization tools

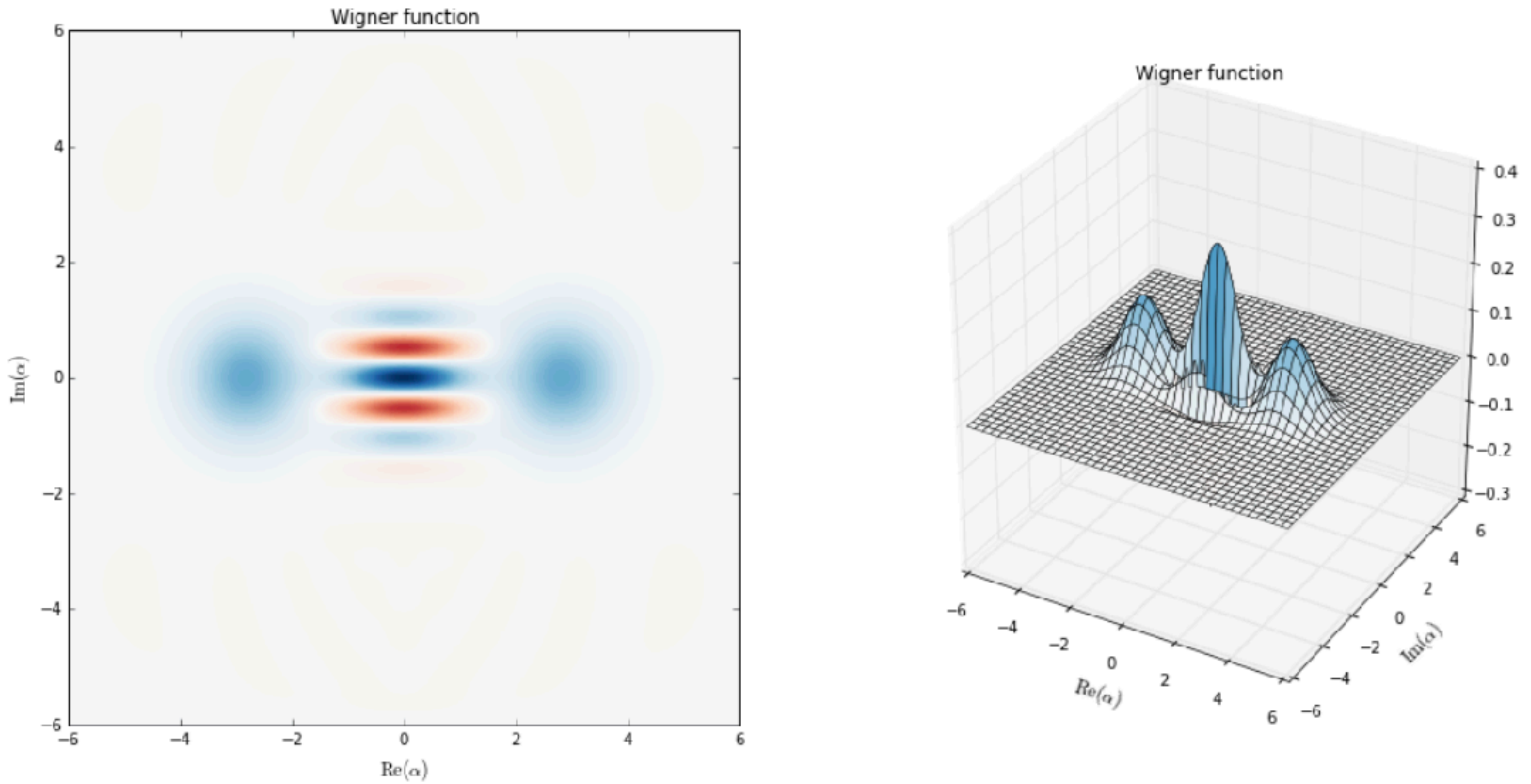
The **Qu**antum **T**oolbox **i**n **P**ython

Superposition of coherent states

qutip.org/tutorials

In [9]:
```
psi = (coherent(N, -2.0) + coherent(N, 2.0)) / np.sqrt(2)
plot_wigner_2d_3d(psi)
```

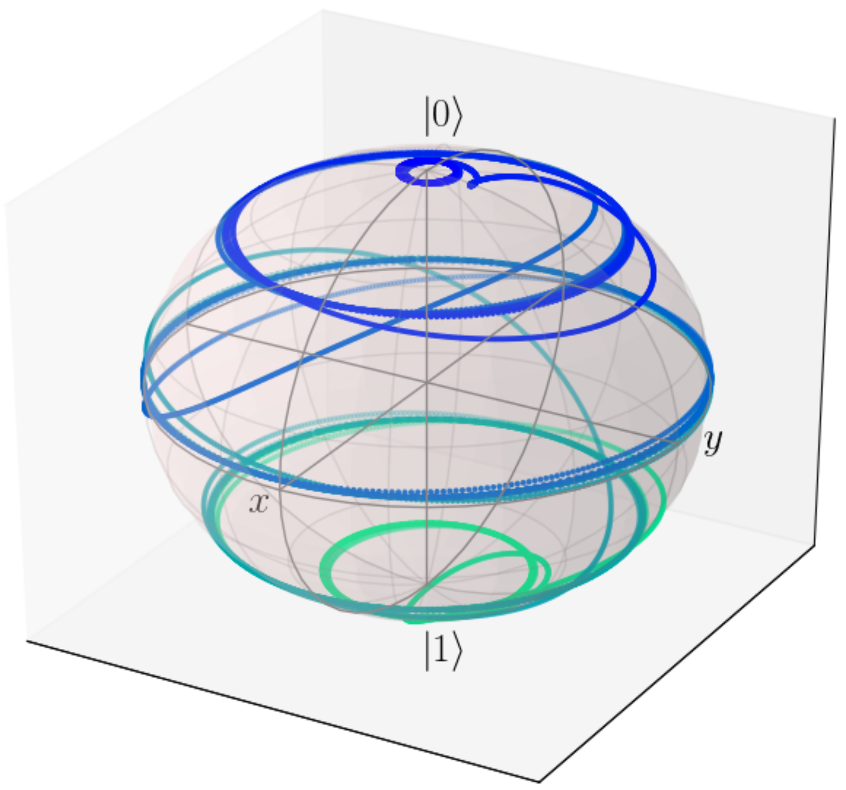
Out[9]:



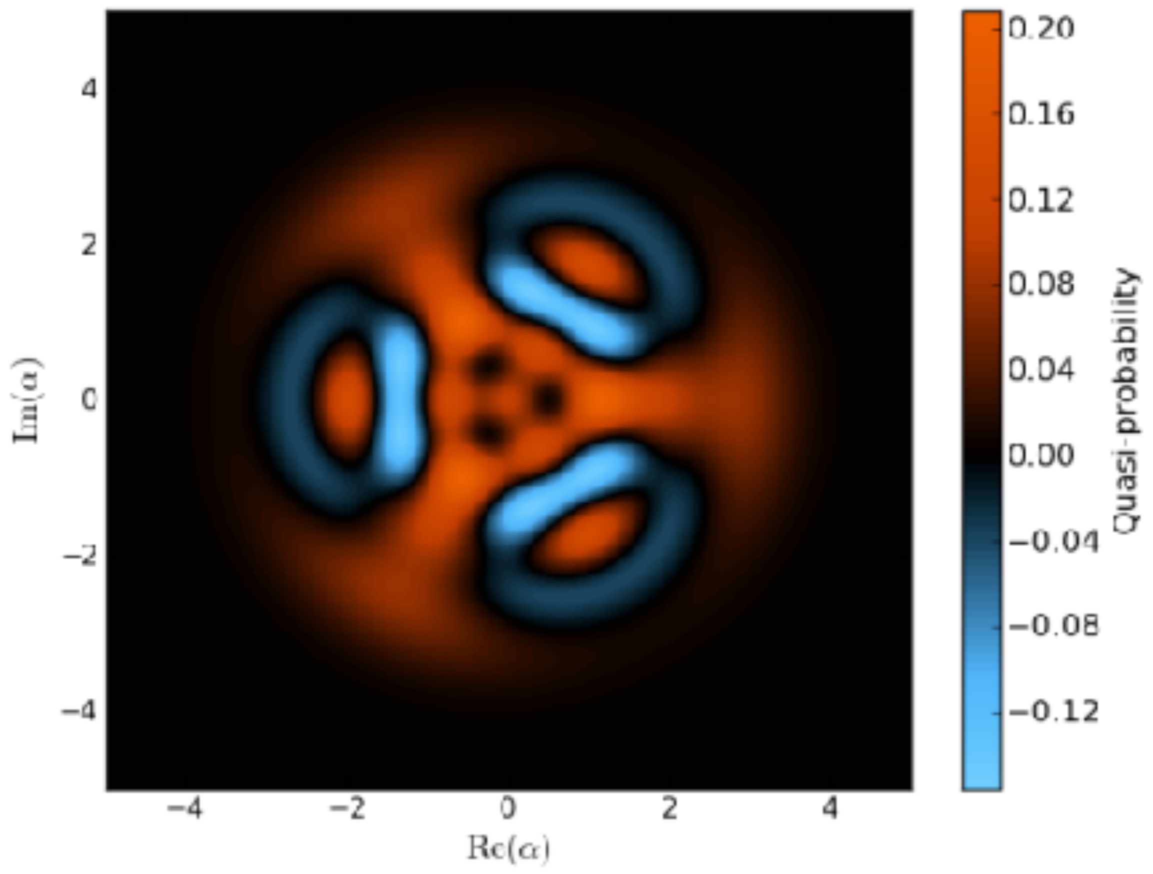
QuTiP: Ad-hoc visualization tools

The **Qu**antum **T**oolbox **i**n **P**ython

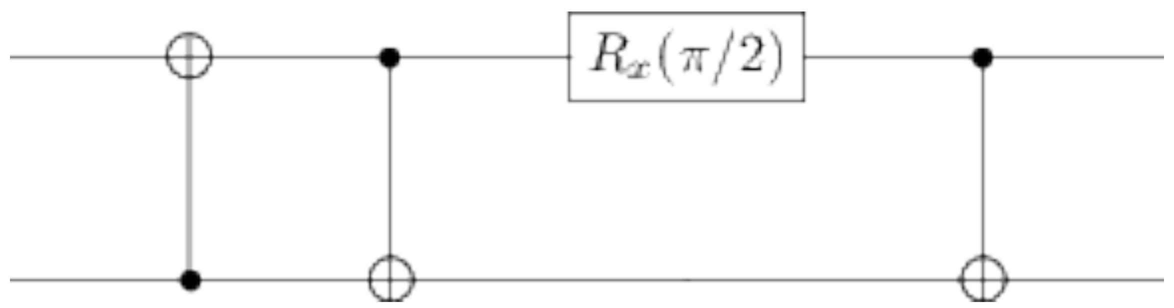
Bloch Sphere



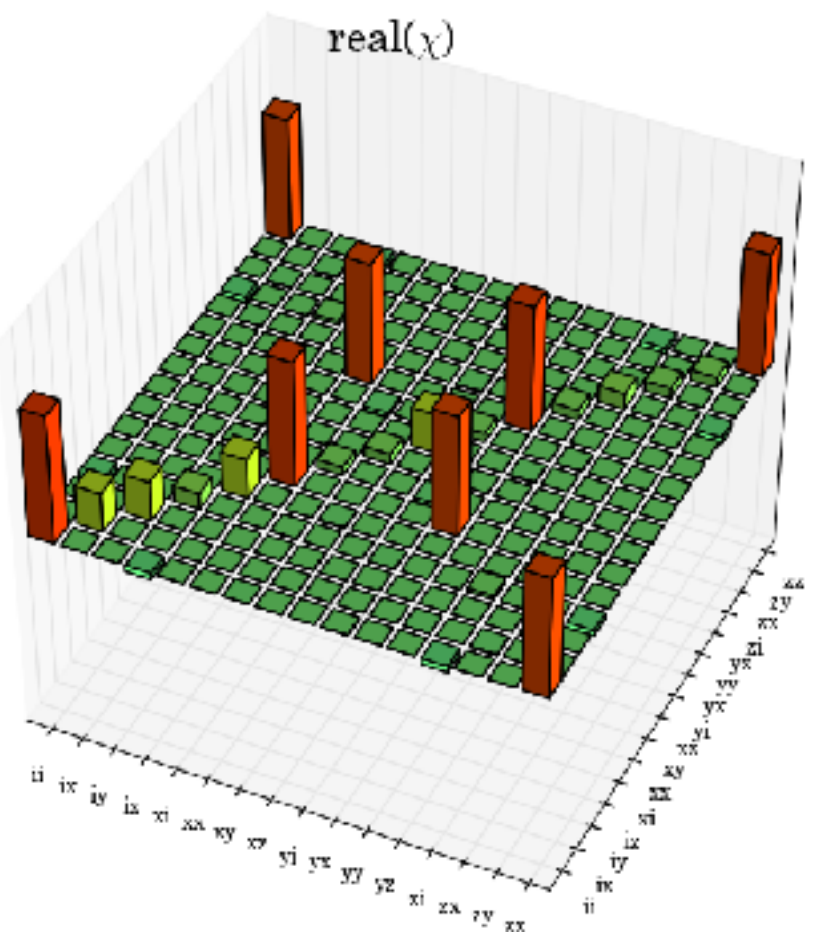
Wigner Function



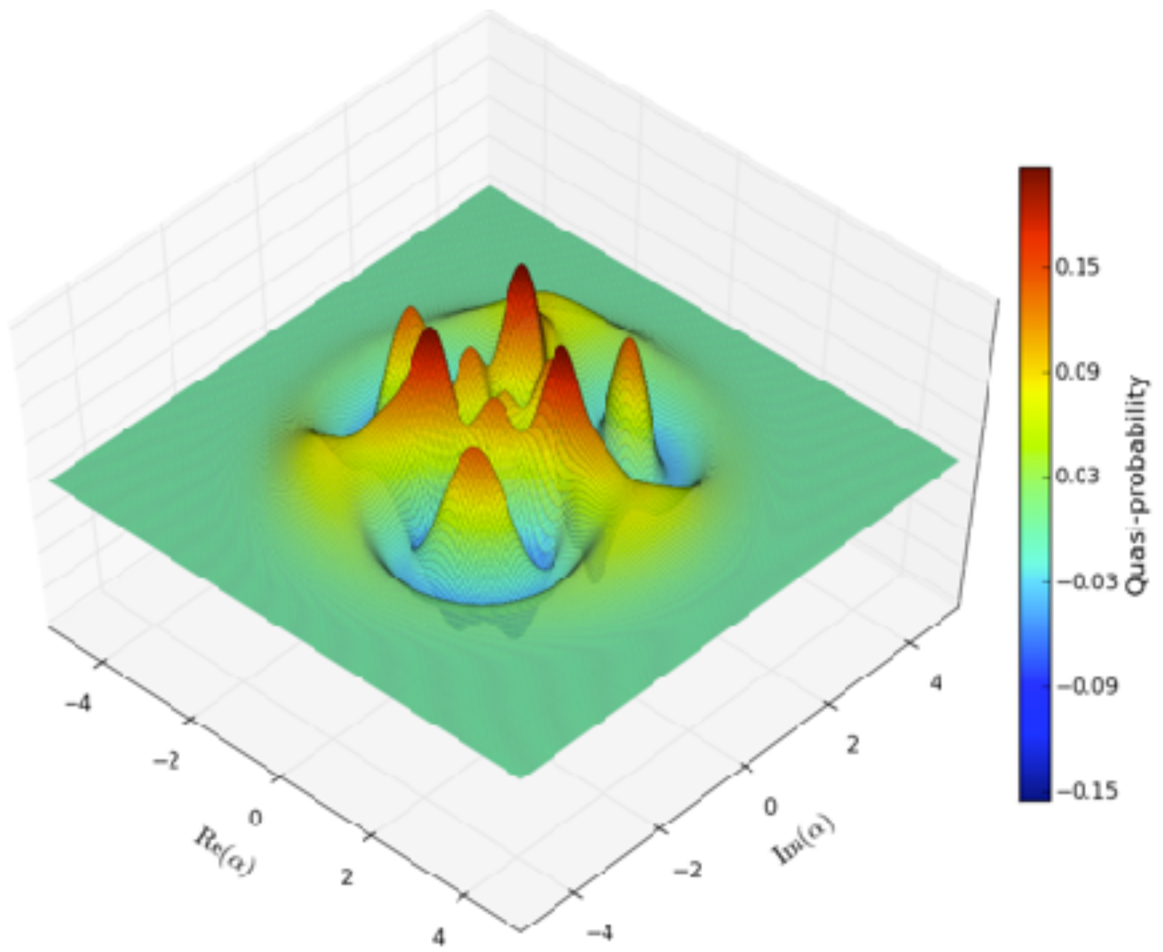
Quantum Circuits



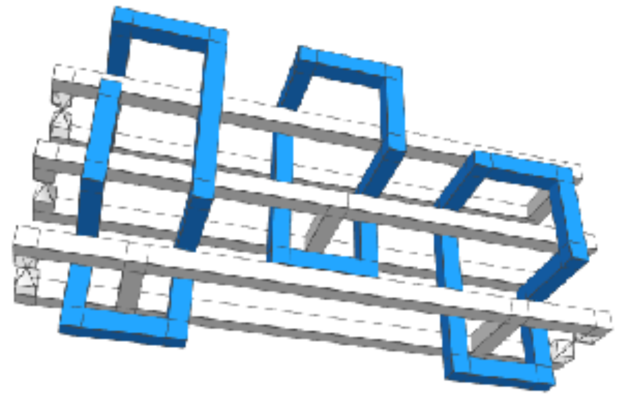
State Tomography



3D Wigner Function



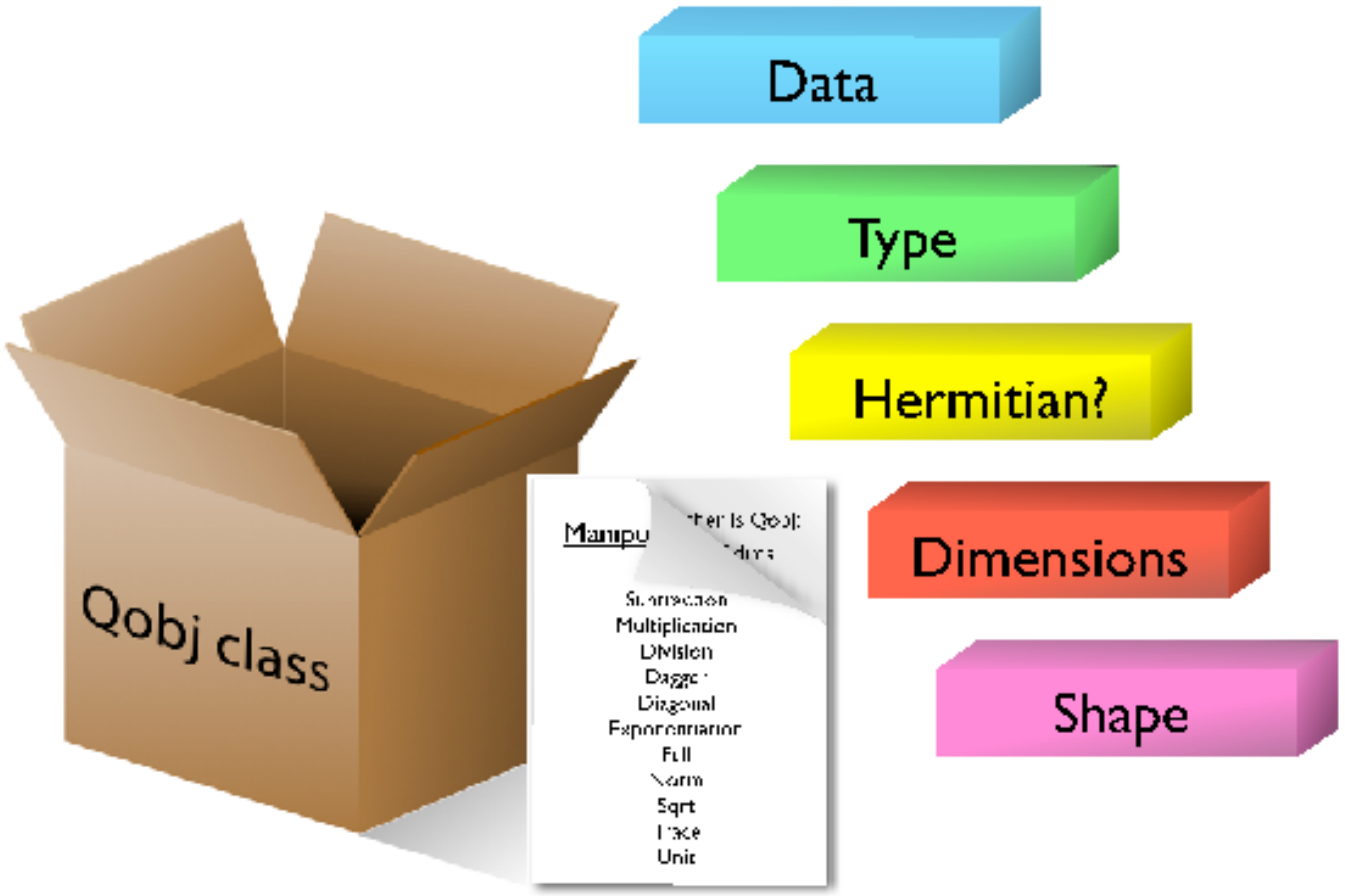
Surface Code Topological Circuits*



*open pull request

QuTiP: The `QObj` class

The **Qu**antum **T**oolbox **i**n **P**ython



J. R. Johansson, P. D. Nation, and F. Nori, Comp. Phys. Comm. **183**, 1760–1772 (2012)
QuTiP: An open-source Python framework for the dynamics of open quantum systems

QuTiP: The `QObj` class

The **Qu**antum **T**oolbox **i**n **P**ython

- State and operators are declared as `QObj`
- **Generate states** and operators
- Algebra (bosonic)

$$AB - BA \neq 0$$

$$\mathscr{E}(\rho) = A\rho B^{\dagger} \longrightarrow \mathscr{D} = B^* \otimes A$$

$$\dot{\rho} = \mathscr{D}\rho$$

```
>> q = Qobj([1], [0])
Quantum object: dims = [[2], [1]],
shape = (2, 1), type = ket
```

$$\begin{pmatrix} 1.0 \\ 0.0 \end{pmatrix}$$

```
>> d = destroy(2)
Quantum object: dims = [[2], [2]],
shape = (2, 2), type = oper,
isherm = False
```

$$\begin{pmatrix} 0.0 & 1.0 \\ 0.0 & 0.0 \end{pmatrix}$$

```
>> q.dag()
Quantum object: dims = [[1], [2]],
shape = (1, 2), type = bra
```

$$\begin{pmatrix} 1.0 & 0.0 \end{pmatrix}$$

QuTiP: Numerical Solvers

The **Qu**antum **T**oolbox **i**n **P**ython

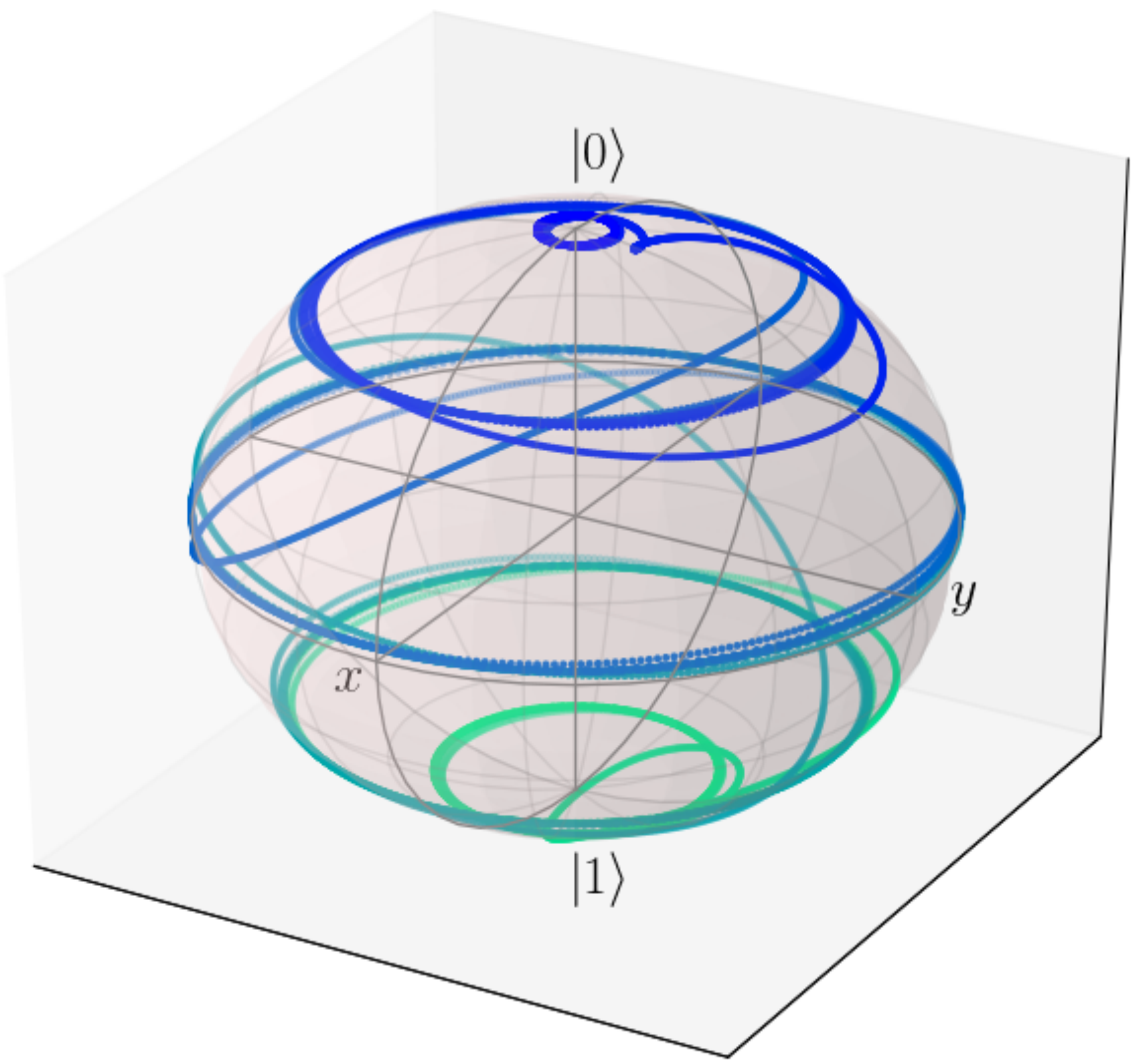
The **`Result`** class stores the expectation values of the operators passed to the solver.

```
result_ref = mesolve(H, rho0, times, c_ops, e_ops)
plot_expectation_values(result_ref, y_labels = "E[a'a], …,)
```

Simple Example: A driven, damped single mode cavity.

Solvers:

- **`mesolve`**: Lindblad master equations
- **`mcsolve`**: Monte-Carlo trajectory
- **`Floquet_modes`**: Floquet theory
- **bloch_redfield:** Bloch-Redfield master equation
- **`ssesolve`**: Stochastic Schrödinger equation
- **`smesolve`:** Stochastic master equations



QuTiP lead team: Acknowledgements and funding

Shahnawaz Ahmed
Chalmers, Sweden
(RIKEN, Japan)

Alex Pitchford
Aberystwyth University
United Kingdom

Eric Giguère
U. de Sherbrooke
Canada

Dr. Neill Lambert
RIKEN, Japan

Prof. Franco Nori
RIKEN, Japan
U. of Michigan (USA)

RIKEN

日本学術振興会
Japan Society for the Promotion of Science

JST Japan Science and Technology Agency

THE ROYAL SOCIETY

JOHN TEMPLETON FOUNDATION

NUMFOCUS
OPEN CODE = BETTER SCIENCE

Google Summer of Code

UNIVERSITÉ DE SHERBROOKE

PRIFYSGOL ABERYSTWYTH UNIVERSITY

@NathanShammah

GitHub: nathanshammah

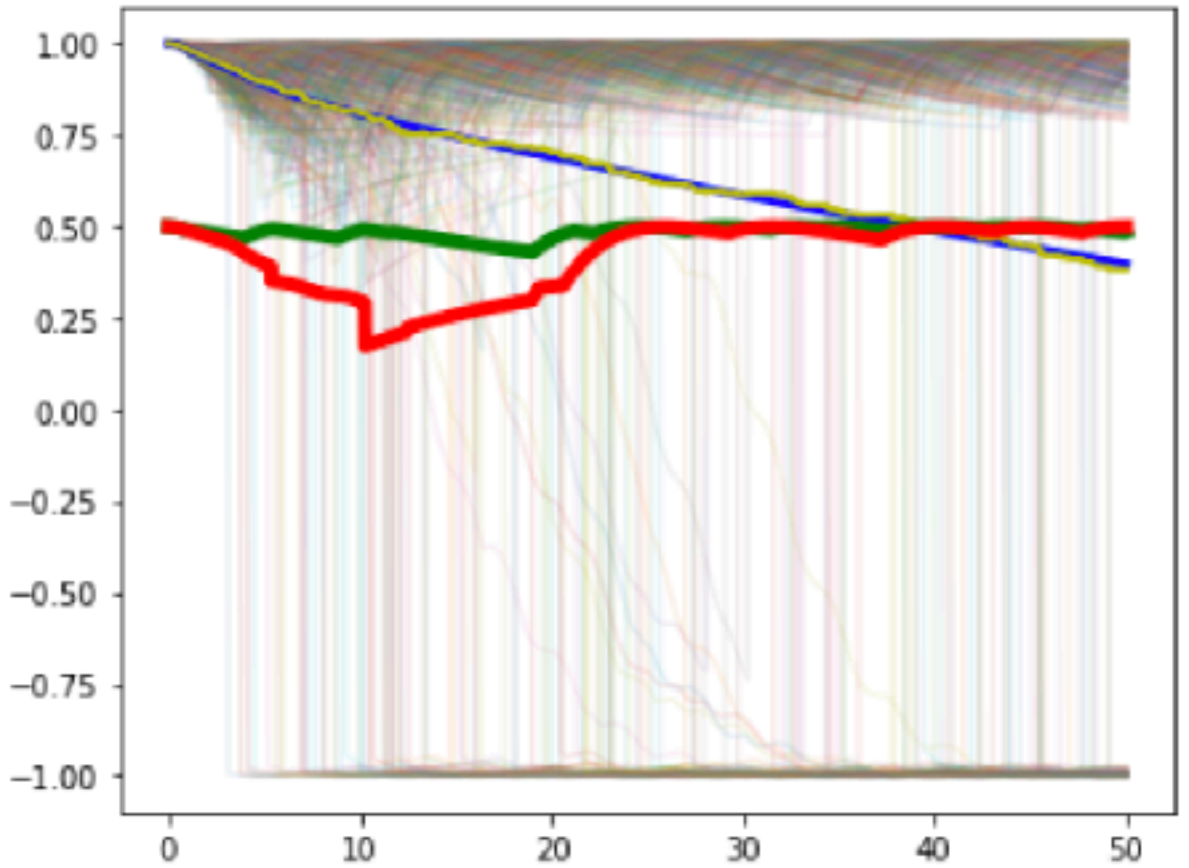
medium.com/quantum-tech

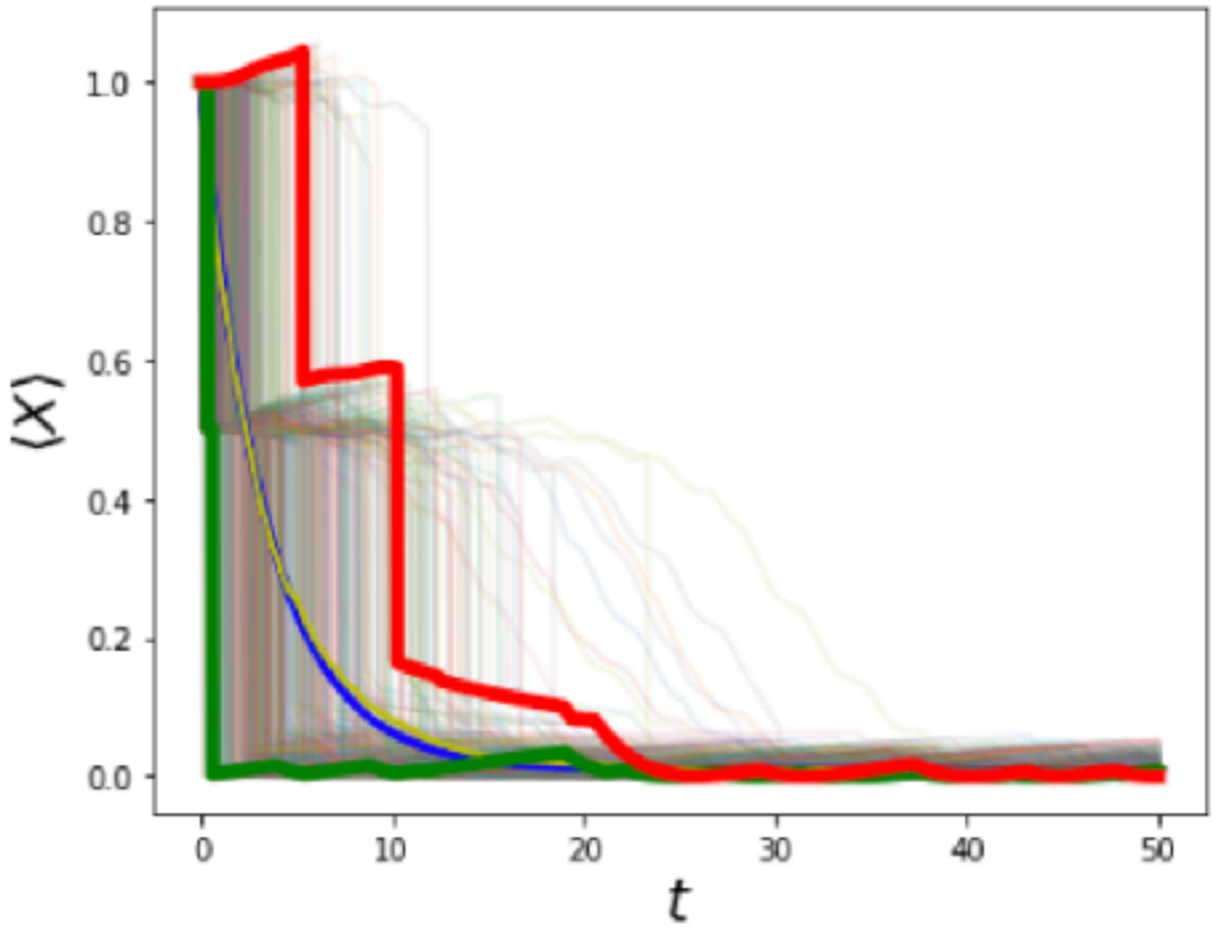
Consulting: quantika.co

QuTiP 4.4.0: Major updates

The **Qu**antum **T**oolbox **i**n **P**ython

Faster, Flexible Quantum Trajectories (Monte Carlo, `mcsolve`)

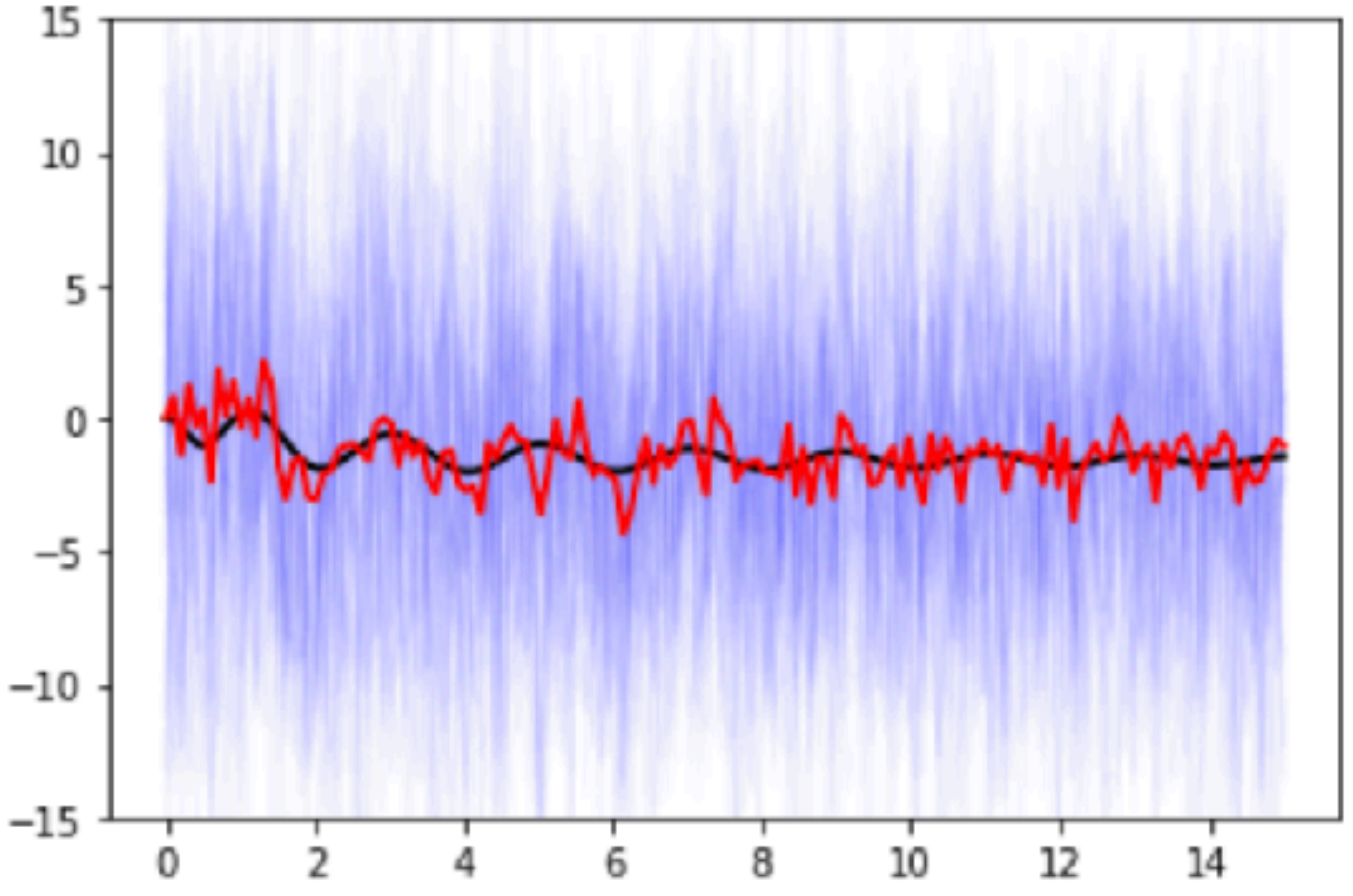




50x faster

Time-dependent jumps

More General Stochastic Master Equations (`smesolve`)



Several integration methods implemented

Quantum Information Processing

User-defined Quantum Gates

`molmer_sorensen`

Quantum Hellinger distance

Bug Fixes More info: qutip.org

QuTiP: A Growing Ecosystem

What's going on

- 2018: Joined NumFOCUS, foundation for scientific code (NumPy)

NUMFOCUS
OPEN CODE = BETTER SCIENCE

- 2019: Participating to Google Summer of Code 2019:

Student Applications opened March 26th. Closed April 9th. 3 students working on summer projects.

https://github.com/qutip/qutip/wiki//Google-Summer-of-Code-2019

2019: Applied to 1st Google Season of Docs 2019: Technical writers projects for documentation.

Google
Summer of Code

- 2018-2019: Reaching out to the sci-dev community.

EuroScipy 2018
July 2018
Trento, Italy

PyData 2018
November 2018
Warsaw, Poland

FOSDEM'19 (Quantum Computing)
February 2, 2019,
Brussels, Belgium

EuroScipy 2019
September 2019
Bilbao, Spain

1st QuTiP developers workshop
February 19-21, 2019
RIKEN, Wako, Japan

- 2018-2019: A growing QuTiP ecosystem of satellite libraries:

`piqs`
QuTiP library
Now a qutip module

`krotov`
QuTiP-based quantum
optimal control library

`matsubara`
A qutip plugin for
non-Markovian dynamics

`qupulse`
QuTiP-integrated
hardware control

QuTiP ecosystem:
Like AstroPy,
but for Quantum

QuTiP ongoing expansion (2019)

qutip.nisq

Model noise in quantum information processing (QIP)

QuTiP's QIP module represents ideal quantum circuits.

Student: Boxi Li (ETH Zurich)

Objectives:
- Go beyond gates as instantaneous unitary transformation
- Noise model for realistic devices
- Noise model for dissipative dynamics

First results:
- Added Mølmer-Sørensen gate
- Allowed user-defined gates
- Added optical pulses for gate shaping

Code: Boxi Li. Github: BoxiLi
Mentors: Alex Pitchford,
Neill Lambert,
Shahnawaz Ahmed,
Nathan Shammah

Google Summer of Code

https://gsoc2019-boxili.blogspot.com

qutip.lattice

Model lattices in QuTiP

A qutip.lattice module implementing Hamiltonians for paradigmatic 1D lattice models.

Student: Saumya Biswas (U. Oregon)

Objectives:
Single particle picture,
- SSH model (topological properties)
Spin chains and bosonic lattices
- Bose-Hubbard model

Complement existing libraries:
- QuSpin
- pythontb

Code: Saumya Biswas
Mentors: Clemens Gneiting,
Eric Giguere,
Shahnawaz Ahmed,
Nathan Shammah

Google Summer of Code

https://latticemodelfunctions.blogspot.com

qutip.tiqs

Translational invariant Lindblad dynamics

Driven-dissipative systems
Liouvillian spectrum

Idea
Exploit the symmetries of the dynamics
Spin chains and bosonic lattices
Liouvillian spectrum

$$\dot{\rho} = \mathscr{L}\rho$$

$$[\mathscr{L}, S] = 0$$

F. Minganti, *et.al.* Phys. Rev. A 98, 042118 (2018)

Code: Fabrizio Minganti. Github: fminga

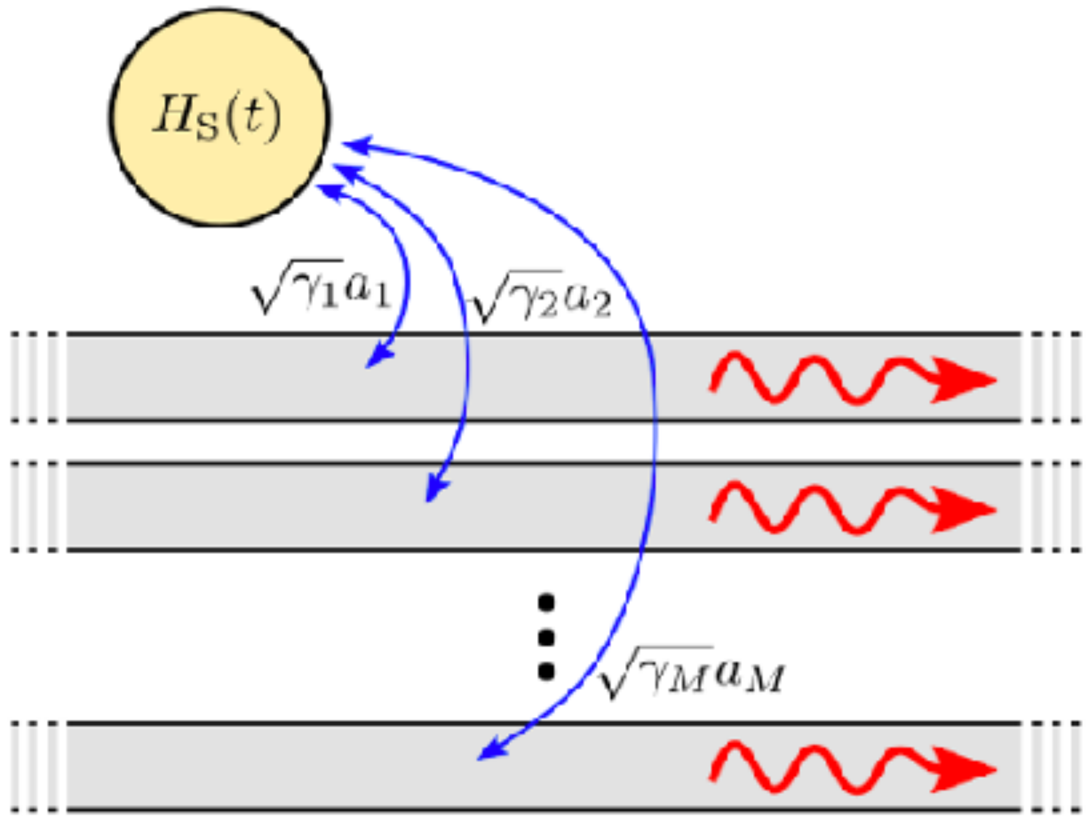
Partly Deployed → **In Deployment** → **In Development**

QuTiP recent additions (2018-2019)

qutip.scattering

Model nonlinear photon scattering
in multiple waveguides

How photons scatter into the waveguide
when the system is driven
with some excitation field



K.A. Fischer, *et.al.* (2017), arXiv: 1710.02875

Code: Ben Bartlett. Github: bencbartlett

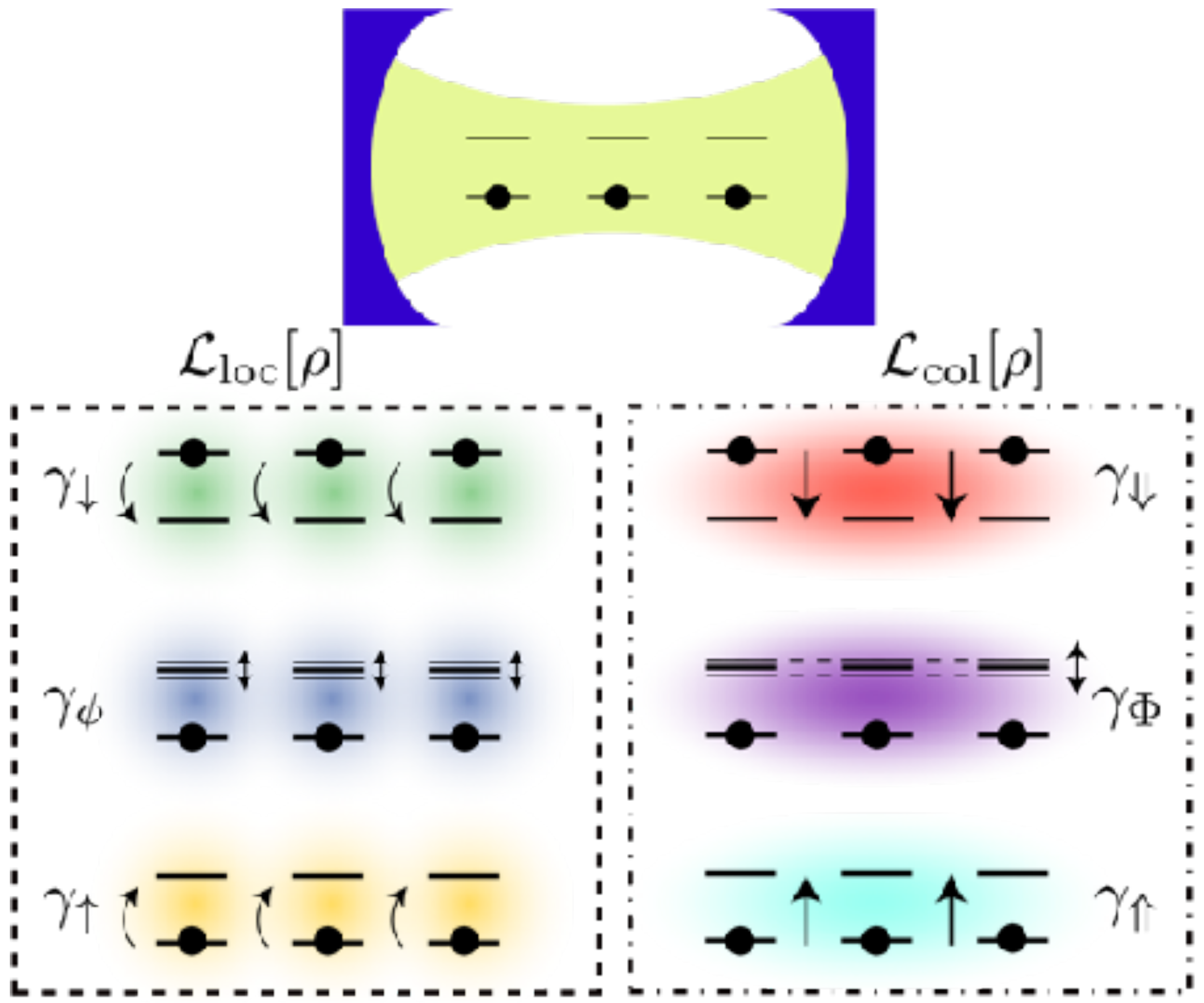
qutip.piqs

Efficiently model local dissipation

Collective dissipation

vs. Local dissipation

vs. Coherent coupling



N. Shammah *et al.*, Phys Rev A **98**, 063815 (2018)

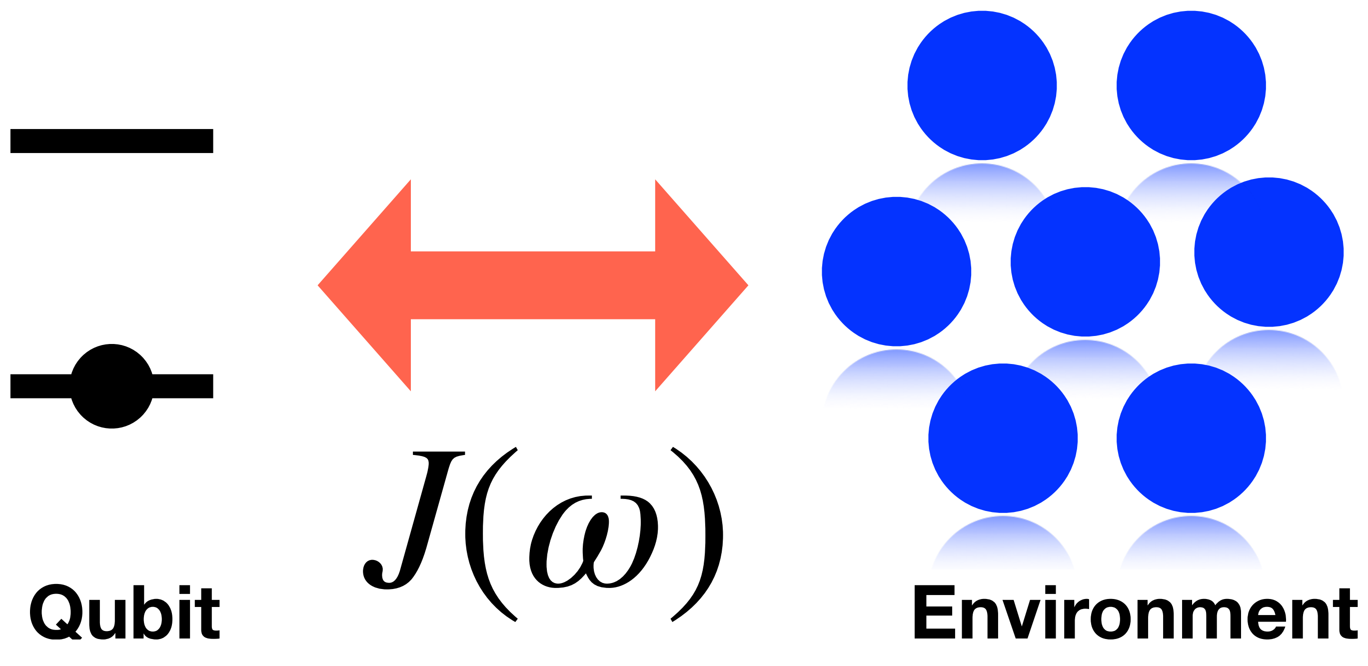
Code: Nathan Shammah and Shahnawaz Ahmed

qutip.nonmarkov

The environment has a memory

Non-Markovian dynamics

Hierarchical Equations of Motion (HEOM).



A. Fruchtman, *et al.*, Sci. Rep. **6**, 28204 (2016)
N. Lambert *et al.*i arXiv: 1903.05892

Code: Shahnawaz Ahmed and Neill Lambert

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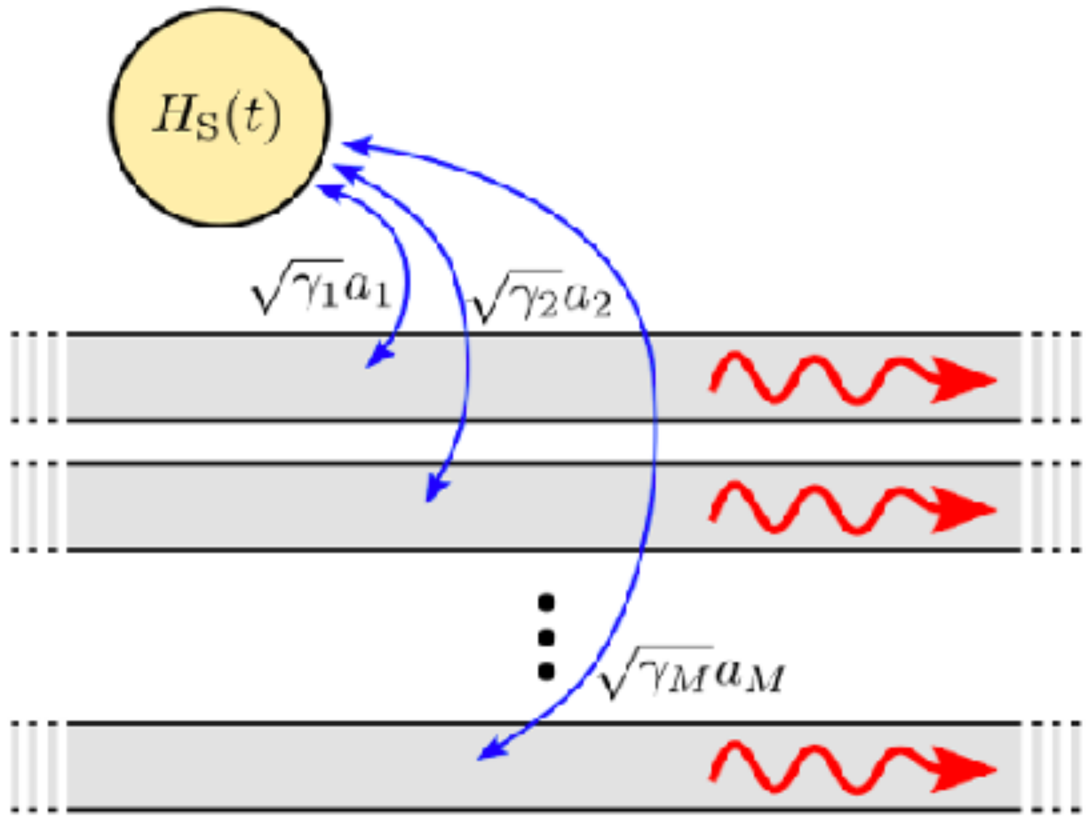

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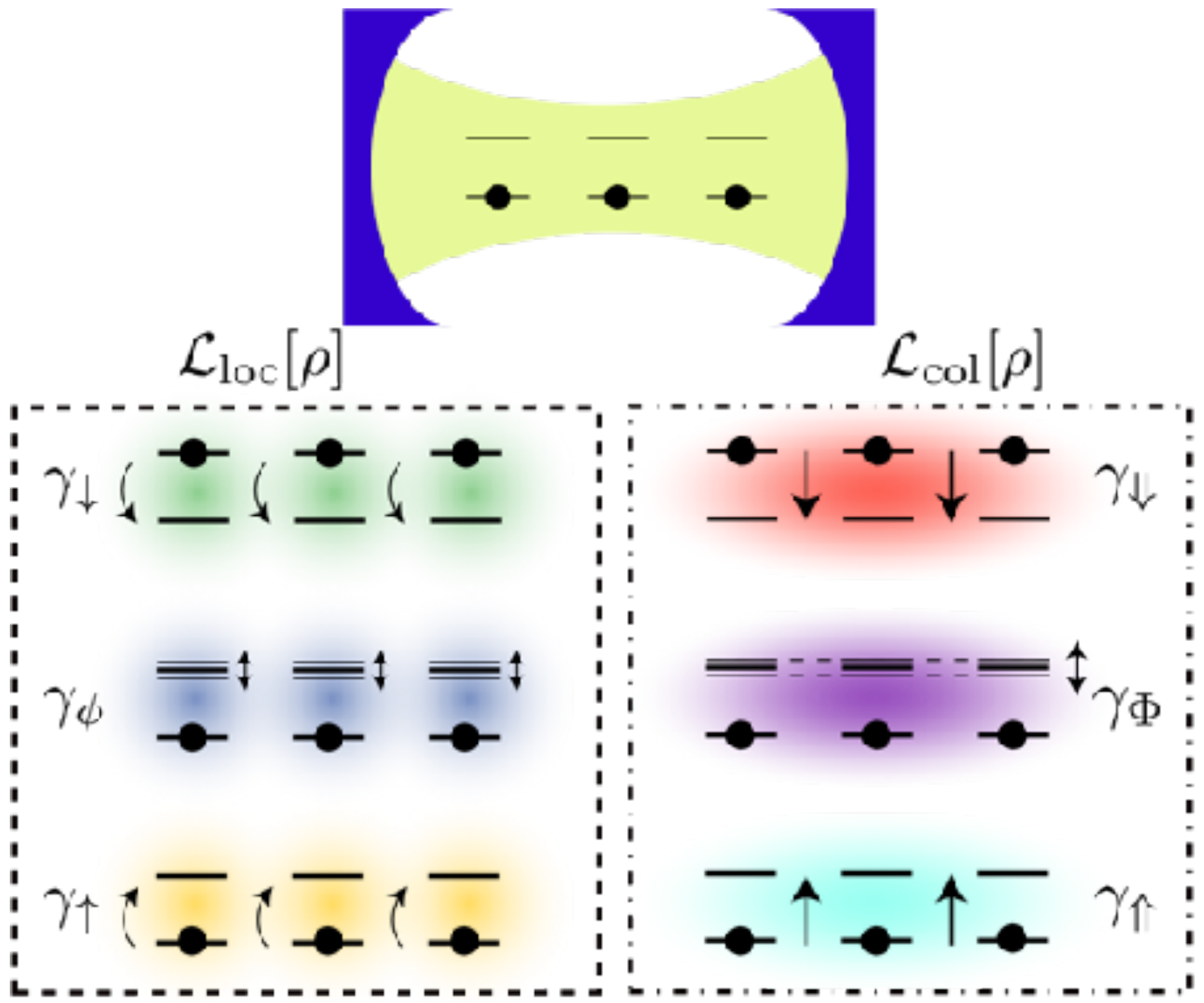

N. Shammah *et al.*, Phys Rev A **98**, 063815 (2018)

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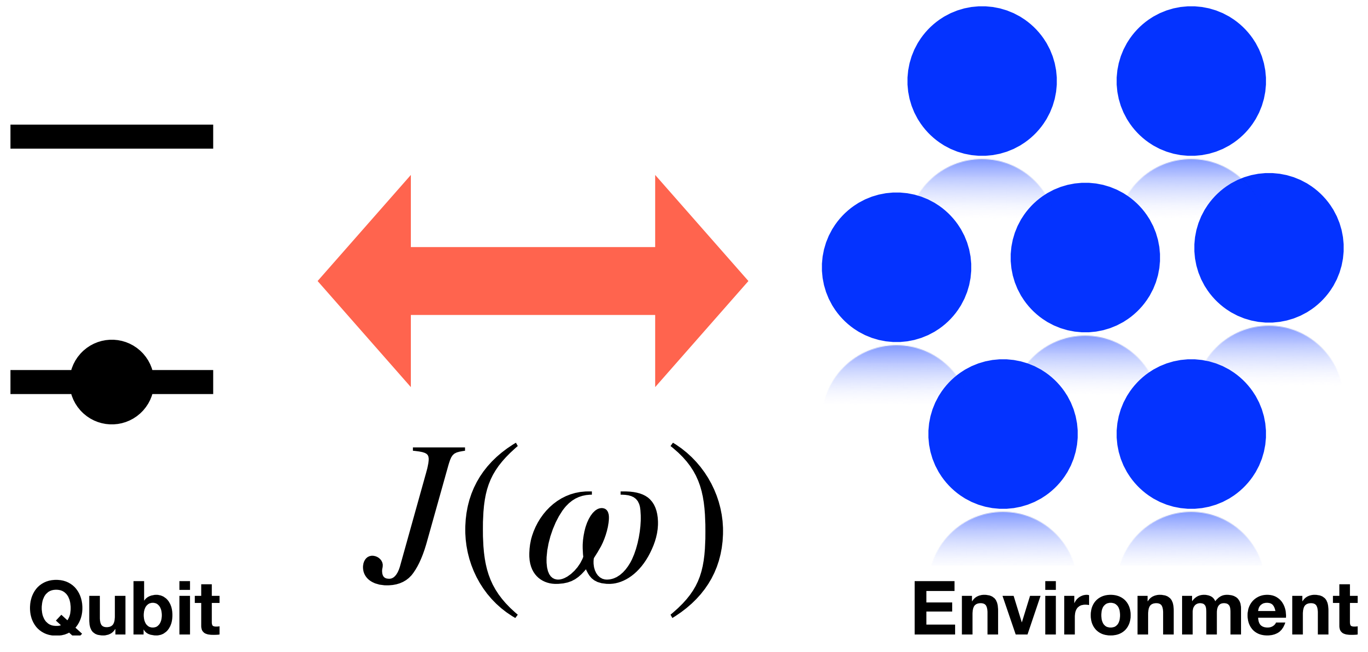

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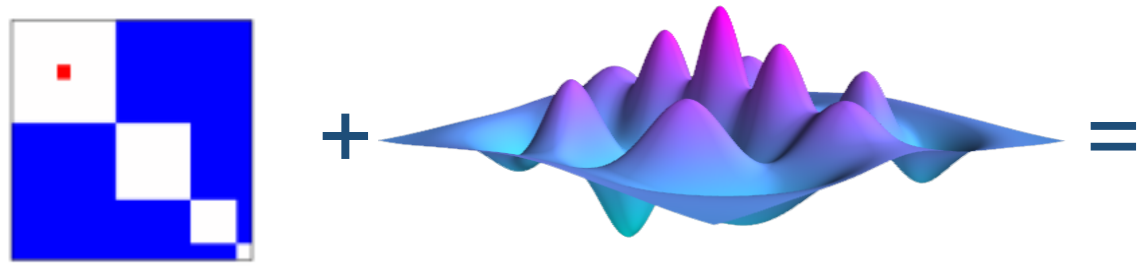
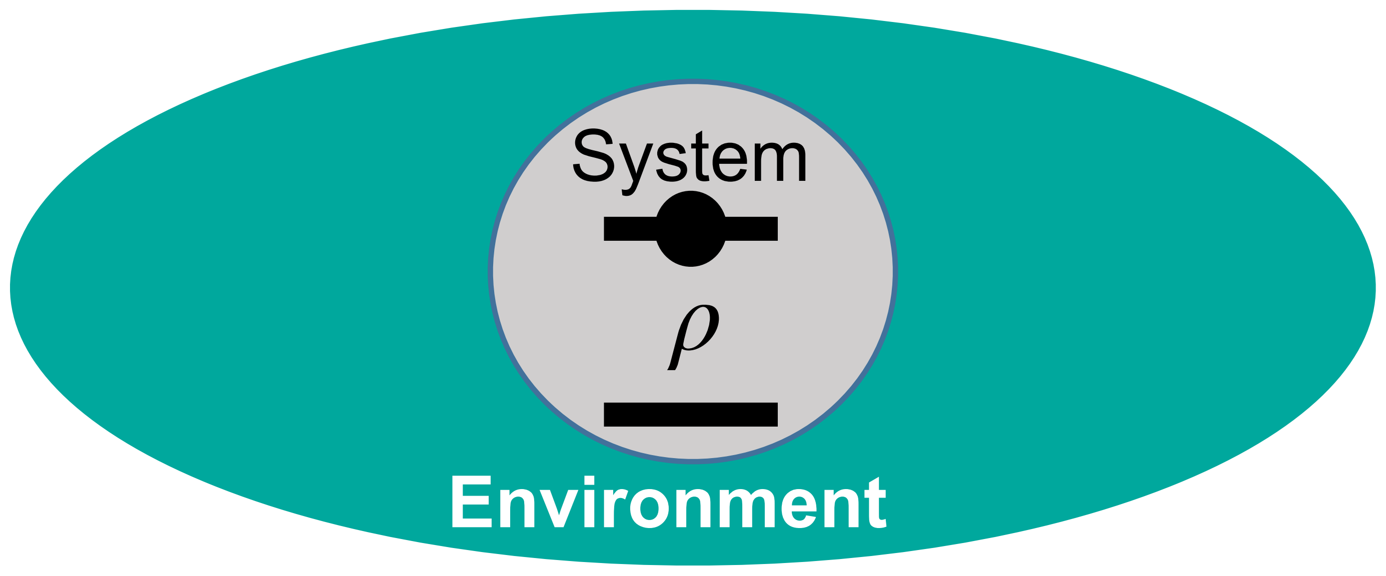


import qutip.piqs

Open quantum systems: Lindblad master equation

In general, the Liouvillian space grows exponentially as 4^N



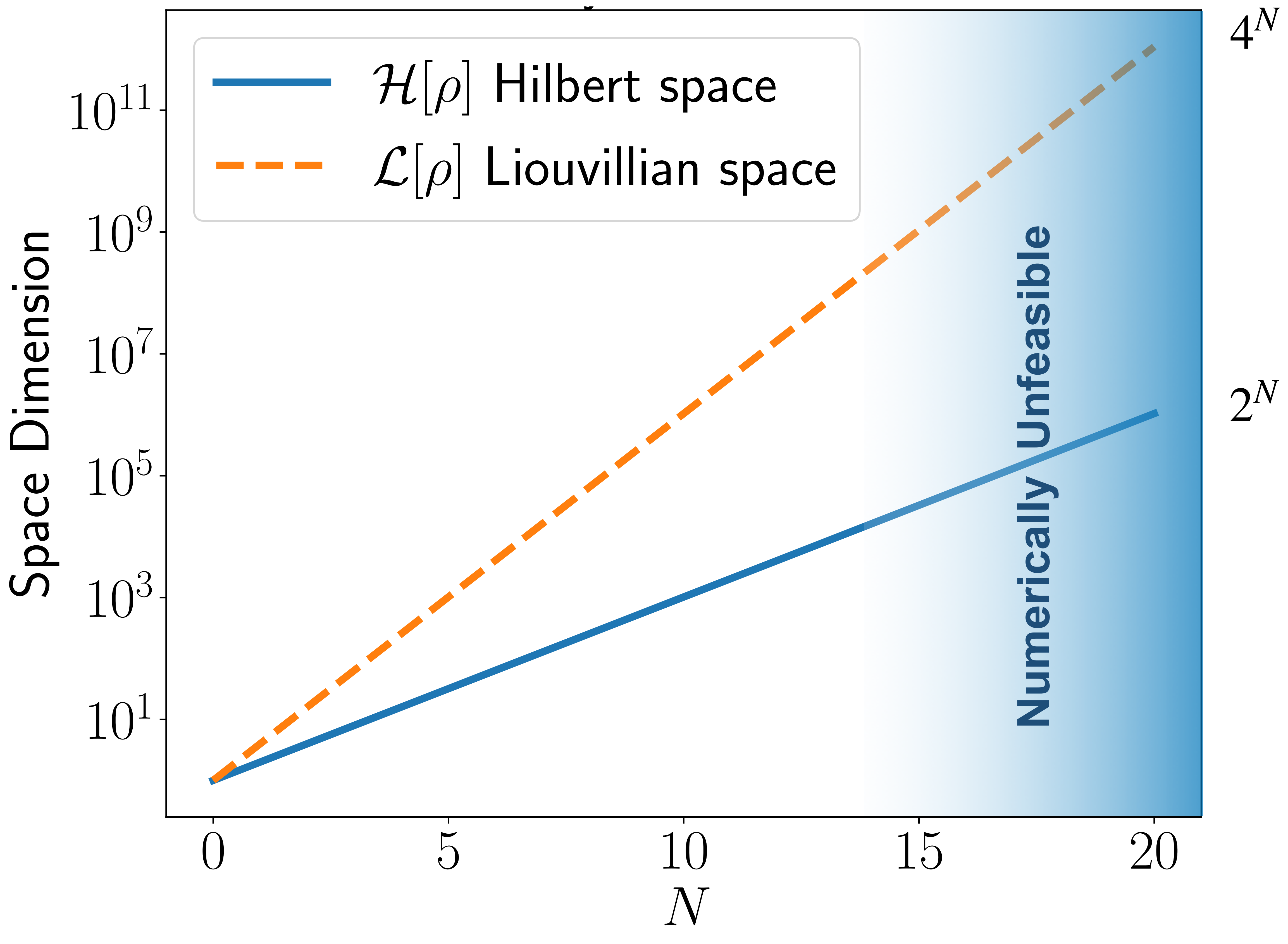
$$\frac{d}{dt}\rho = -\frac{i}{\hbar}[H,\rho] + \sum_{i,j}^{N} \gamma_{ij} \left(L_i \rho L_j^\dagger - \frac{1}{2} L_i^\dagger L_j \rho - \frac{1}{2} \rho L_i^\dagger L_j \right)$$

Approximations:

- **Born**
- **Markov**
- **Rotating wave**

$$\frac{d}{dt}\overrightarrow{\rho} = \mathscr{L}\overrightarrow{\rho}$$

Liouvillian



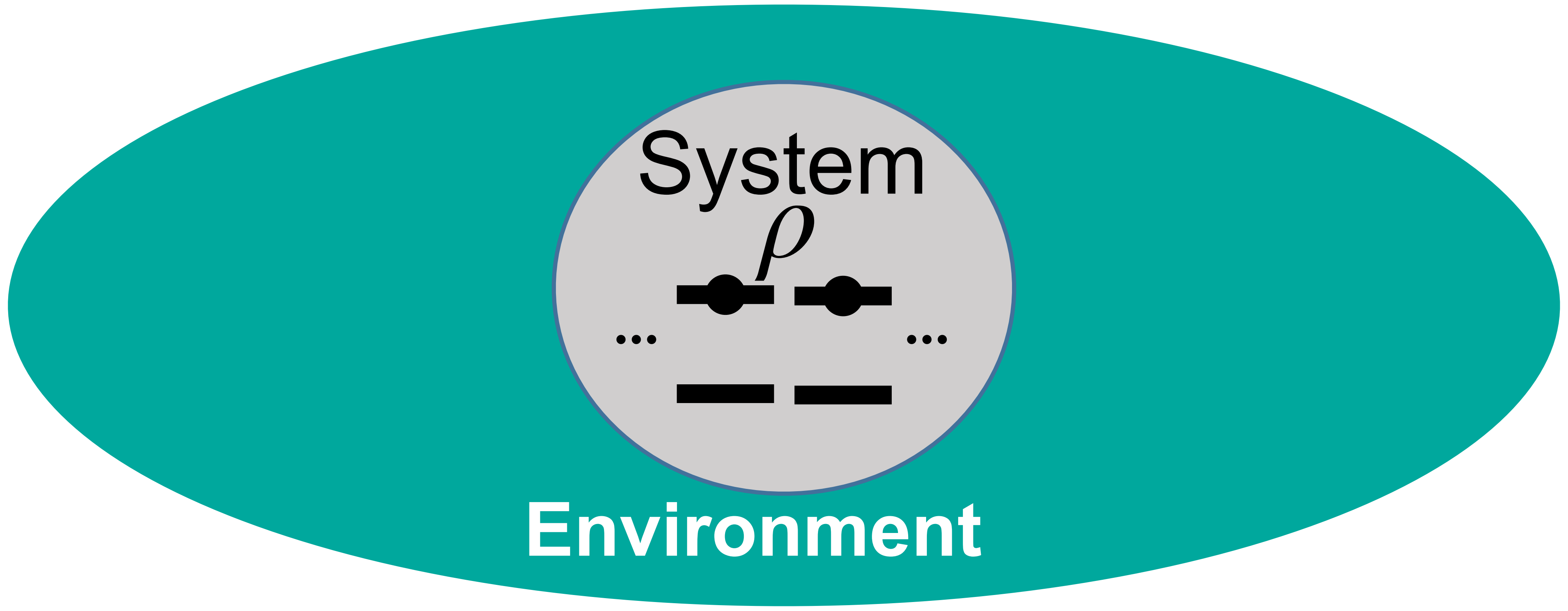
Qubits Number

`qutip.piqs`: Exploiting permutational symmetry

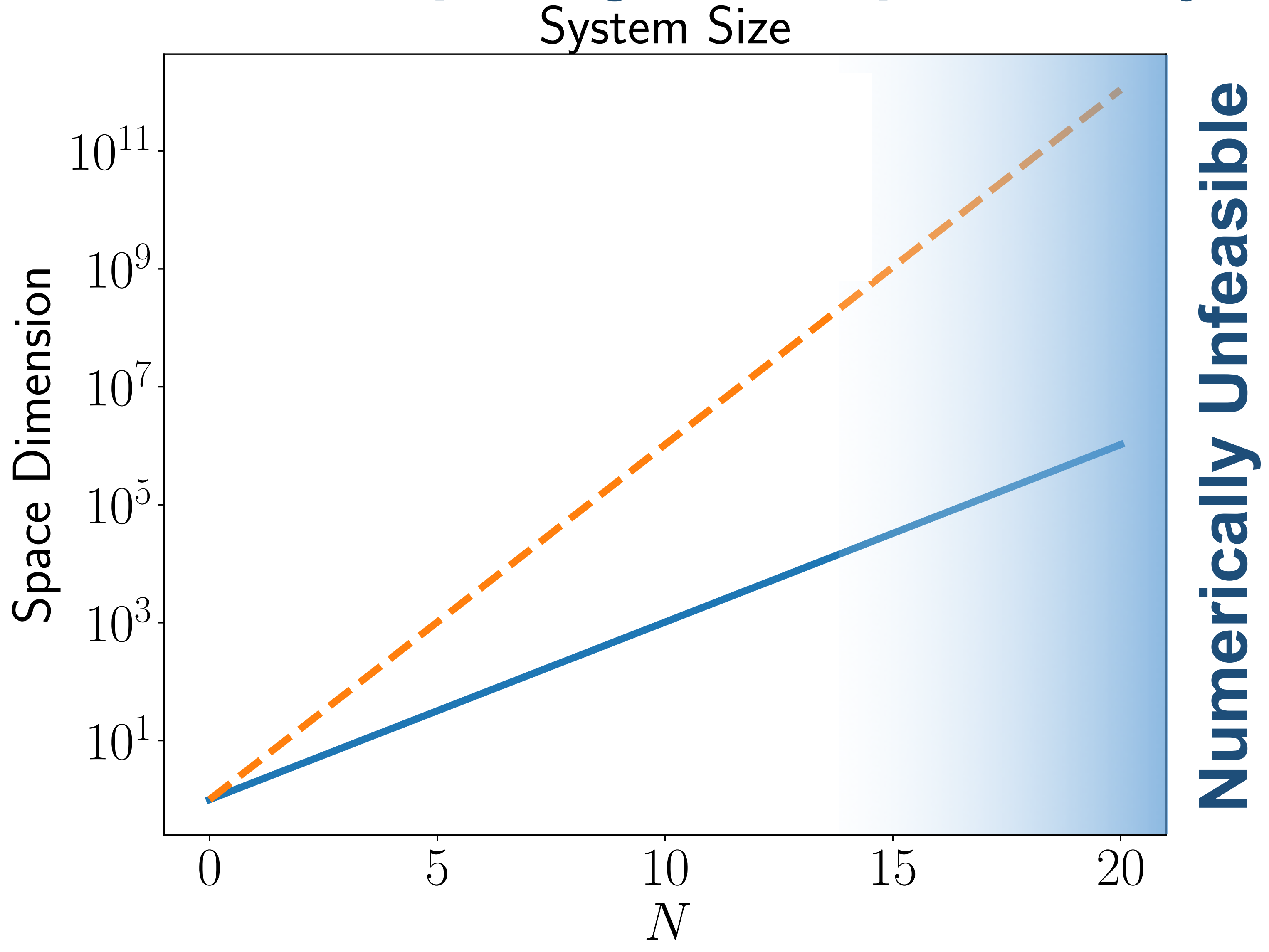
About a recent QuTiP module simulating the dynamics of *many* qubits

$$i\hbar\frac{d}{dt}\rho = [H,\rho] + \gamma\sum_{i}^{N}\left(L_i\rho L_i^\dagger - \frac{1}{2}L_i^\dagger L_i\rho - \frac{1}{2}\rho L_i^\dagger L_i\right)$$

Problem



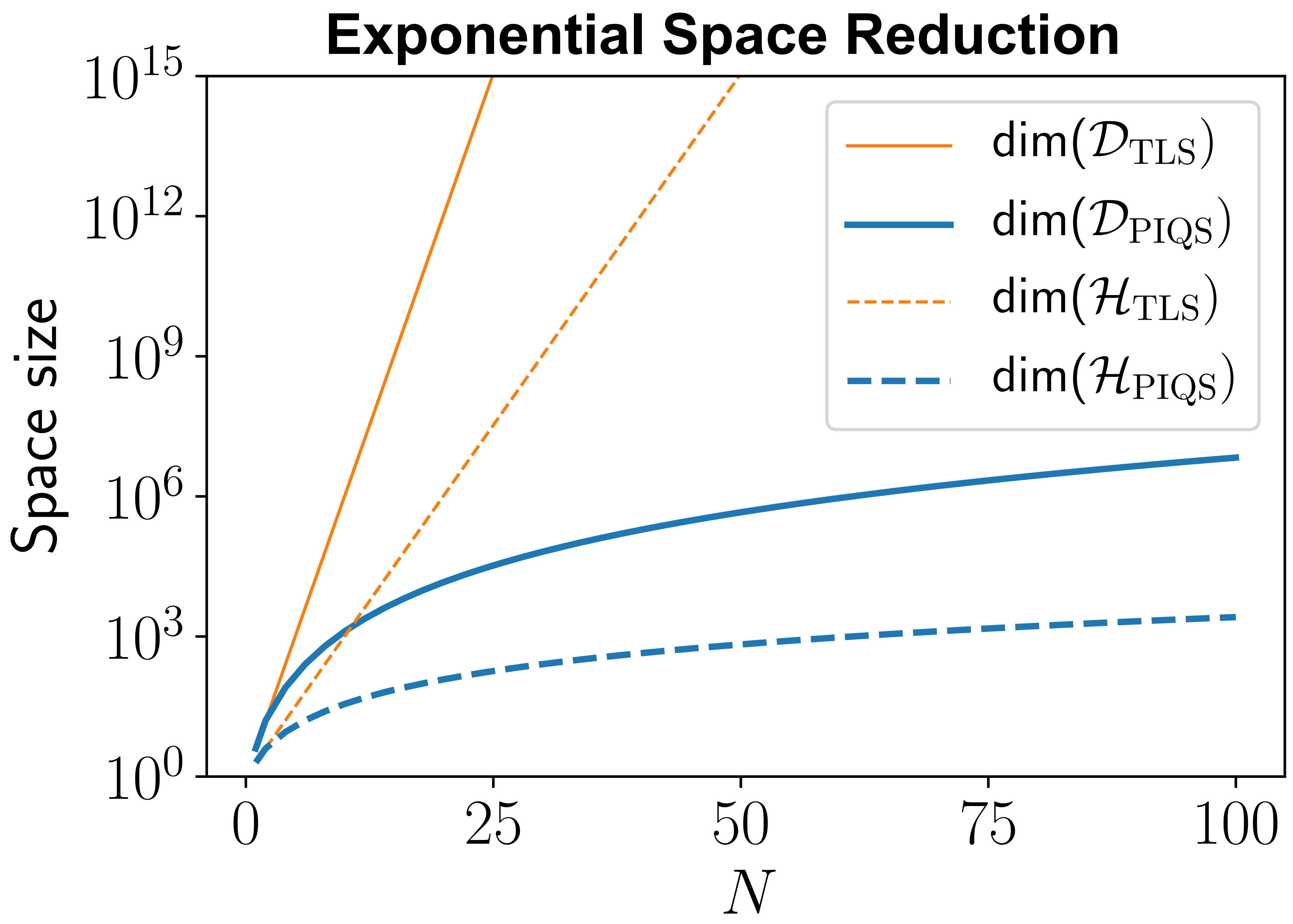
The Liouvillian space grows exponentially as 4^N



Solution

Permutational symmetry

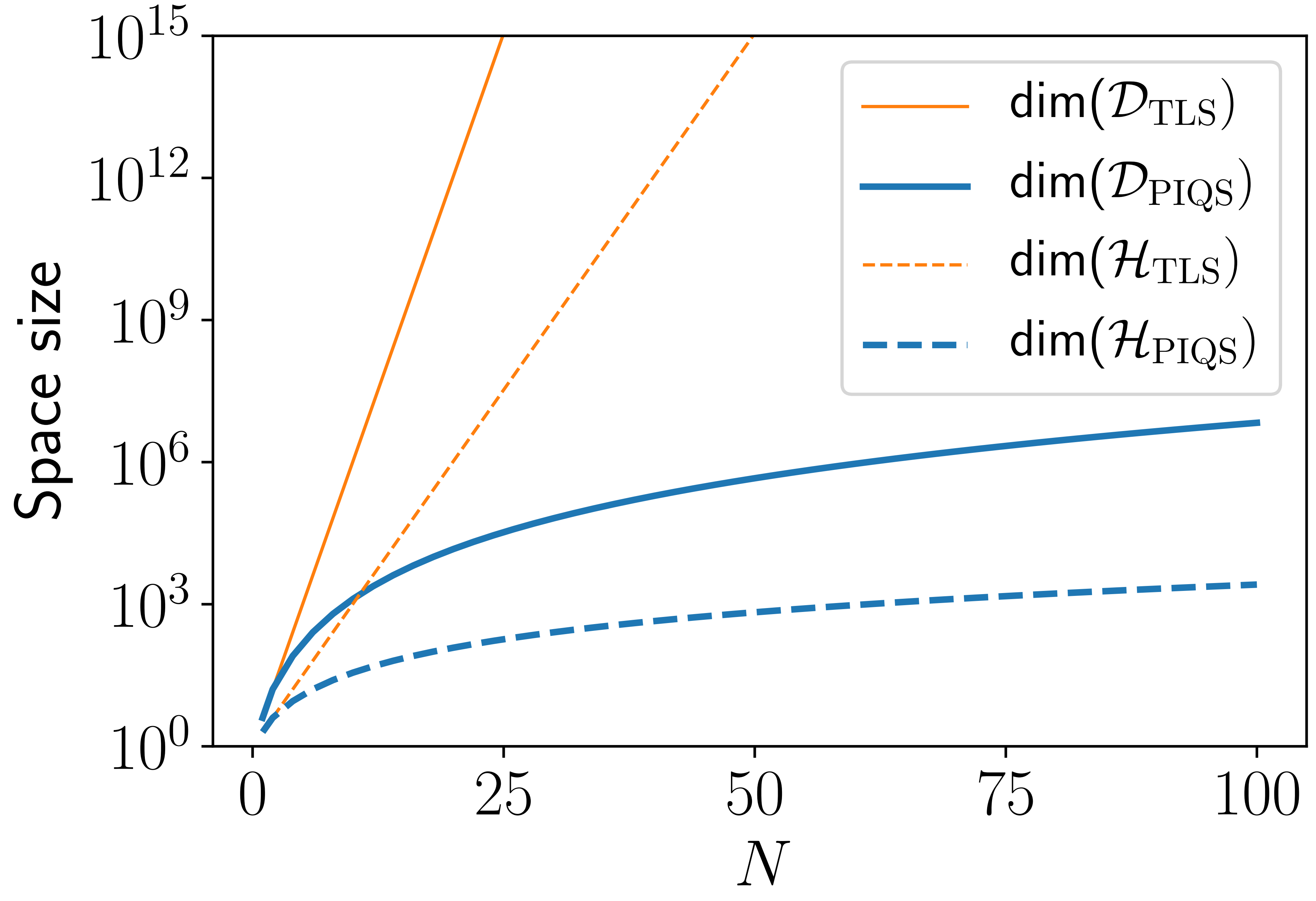
The Su(2) Pauli operators can be written using Su(4) generators in Liouvillian space **obtaining an exponential size reduction.**



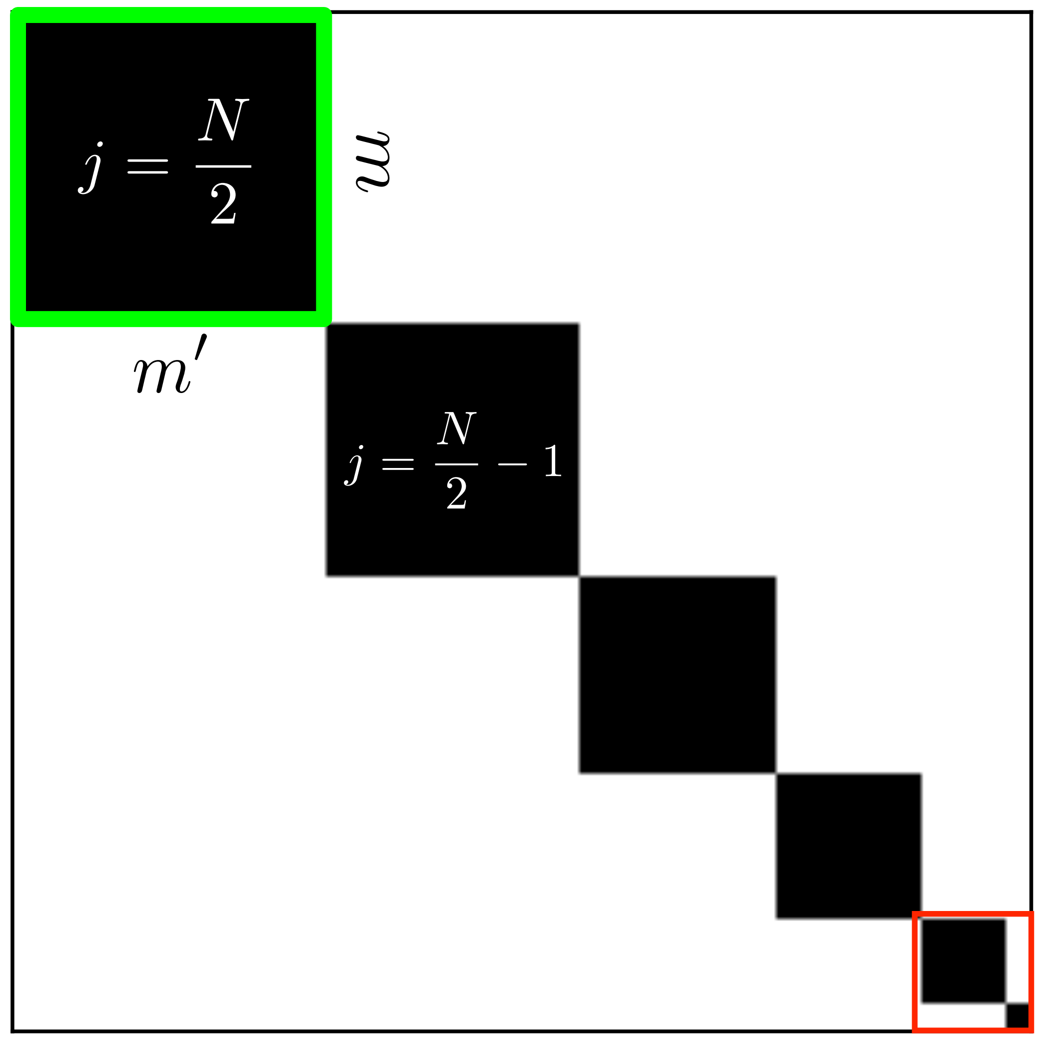
Numerical method in the Dicke space

Homogeneous local dissipation can be included in the dynamical model

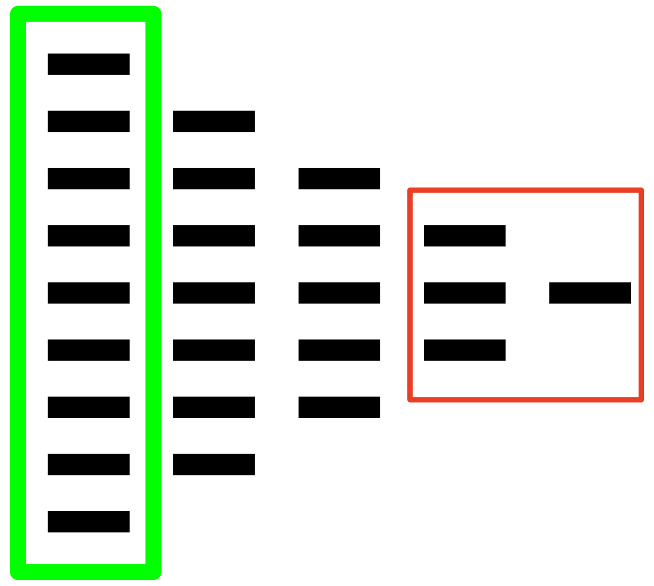
Exponential Space Reduction



Block-Diagonal Density Matrix



Dicke space visualization



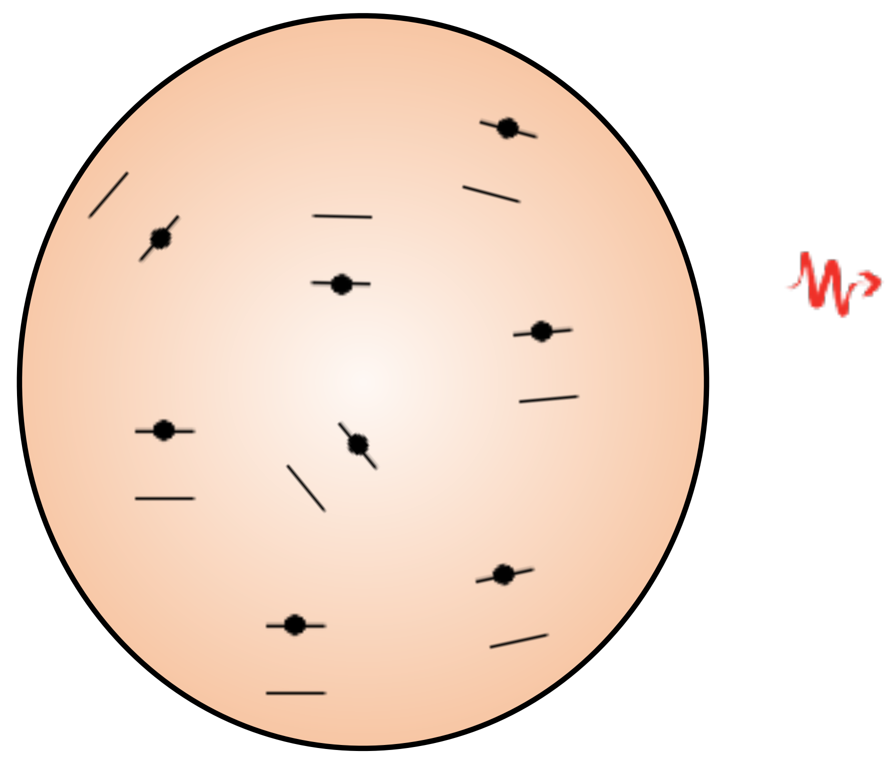
Number of Dicke States

$$n_{\mathrm{DS}}(N) = \sum_{j=0}^{N/2}(2j+1) = \left(\frac{N}{2}+1\right)^2$$

B. A. Chase and J. M. Geremia, Phys. Rev. A **78**, 052101 (2008)
F. Damanet, D. Braun, and J. Martin, Rev. A **94**, 033838 (2016)
N. Shammah et al. Phys. Rev. A (2017)

Dicke superradiance: light emission from an atomic cloud

Probing the dynamics of *many* excited qubits / two-level systems



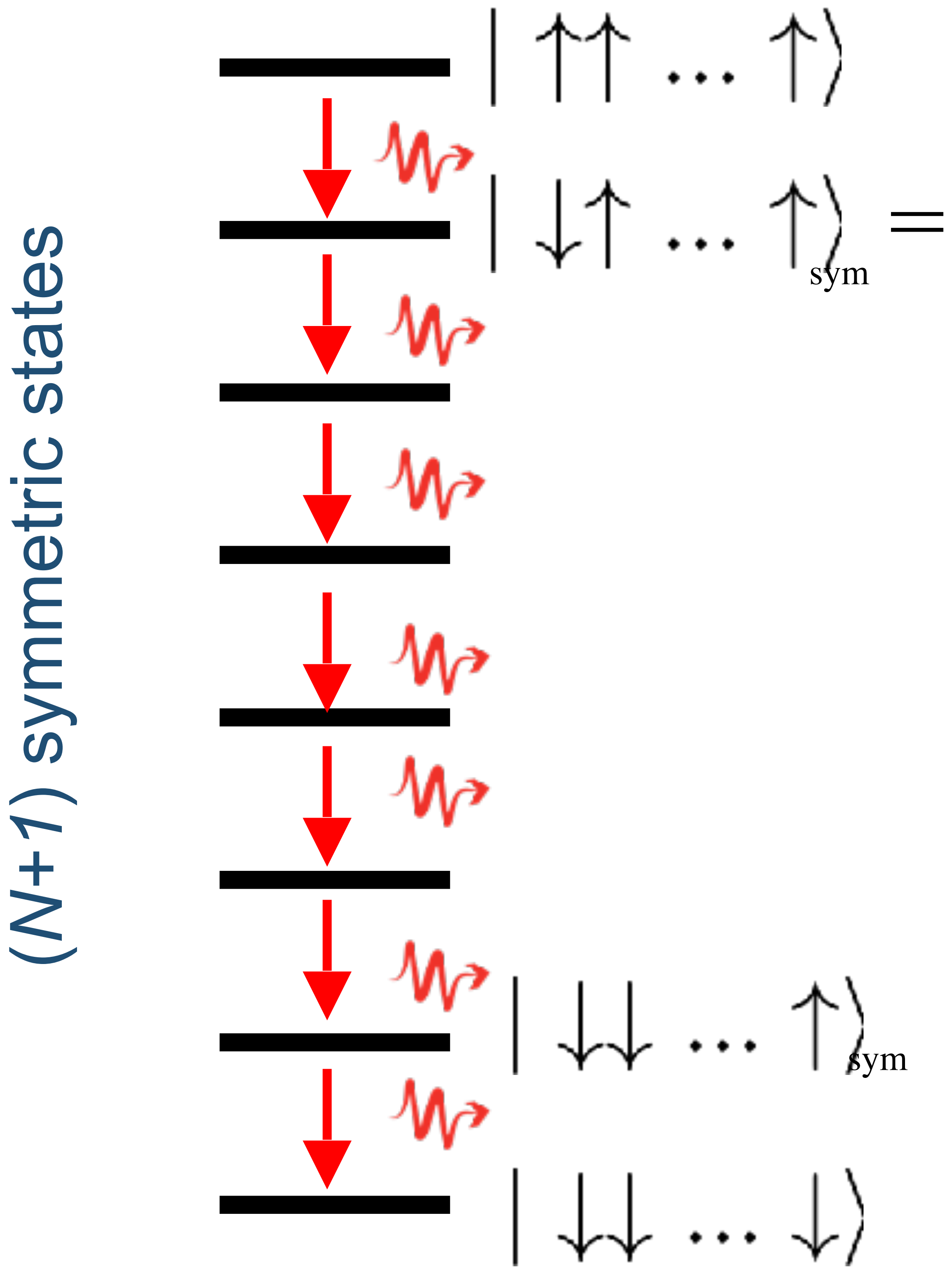
$t = 0$ $\quad\quad |\uparrow\uparrow \ldots \uparrow\rangle$

$$|\downarrow\uparrow \ldots \uparrow\rangle_{\text{sym}} = \frac{1}{\sqrt{N}}\left(|\downarrow\uparrow \ldots \uparrow\rangle + |\uparrow\downarrow \ldots \uparrow\rangle + \ldots\right)$$

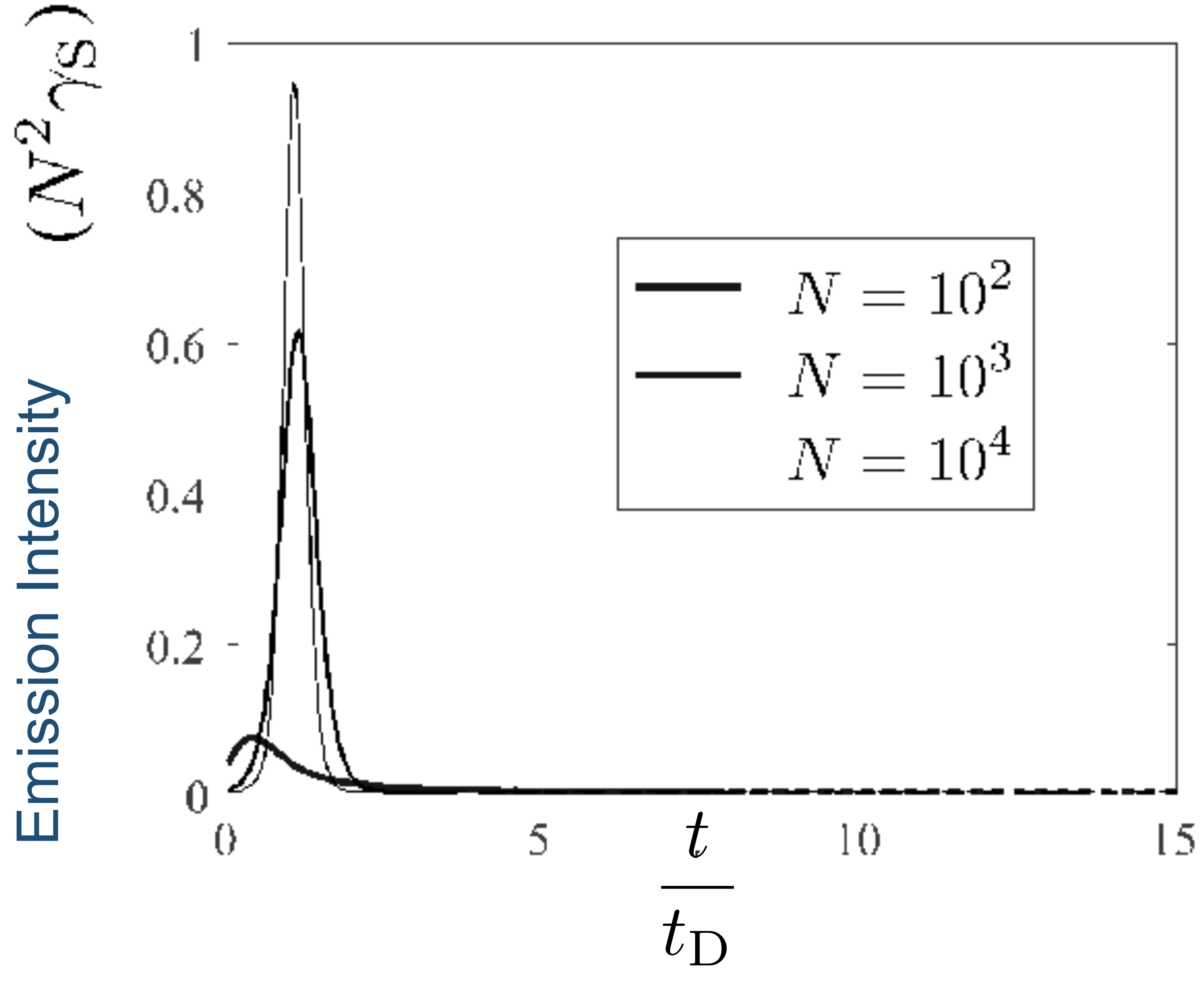
Superradiance emerges from collective coupling

Ideal case

$$\dot{\rho} = -i[\omega_0 J_z\,, \rho] + \gamma\,(J_-\rho\sigma_+ - J_+J_-\rho - \rho J_+J_-)$$



$$|\downarrow\uparrow\dots\uparrow\rangle_{\text{sym}} = \frac{1}{\sqrt{N}}\left(|\downarrow\uparrow\dots\uparrow\rangle + |\uparrow\downarrow\dots\uparrow\rangle + \dots\right)$$



$$t_D = \frac{\log N}{(N\gamma_S)}$$

M. Gross & S. Haroche, Physics Reports **93**, 301 (1982)

Superradiance in the symmetric limit

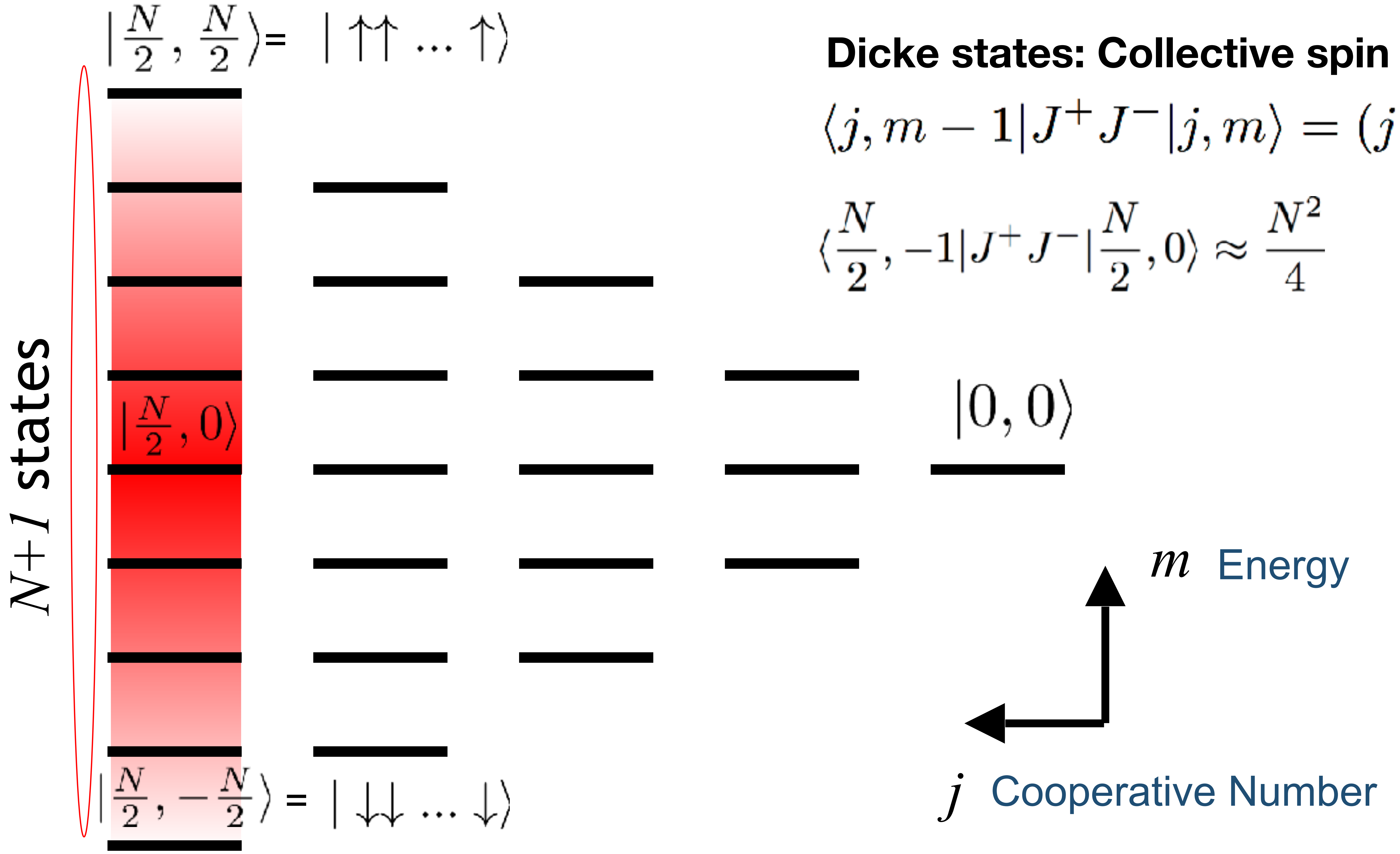
Standard study confined in a limited sector of the Hilbert space

$$\dot{\rho} = -i[\omega_0 J_z, \rho] + \gamma\,(J_- \rho \sigma_+ - J_+ J_- \rho - \rho J_+ J_-)$$

Dicke states: Collective spin states

$$\langle j, m-1 | J^+ J^- | j, m\rangle = (j-m)(j+m+1)$$

$$\langle \frac{N}{2}, -1 | J^+ J^- | \frac{N}{2}, 0\rangle \approx \frac{N^2}{4}$$

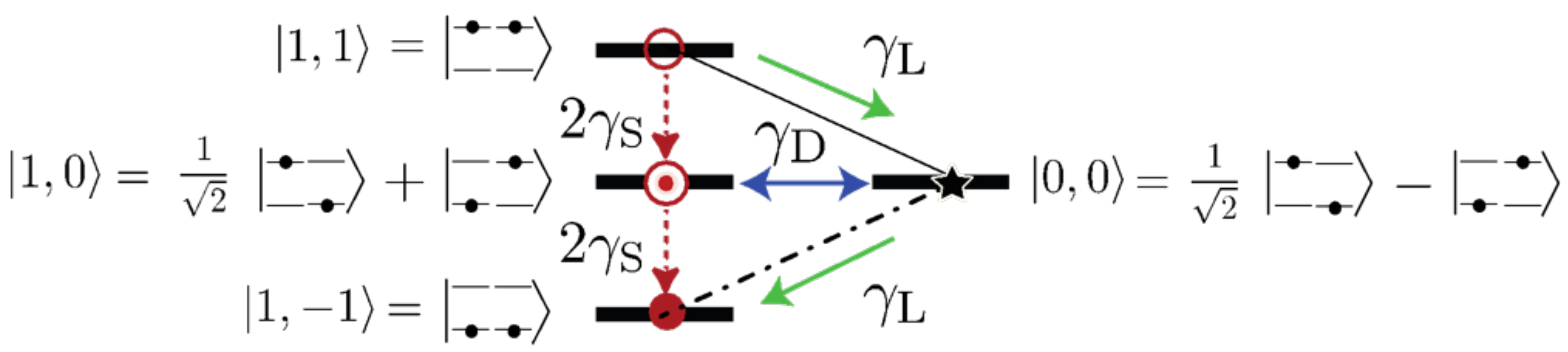


M. Gross & S. Haroche, Physics Reports **93**, 301 (1982)

The Dicke space generalizes the singlet-triplet system

Local processes connect the singlet (dark) state

$$\dot{\rho} = i\omega_0[\rho, J_z] + \boxed{\frac{\gamma_S}{2}\mathcal{L}_{J_-}[\rho]} + \boxed{\sum_{n=1}^{N}\frac{\gamma_L}{2}\mathcal{L}_{J_{-,n}}[\rho]} + \boxed{\sum_{n=1}^{N}\frac{\gamma_D}{2}\mathcal{L}_{J_{z,n}}[\rho]}$$



N=2

N Shammah *et al.*, Superradiance with local phase-breaking mechanisms, Phys. Rev. A 2017, arXiv 1704.07066

Analytical techniques

Hierarchy Truncation: Higher-moment correlations are neglected

Mean-field approximation $N \gg 2$

Factorization

$$\langle \dot{J}_z^2 \rangle = \gamma_S \left(\langle J^2 \rangle + \langle J_z \rangle - 3\langle J_z^2 \rangle + 2\underline{\langle J_z \rangle \langle J_z^2 \rangle} - 2\underline{\langle J_z \rangle \langle J^2 \rangle} \right)$$

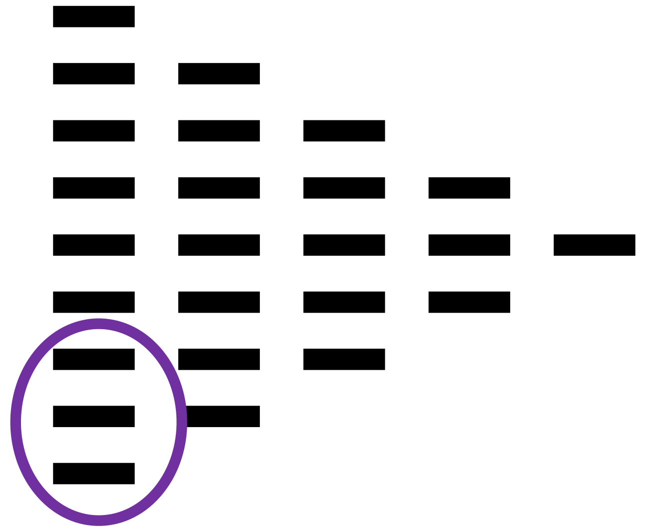
$$\boxed{\langle J_z^3 \rangle \simeq \langle J_z^2 \rangle \langle J_z \rangle} \qquad \boxed{\langle J_z J^2 \rangle \simeq \langle J_z \rangle \langle J^2 \rangle}$$

Bosonic approximation: Holstein-Primakoff (not suitable for local processes)

I. Holstein-Primakoff

$$H_{\text{int}} = g(J_+ + J_-)(a + a^\dagger) \approx \underline{\sqrt{2j}}\, g(b + b^\dagger)(a + a^\dagger)$$

$$J_z = b^\dagger b - j \qquad J_- = \sqrt{2j}\sqrt{1 - \frac{b^\dagger b}{2j}}\, b \qquad [b, b^\dagger] = 1 \qquad \frac{\langle b^\dagger b \rangle}{2j} \ll 1$$

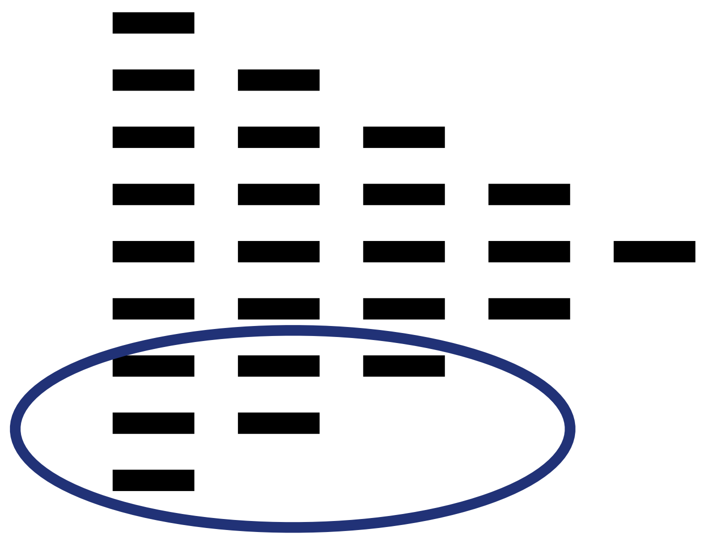


Bosonic approximation: Polariton Modes valid in the dilute regime

II. Dilute regime without cooperative number conservation

$$H_{\text{int}} = \underline{\sqrt{N}}\,\mathbf{d} \cdot \mathbf{E}(b_0 + b_0^\dagger)$$

$$b_p^\dagger = \frac{1}{\sqrt{N}} \sum_n f_n^p J_{+,n}, \quad p \in [1, N] \qquad [b_p, b_q^\dagger] = \delta_{p,q}$$

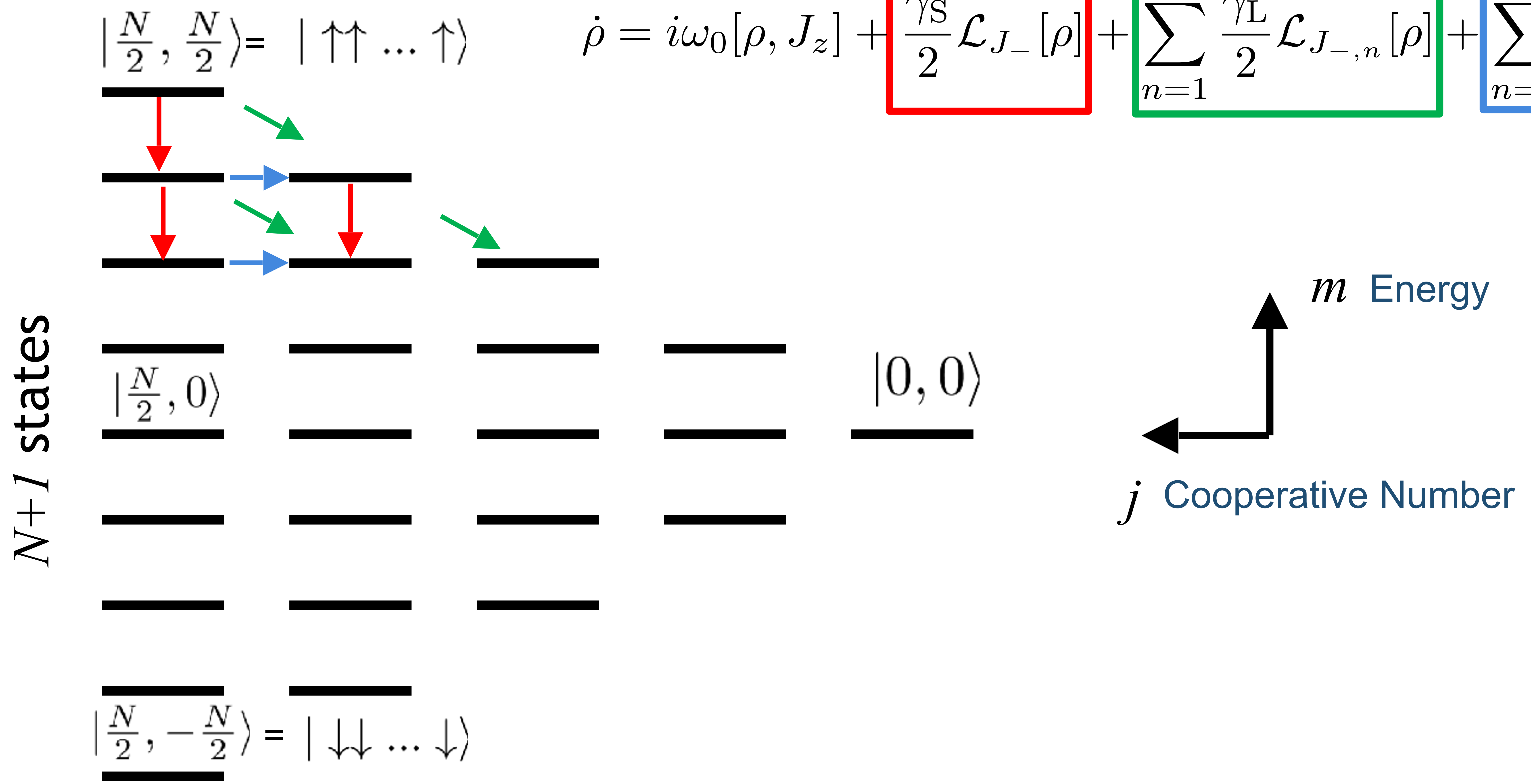


N Shammah *et al.*, Superradiance with local phase-breaking mechanisms, Phys. Rev. A 2017, arXiv 1704.07066

Superradiance and phase-breaking effects

Exploring inner states by including local incoherent processes

$$\dot{\rho} = i\omega_0[\rho, J_z] + \boxed{\frac{\gamma_S}{2}\mathcal{L}_{J_-}[\rho]} + \boxed{\sum_{n=1}^{N}\frac{\gamma_L}{2}\mathcal{L}_{J_{-,n}}[\rho]} + \boxed{\sum_{n=1}^{N}\frac{\gamma_D}{2}\mathcal{L}_{J_{z,n}}[\rho]}$$



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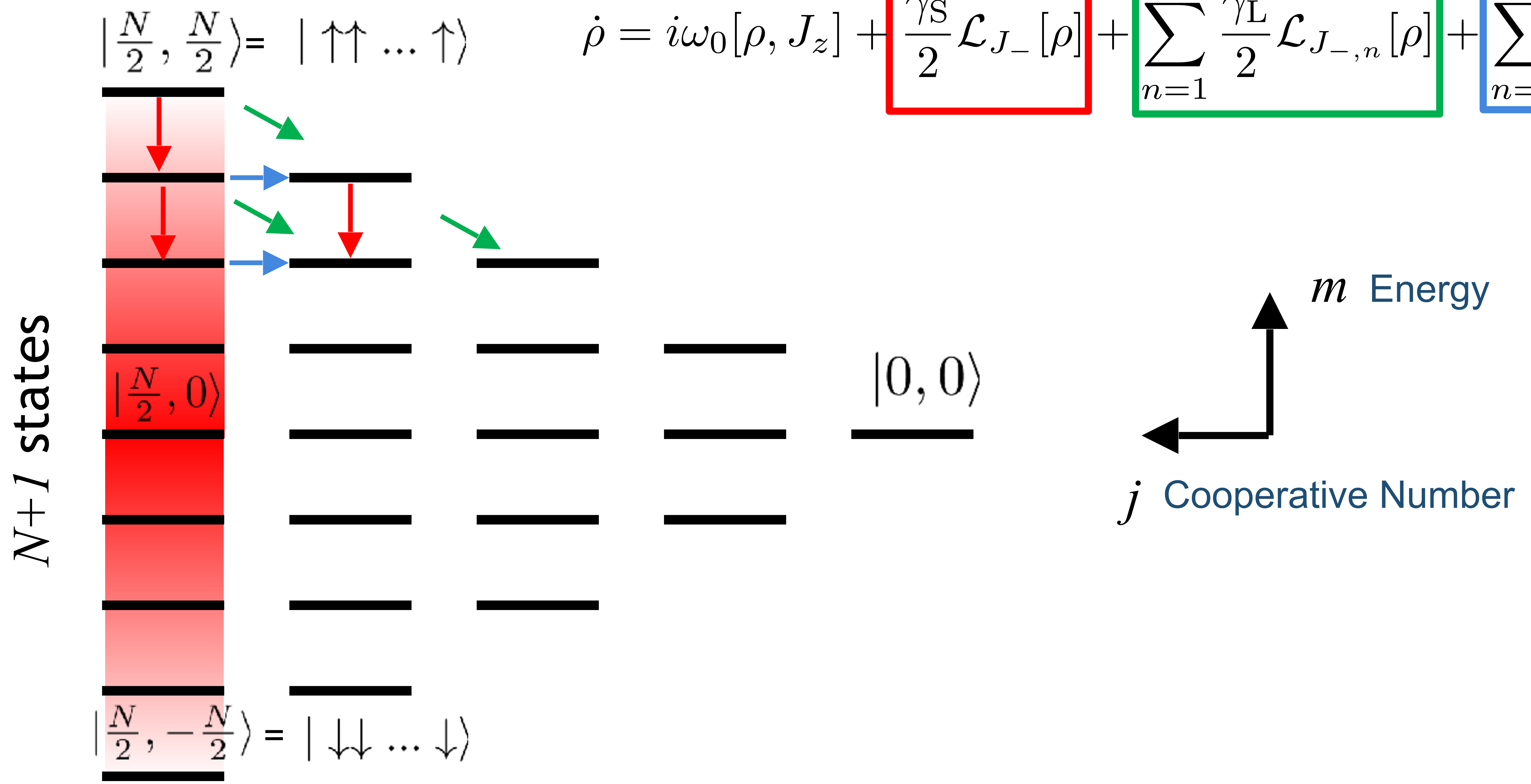

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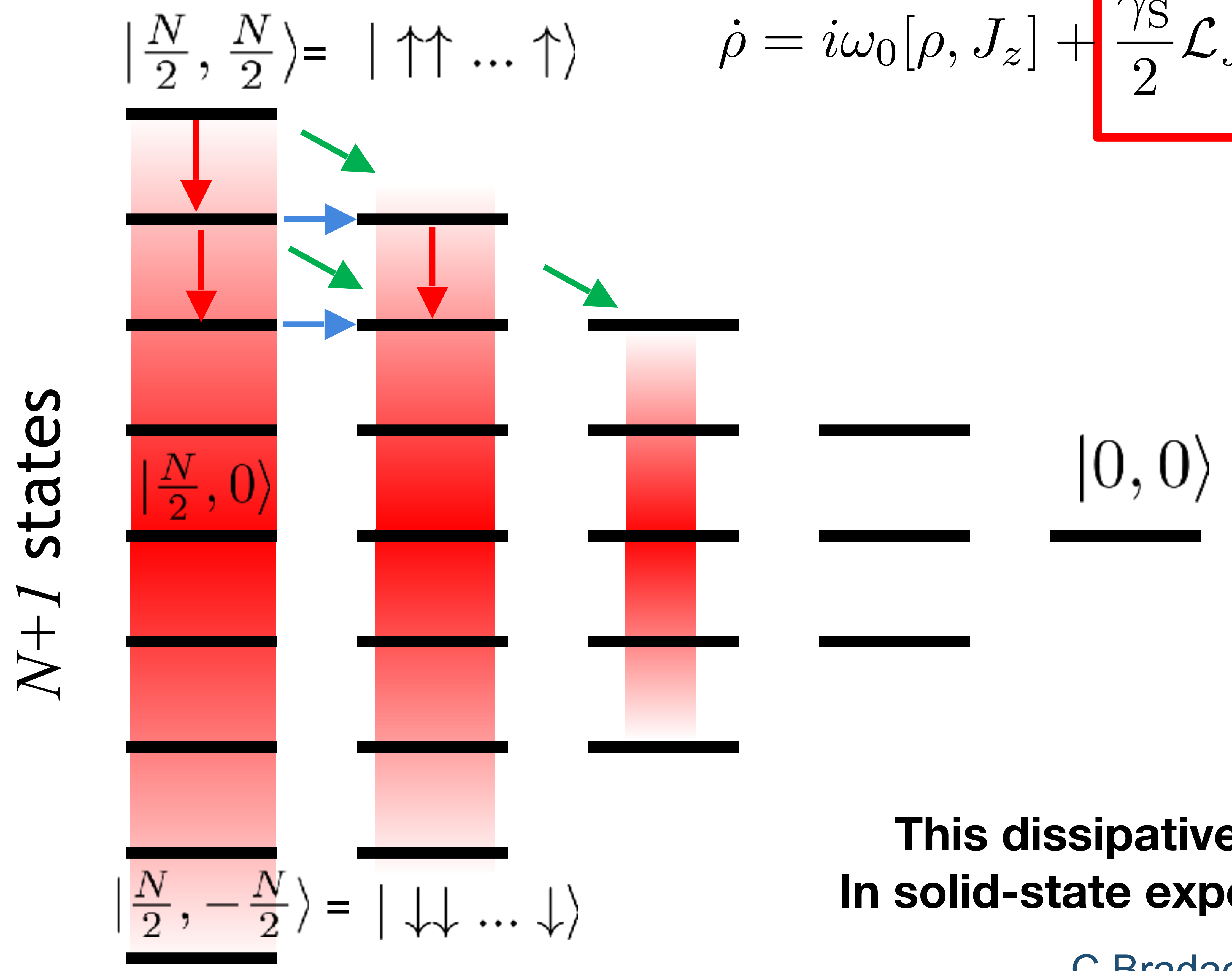
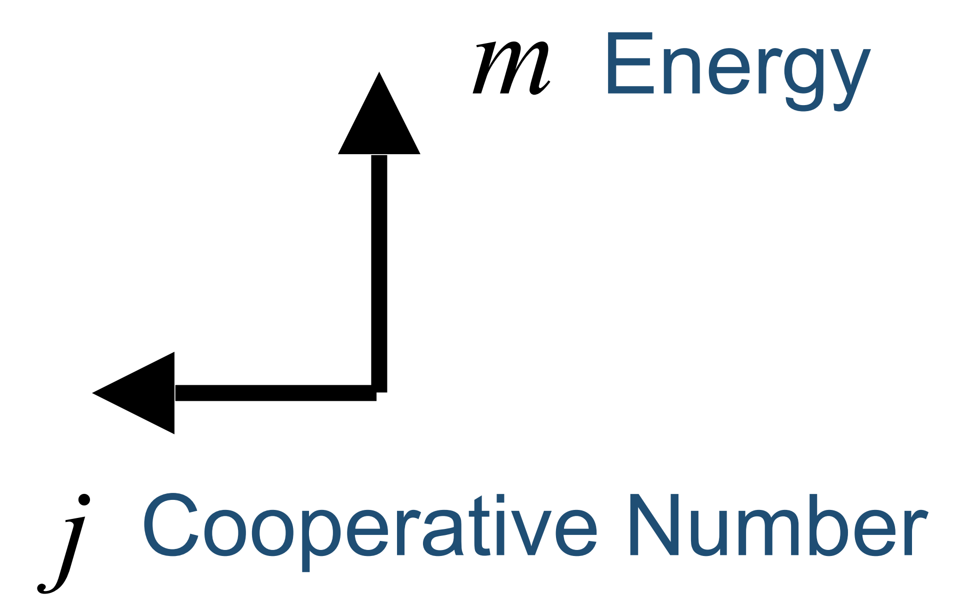




This dissipative superradiant regime has been observed
In solid-state experiments with NV-centers and color centers

C Bradac *et al.*, Nature Communications, 2017
A Angerer *et al.*, Nature Physics, 2018

N Shammah *et al.*, Superradiance with local phase-breaking mechanisms, Phys. Rev. A 2017, arXiv 1704.07066

Exploiting permutational symmetry

Generalization to all-to-all Hamiltonians: Validity and Limitations

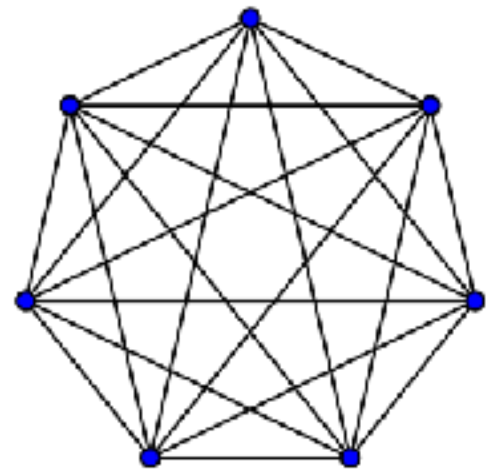
$$\frac{d}{dt}\rho \boxed{= -\frac{i}{\hbar}[H,\rho]} + \boxed{\boxed{\frac{\gamma_\Downarrow}{2}\mathcal{L}_{J_-}[\rho] + \frac{\gamma_\Phi}{2}\mathcal{L}_{J_z}[\rho] + \frac{\gamma_\Uparrow}{2}\mathcal{L}_{J_+}[\rho]} + \boxed{\sum_{n=1}^{N}\left(\frac{\gamma_\downarrow}{2}\mathcal{L}_{J_{-,n}}[\rho] + \frac{\gamma_\phi}{2}\mathcal{L}_{J_{z,n}}[\rho] + \frac{\gamma_\uparrow}{2}\mathcal{L}_{J_{+,n}}[\rho]\right)} + \boxed{\frac{w}{2}\mathcal{L}_{a^\dagger}[\rho] + \frac{\kappa}{2}\mathcal{L}_{a}[\rho]}}$$

TLS Pauli operators $[J_{x,n}, J_{y,n'}] = iJ_{z,n}\delta_{n,n'}$

Lindblad Superoperator $\mathcal{L}_A[\rho] = 2A\rho A^\dagger - A^\dagger A\rho - \rho A^\dagger A$

- Unitary Dynamics
- Non-unitary Dynamics
  - Collective Dissipation
  - Local Dissipation
  - Bosonic Dissipation

Range of applicability



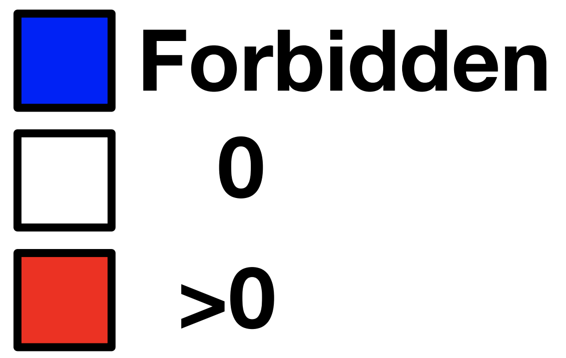
Hamiltonian
Collective spin operators only
Complete graph (fully connected).
Constant edges weight: no lattice distance.

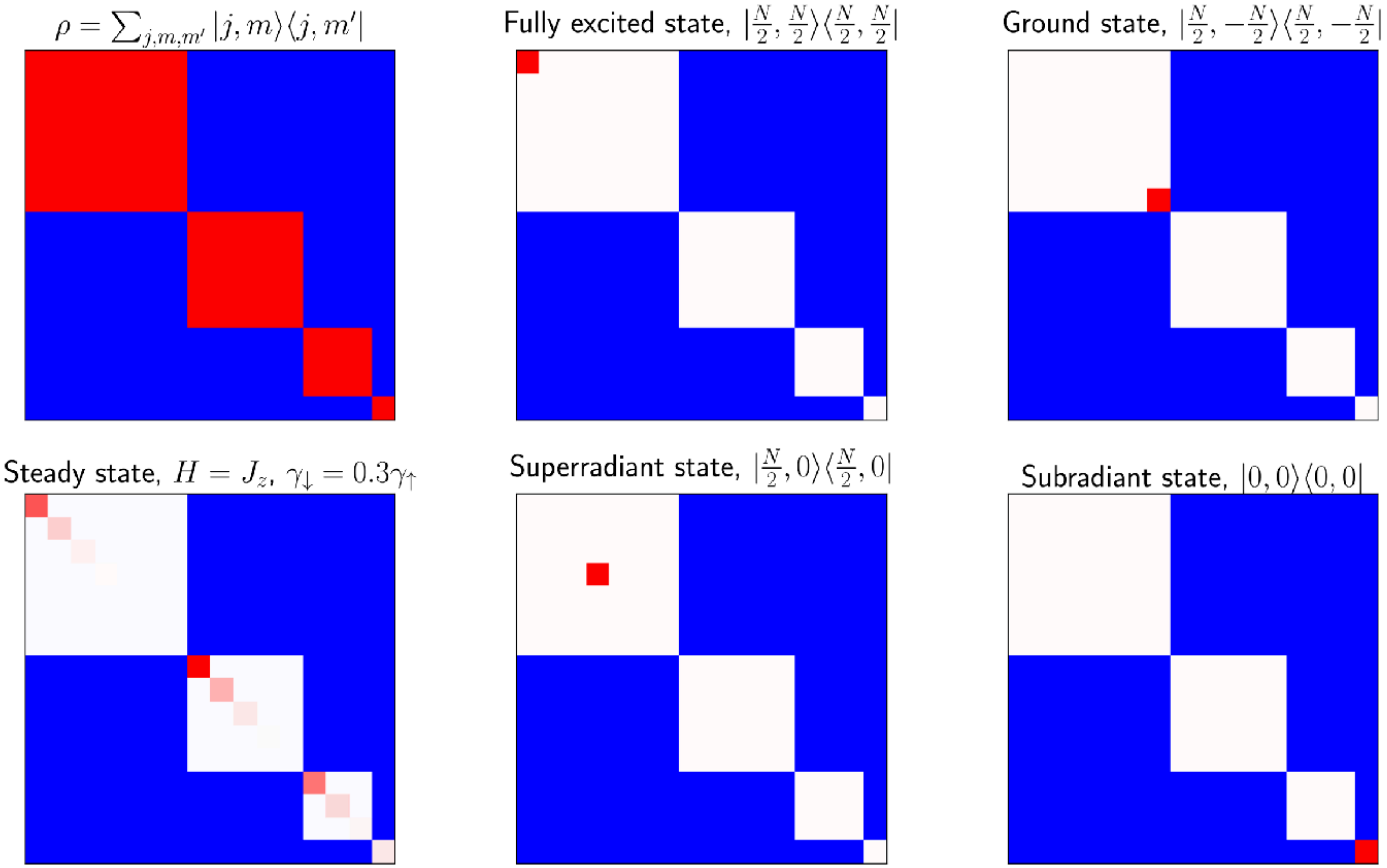
Dissipation
Homogeneous local couplings.

States
Limited to identical qubit states.

Dicke state basis as a visualization tool

Density matrices of collective quantum states





Can also study:
GHZ state
CSS states
Thermal states

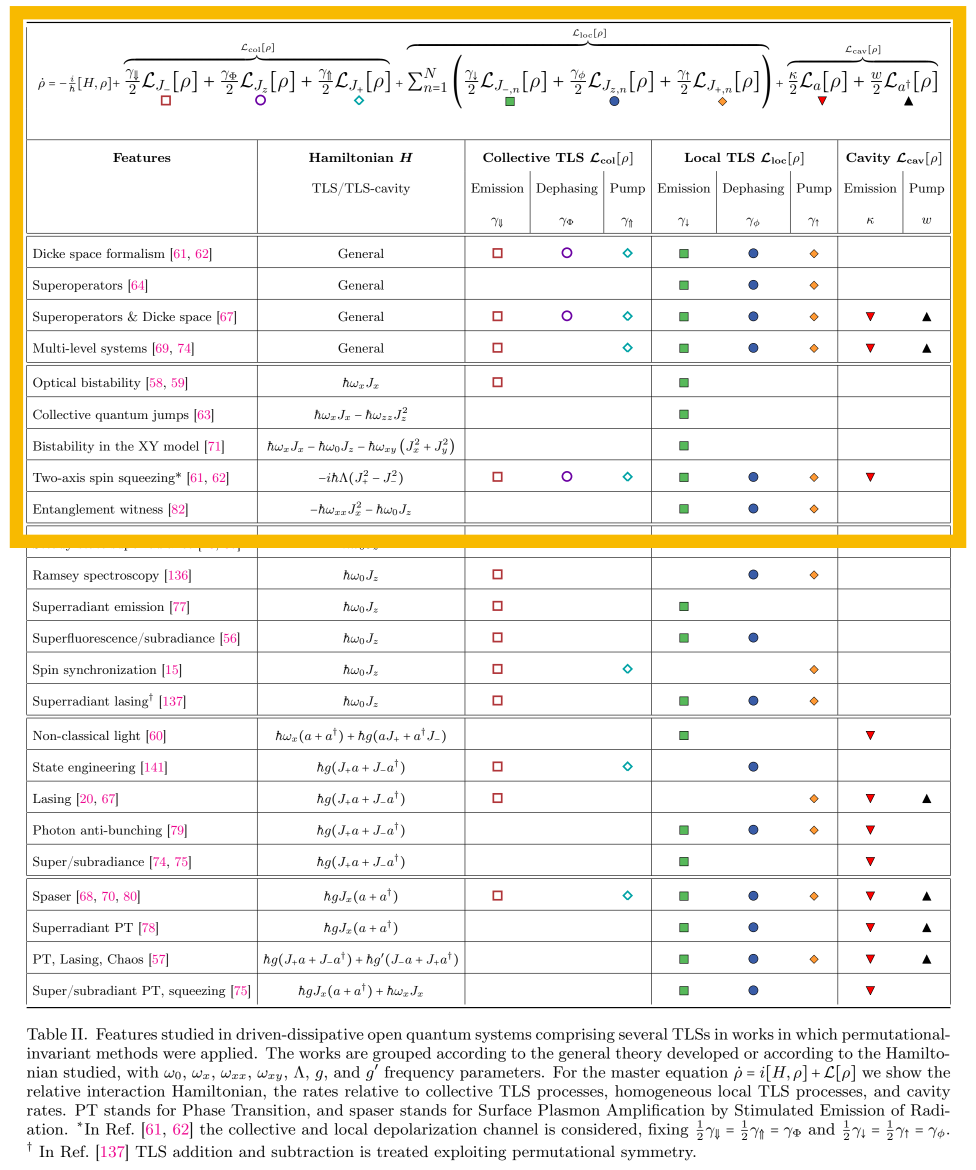
Literature overview

Permutational symmetry in Lindblad dynamics

N. Shammah, S. Ahmed, N. Lambert, S. De Liberato, and F. Nori, PRA **98**, 063815 (2018), arXiv:1805.05129

$$\dot{\rho} = -\frac{i}{\hbar}[H,\rho] + \overbrace{\frac{\gamma_\Downarrow}{2}\underset{\square}{\mathcal{L}_{J_-}}[\rho] + \frac{\gamma_\Phi}{2}\underset{\bigcirc}{\mathcal{L}_{J_z}}[\rho] + \frac{\gamma_\Uparrow}{2}\underset{\diamond}{\mathcal{L}_{J_+}}[\rho]}^{\mathcal{L}_{\mathrm{col}}[\rho]} + \overbrace{\sum_{n=1}^{N}\left(\frac{\gamma_\downarrow}{2}\underset{\blacksquare}{\mathcal{L}_{J_{-,n}}}[\rho] + \frac{\gamma_\phi}{2}\underset{\bullet}{\mathcal{L}_{J_{z,n}}}[\rho] + \frac{\gamma_\uparrow}{2}\underset{\blacklozenge}{\mathcal{L}_{J_{+,n}}}[\rho]\right)}^{\mathcal{L}_{\mathrm{loc}}[\rho]} + \overbrace{\frac{\kappa}{2}\underset{\blacktriangledown}{\mathcal{L}_{a}}[\rho] + \frac{w}{2}\underset{\blacktriangle}{\mathcal{L}_{a^\dagger}}[\rho]}^{\mathcal{L}_{\mathrm{cav}}[\rho]}$$

Features	Hamiltonian H	Collective TLS $\mathcal{L}_{\mathrm{col}}[\rho]$			Local TLS $\mathcal{L}_{\mathrm{loc}}[\rho]$			Cavity $\mathcal{L}_{\mathrm{cav}}[\rho]$	
	TLS/TLS-cavity	Emission	Dephasing	Pump	Emission	Dephasing	Pump	Emission	Pump
		$\gamma_\Downarrow$	γ_Φ	$\gamma_\Uparrow$	$\gamma_\downarrow$	γ_ϕ	$\gamma_\uparrow$	κ	w
Dicke space formalism [61, 62]	General	□	○	◇	■	●	◆		
Superoperators [64]	General				■	●	◆		
Superoperators & Dicke space [67]	General	□	○	◇	■	●	◆	▼	▲
Multi-level systems [69, 74]	General	□		◇	■	●	◆	▼	▲
Optical bistability [58, 59]	$\hbar\omega_x J_x$	□			■				
Collective quantum jumps [63]	$\hbar\omega_x J_x - \hbar\omega_{zz} J_z^2$				■				
Bistability in the XY model [71]	$\hbar\omega_x J_x - \hbar\omega_0 J_z - \hbar\omega_{xy}\left(J_x^2 + J_y^2\right)$				■				
Two-axis spin squeezing* [61, 62]	$-i\hbar\Lambda(J_+^2 - J_-^2)$	□	○	◇	■	●	◆	▼	
Entanglement witness [82]	$-\hbar\omega_{xx} J_x^2 - \hbar\omega_0 J_z$				■	●	◆		

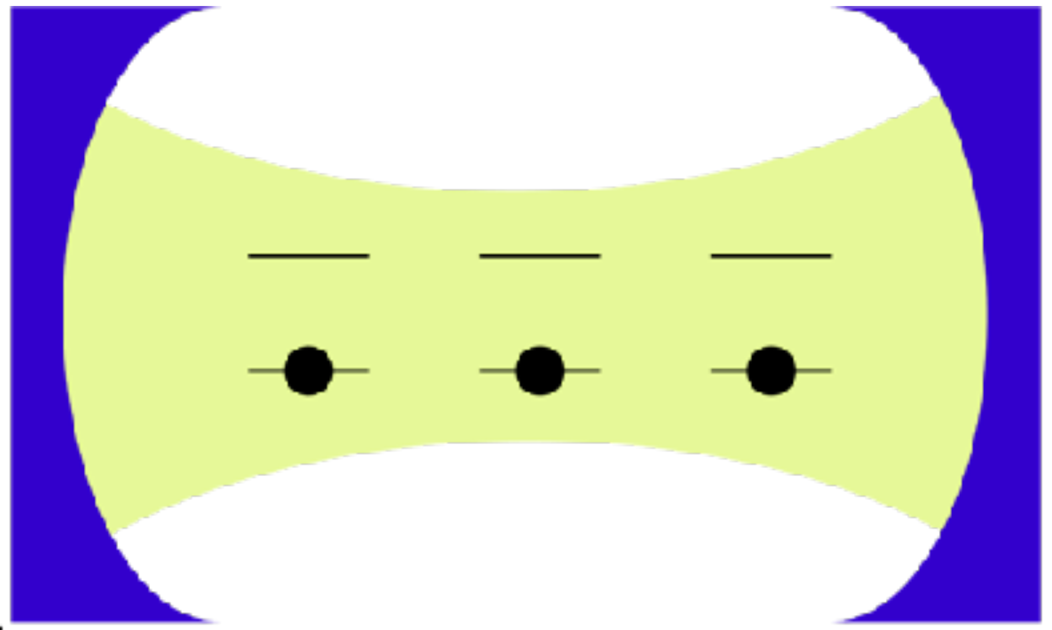


Features	Hamiltonian H	$\gamma_\Downarrow$	γ_Φ	$\gamma_\Uparrow$	$\gamma_\downarrow$	γ_ϕ	$\gamma_\uparrow$	κ	w
Ramsey spectroscopy [136]	$\hbar\omega_0 J_z$	□				●	◆		
Superradiant emission [77]	$\hbar\omega_0 J_z$	□			■				
Superfluorescence/subradiance [56]	$\hbar\omega_0 J_z$	□			■	●			
Spin synchronization [15]	$\hbar\omega_0 J_z$	□		◇			◆		
Superradiant lasing† [137]	$\hbar\omega_0 J_z$	□			■	●	◆		
Non-classical light [60]	$\hbar\omega_x(a + a^\dagger) + \hbar g(aJ_+ + a^\dagger J_-)$				■			▼	
State engineering [141]	$\hbar g(J_+a + J_-a^\dagger)$	□		◇		●			
Lasing [20, 67]	$\hbar g(J_+a + J_-a^\dagger)$	□					◆	▼	▲
Photon anti-bunching [79]	$\hbar g(J_+a + J_-a^\dagger)$				■	●	◆	▼	
Super/subradiance [74, 75]	$\hbar g(J_+a + J_-a^\dagger)$				■			▼	
Spaser [68, 70, 80]	$\hbar g J_x(a + a^\dagger)$	□		◇	■	●	◆	▼	▲
Superradiant PT [78]	$\hbar g J_x(a + a^\dagger)$				■	●		▼	▲
PT, Lasing, Chaos [57]	$\hbar g(J_+a + J_-a^\dagger) + \hbar g'(J_-a + J_+a^\dagger)$				■	●	◆	▼	▲
Super/subradiant PT, squeezing [75]	$\hbar g J_x(a + a^\dagger) + \hbar\omega_x J_x$				■	●		▼	

Table II. Features studied in driven-dissipative open quantum systems comprising several TLSs in works in which permutational-invariant methods were applied. The works are grouped according to the general theory developed or according to the Hamiltonian studied, with ω_0, ω_x, ω_{xx}, ω_{xy}, Λ, g, and g' frequency parameters. For the master equation $\dot{\rho} = i[H,\rho] + \mathcal{L}[\rho]$ we show the relative interaction Hamiltonian, the rates relative to collective TLS processes, homogeneous local TLS processes, and cavity rates. PT stands for Phase Transition, and spaser stands for Surface Plasmon Amplification by Stimulated Emission of Radiation. *In Ref. [61, 62] the collective and local depolarization channel is considered, fixing $\frac{1}{2}\gamma_\Downarrow = \frac{1}{2}\gamma_\Uparrow = \gamma_\Phi$ and $\frac{1}{2}\gamma_\downarrow = \frac{1}{2}\gamma_\uparrow = \gamma_\phi$. † In Ref. [137] TLS addition and subtraction is treated exploiting permutational symmetry.

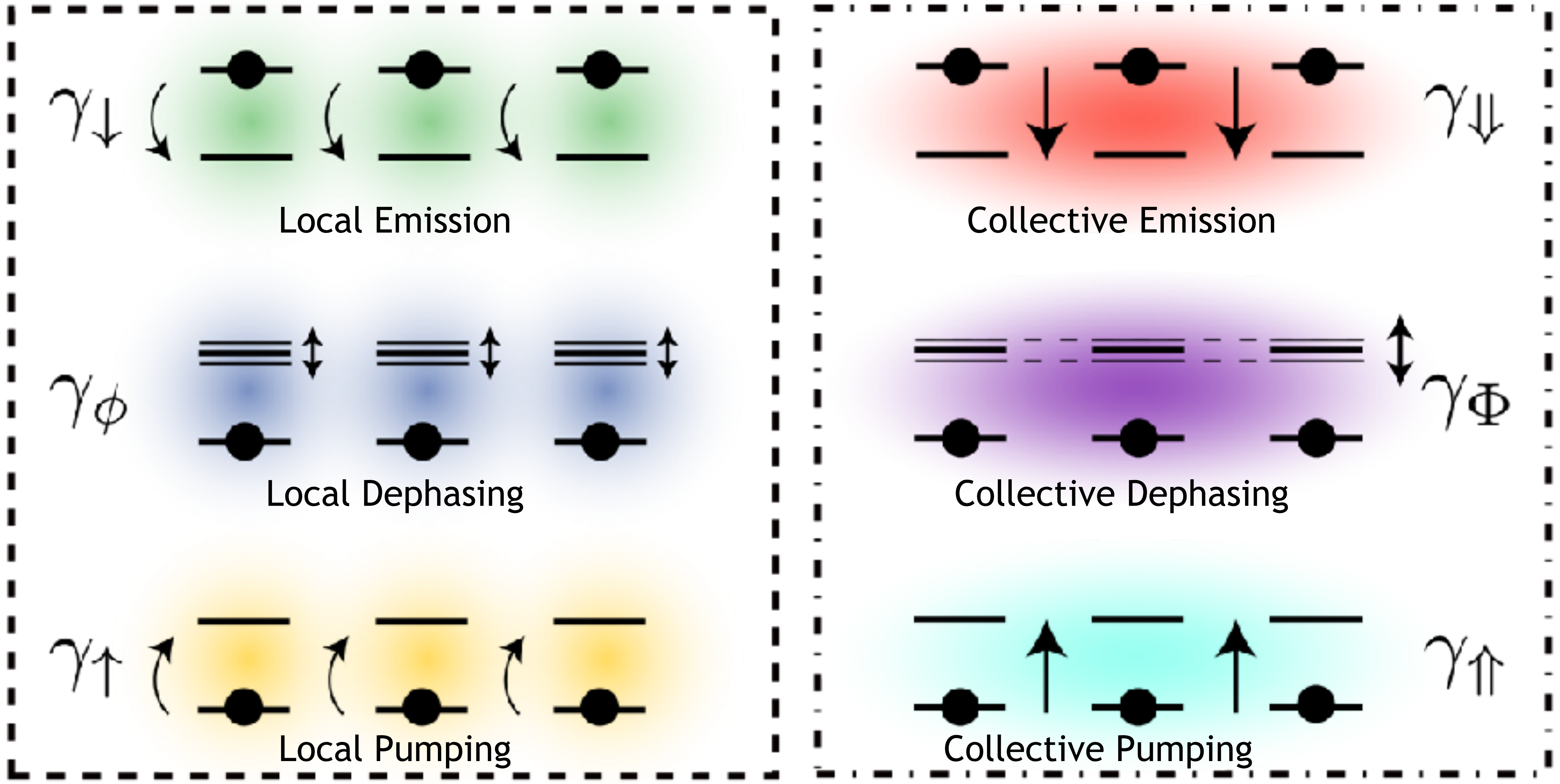
Dissipative open quantum systems simulation

Coherent Dynamics $$i\hbar\frac{d}{dt}\rho = [H,\rho] + \sum_{i,j}^{N}\gamma_{ij}\left(L_i\rho L_j^\dagger - \frac{1}{2}L_i^\dagger L_j\rho - \frac{1}{2}\rho L_i^\dagger L_j\right)$$



$\gamma_{ij} \equiv \gamma\delta_{ij}$ $\qquad$ $\gamma_{ij} \equiv \gamma \;\forall i,j$

Dissipation Processes $\mathcal{L}_{\mathrm{loc}}[\rho]$ Local $\qquad$ $\mathcal{L}_{\mathrm{col}}[\rho]$ Collective



Open quantum systems with local and collective incoherent processes: Efficient numerical simulation using permutational invariance

N. Shammah, S. Ahmed, N. Lambert, S. De Liberato, and F. Nori
Phys Rev A **98**, 063815 (2018) arXiv:1805.05129

Open Quantum Systems

Non-Equilibrium Phase Transitions

Quantum Optics, Spin Squeezing

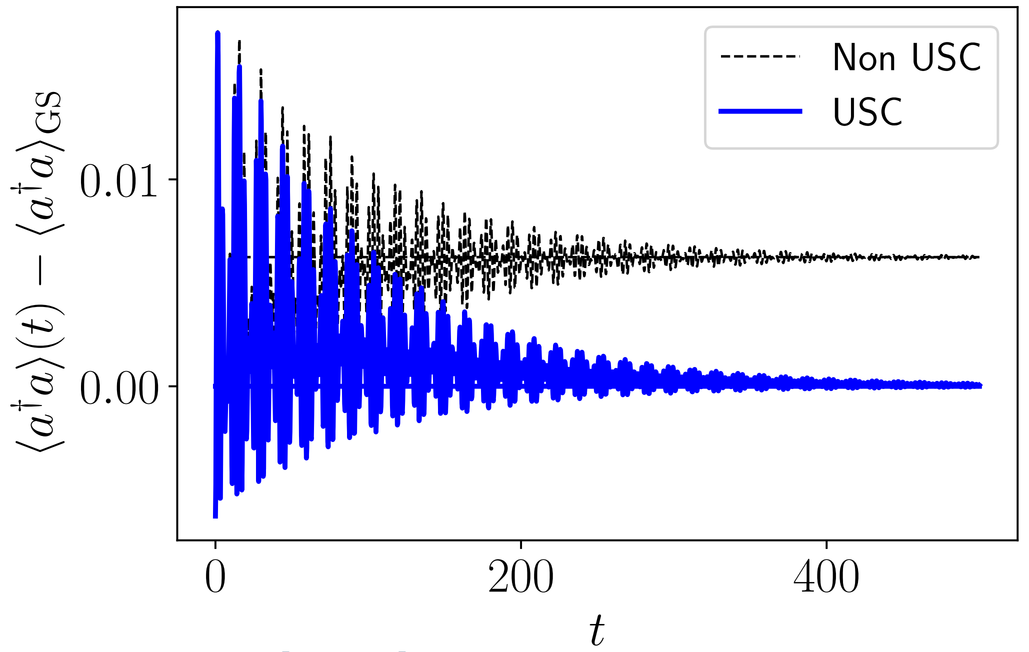
Ultrastrong Coupling of Light and Matter

PIQS: Driven-dissipative two-level system ensembles

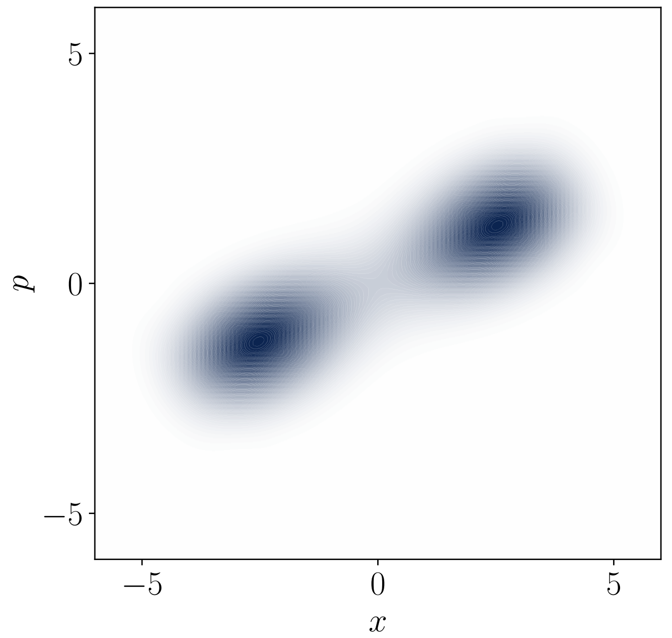
The Permutational Invariant Quantum Solver

$$i\hbar\frac{d}{dt}\rho = [H,\rho] + \gamma_\alpha \sum_i^N \mathscr{L}_{L_i}[\rho]$$

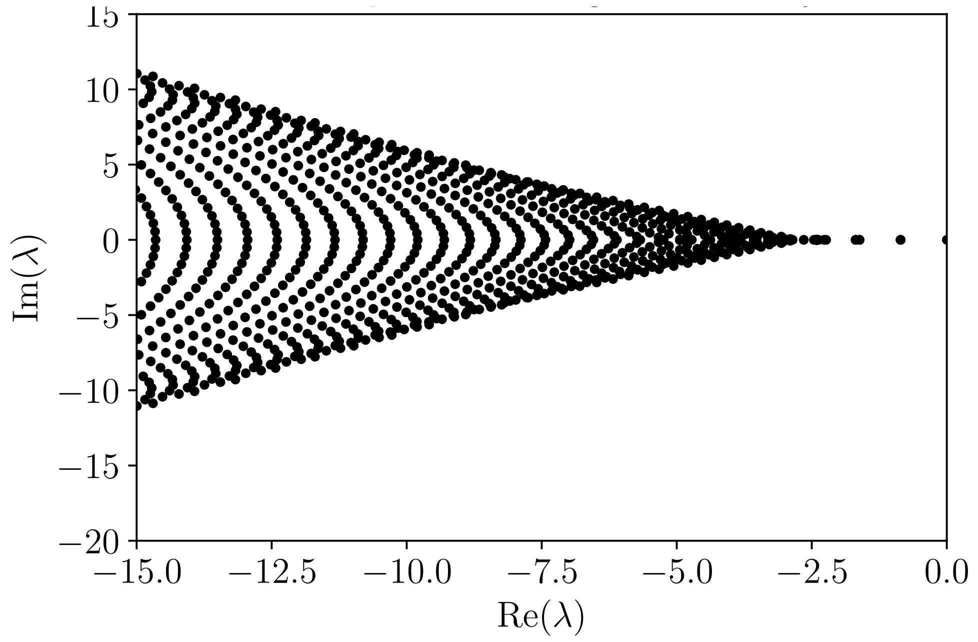
Ultrastrong Coupling (USC)



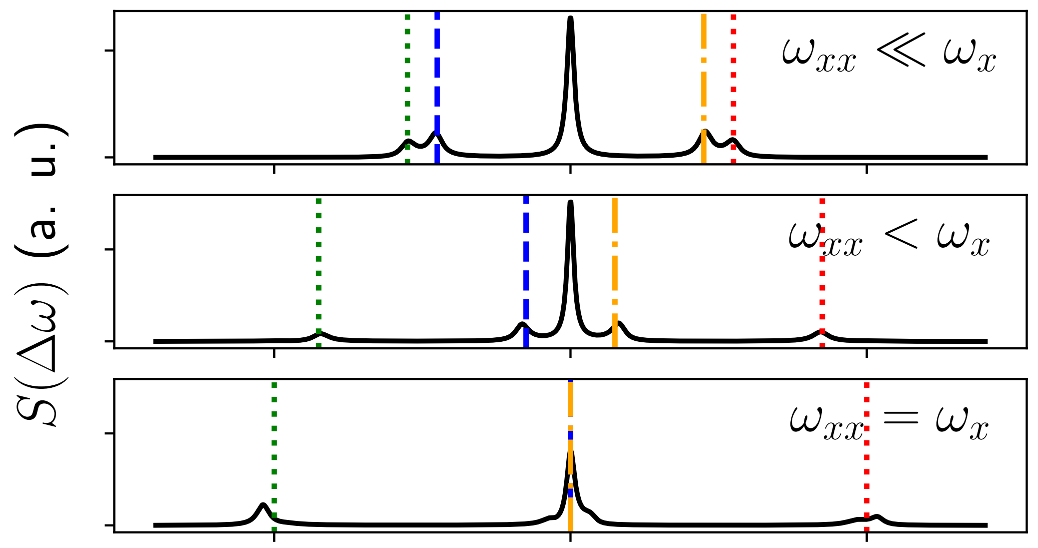
Superradiant PT



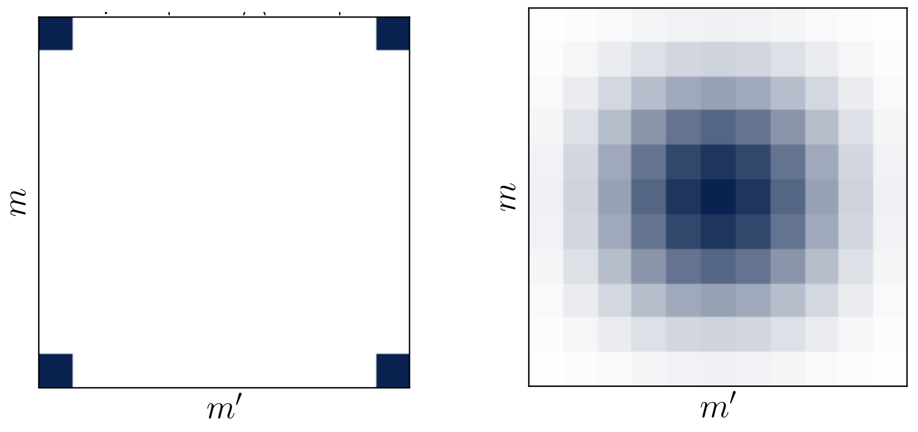
Boundary Time Crystals



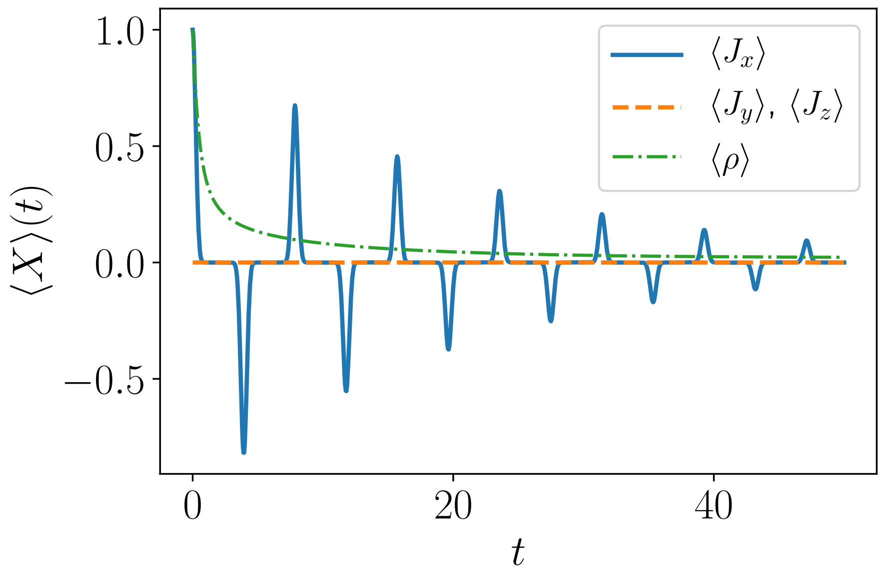
Resonance Fluorescence



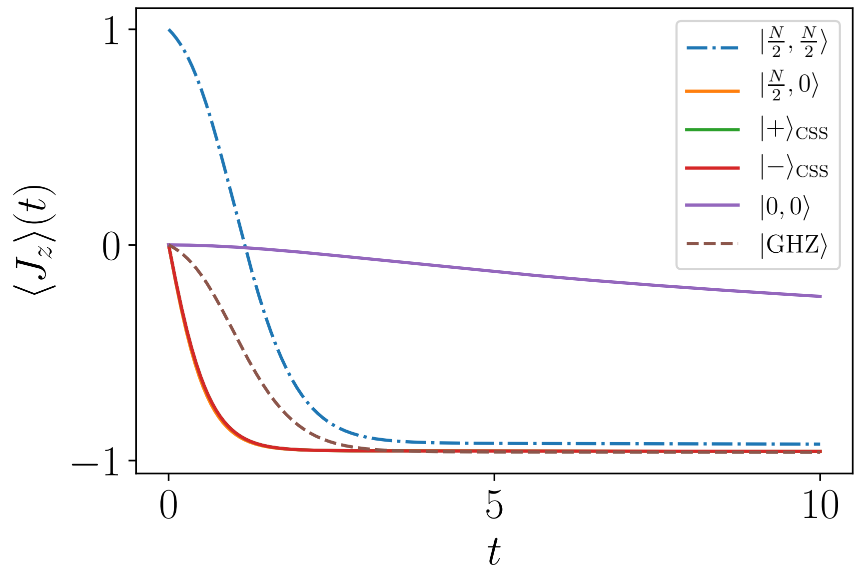
GHZ and coherent spin states



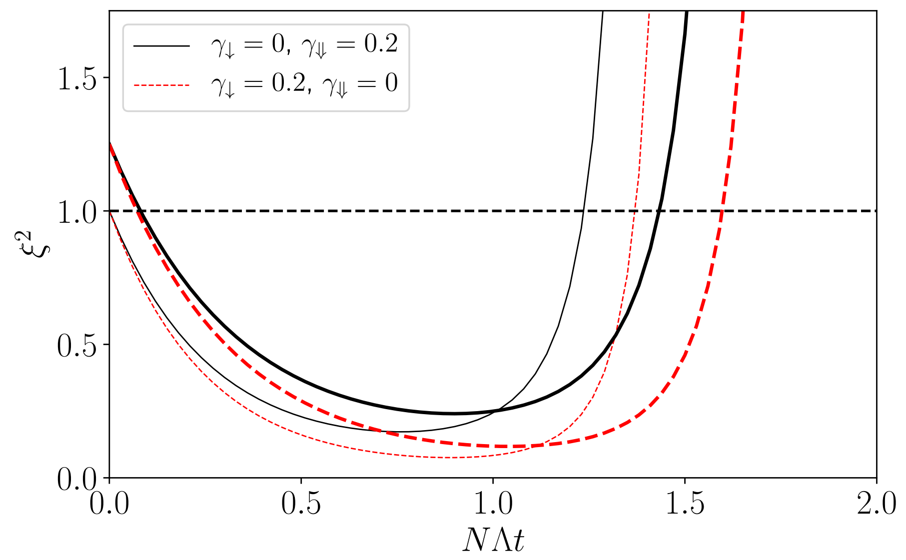
Noisy Heisenberg Model



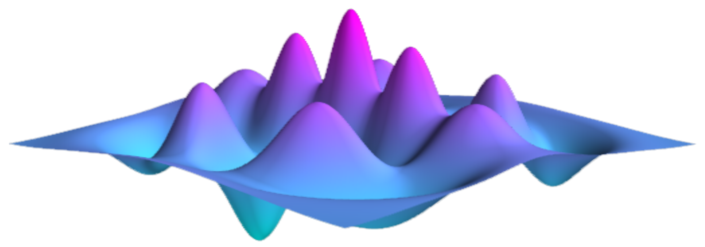
Dicke Superradiance



Spin Squeezing



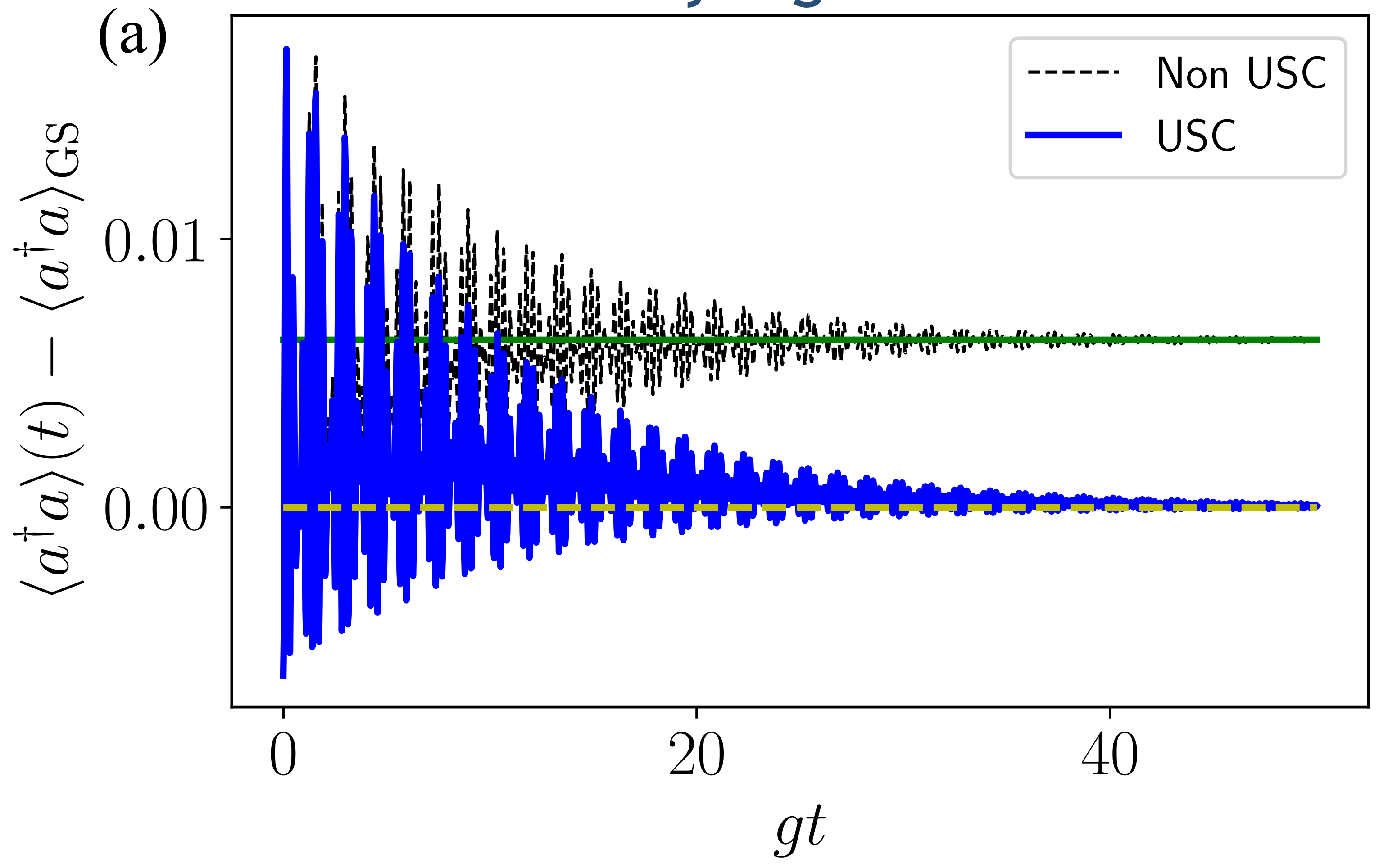
N. Shammah, S. Ahmed, N. Lambert, S. De Liberato, and F. Nori, PRA 98, 063815 (2018) arXiv:1805.05129

 +  = **import** qutip.piqs

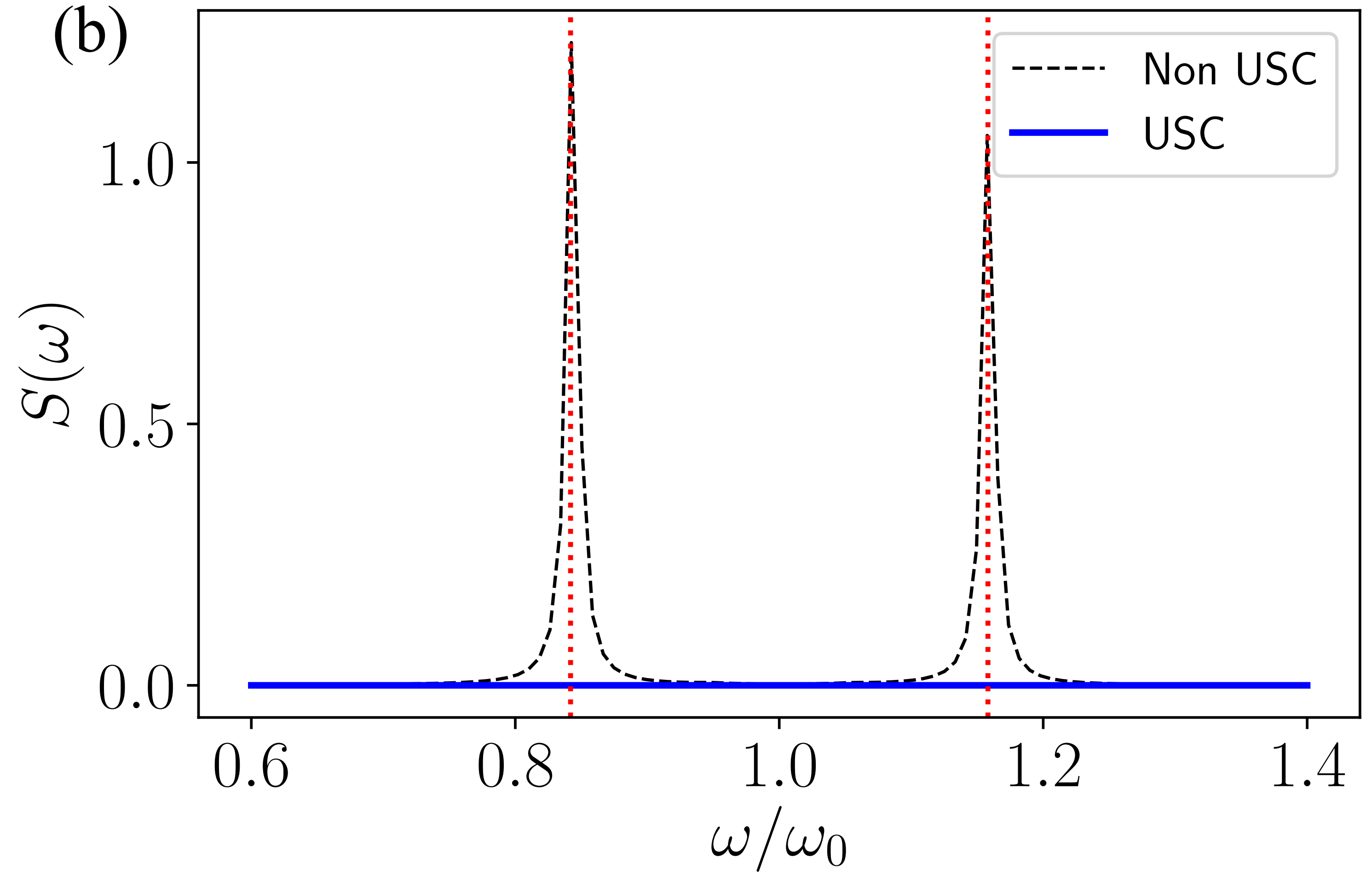
PIQS Example: Ultrastrong Coupling (USC) Regime

Relaxation to the true steady state by deriving consistent operators: no light emission

Extra-cavity Light Field vs. Time



Extra-cavity Light Spectrum vs. Frequency



Hamiltonian

$$\boxed{H = \hbar\omega_0 J_z + \hbar\omega_{\mathrm{cav}} a^\dagger a + \hbar g J_x \left(a + a^\dagger\right)}$$

Dissipation $\mathcal{L}[\rho]$

$$\dot{\rho} = -\frac{i}{\hbar}[H,\rho] + \sum_{r,s>r}\left(\frac{\kappa}{2}|X^{r,s}|^2 + \frac{\gamma_\downarrow}{2}\sum_{n=1}^{N}|J_{x,n}^{r,s}|^2\right)\mathcal{L}[|r\rangle\langle s|](\rho)$$

$$J_{\alpha,n} = \sum_{r,s} J_{\alpha,n}^{r,s}\,|r\rangle\langle s| \qquad \sum_{n=1}^{N}|J_{-,n}^{r,s}|^2_{s>r} = \frac{1}{2}\langle s|\sum_{n=1}^{N}\mathcal{L}_{J_{-,n}}(|r\rangle\langle r|)\,|s\rangle_{s>r}$$

Derive the “dressed” light-matter jump operators for the Lindbladian

Case N=1: F. Beaudoin, J.M. Gambetta, A. Blais, PRA (2011)

Case N>>1: N. Shammah, S. Ahmed, N. Lambert, S. De Liberato, and F. Nori, PRA 98, 2018, arXiv:1805.05129

Steady-state light emission and transport

Dissipation and ultrastrong coupling regime with transport: how to engineer light emission

Electron transport + Light-matter coupling:

Multielectron Ground State Electroluminescence

M. Cirio*, N. Shammah*, N. Lambert, S. De Liberato, and F. Nori

Phys. Rev. Lett. **122**, 190403 (2019) arXiv: 1811.08682

Features

- **Cooperative effect**
- **Robust with *N***

Possible Experiments

- Quantum wells (Faist)
- Quantum dots (Vandersypen, Petta)
- Superconducting circuits
- Hybrid devices (Wallraff)

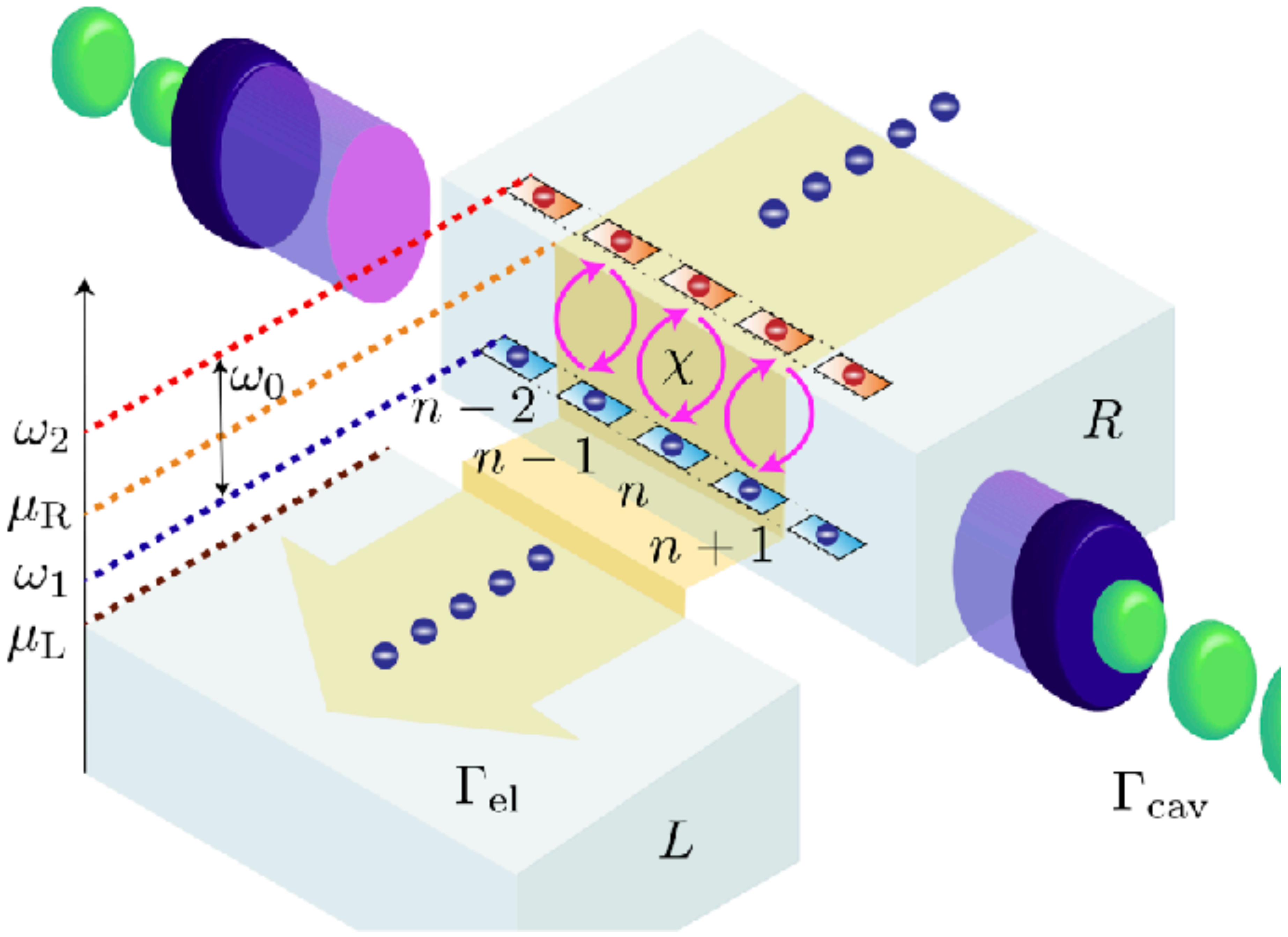


Figure: Solid-state device.

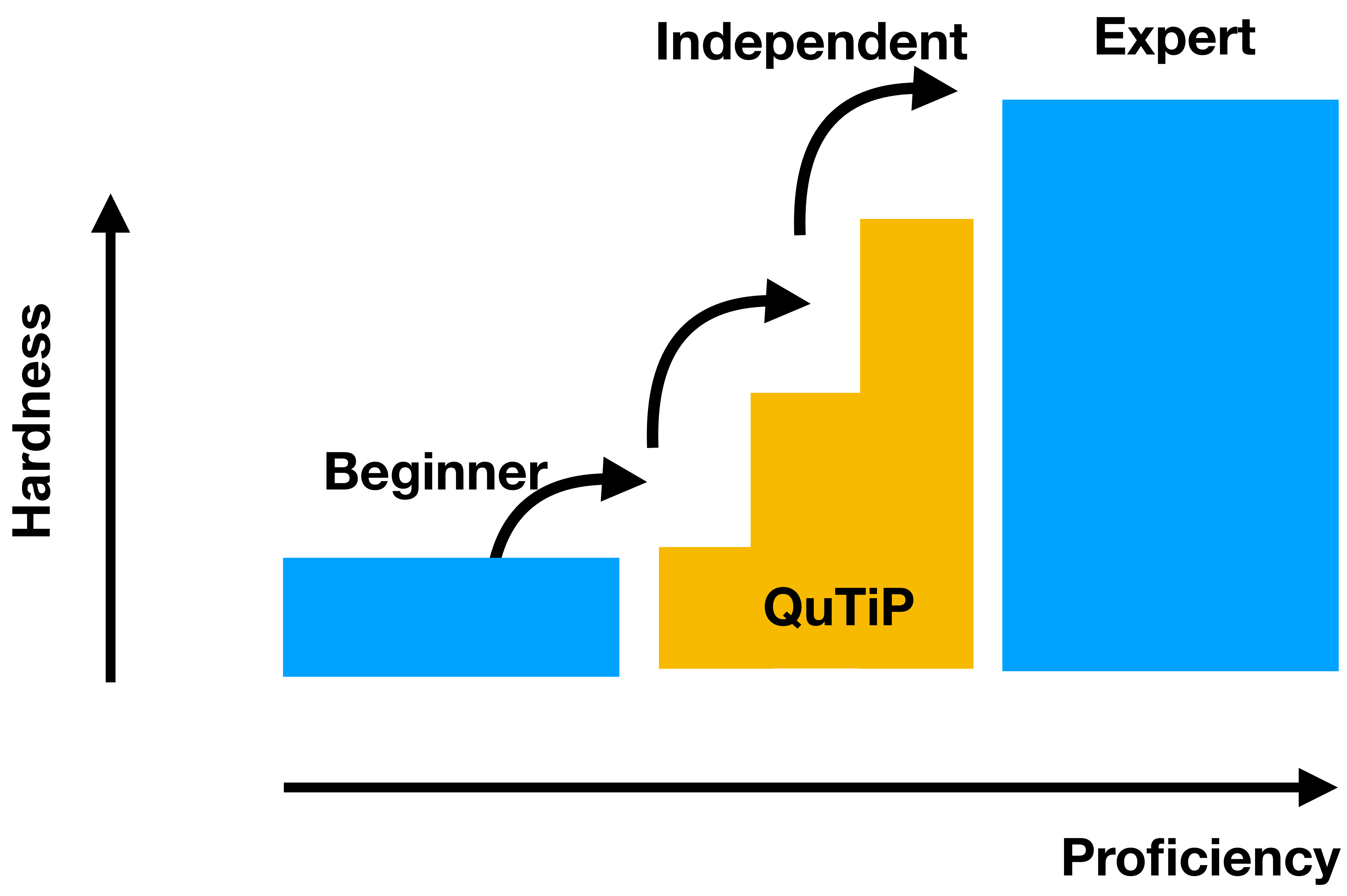
Dr. Mauro Cirio
GSCAEP (China)

Prof. Simone De Liberato
U. of Southampton (UK)

QuTiP: A tool to explore quantum mechanics

The **Qu**antum **T**oolbox **i**n **P**ython

Quantum mechanics



Tutorials at qutip.org/tutorials

Python Introduction

- Quick introduction to Python
- Overview of NumPy Arrays
- Brief Introduction to Matplotlib

For a more in depth discussion see: Lectures on scientific computing with Python.

Basics

- Introduction to QuTiP
- Exponential series
- Groundstates: Jaynes-Cumming model in the ultrastrong coupling regime
- Superoperators, Pauli Basis and Channel Contraction

Visualization

- Visualization demos
- Energy-level diagrams
- Bloch-sphere animation
- Bloch Sphere with Colorbar
- Wigner functions
- Pseudo-probability functions
- Process tomography
- Qubism visualizations

Quantum information processing

- Quantum gates and circuits
- Toffoli gate to CNOT
- Spin Chain Qubit model

Time evolution

- Master equation solver: Qubit dynamics
- Master equation solver: Vacuum Rabi oscillations
- Master equation solver: Spin chain
- Monte-Carlo solver: Trilinear oscillators
- Monte-Carlo solver: Birth and Death of Photons in a Cavity
- Bloch-Redfield master equation solver
- Time-dependent Bloch-Redfield Quantum Dot
- Floquet formalism
- Quasi-steadystate of time-dependent (periodic) systems
- Time-dependent master equation: Landau-Zener transitions
- Time-dependent master equation: Landau-Zener-Stuckelberg inteferometry
- Stochastic master equation: Heterodyne detection
- Stochastic master equation: Inefficient detection
- Stochastic master equation: Jaynes-Cumming model with photocurrent detection
- Stochastic master equation: Feedback control
- Steady state solvers: Optomechanical system
- Homodyned Jaynes-Cummings Emission

Optimal control

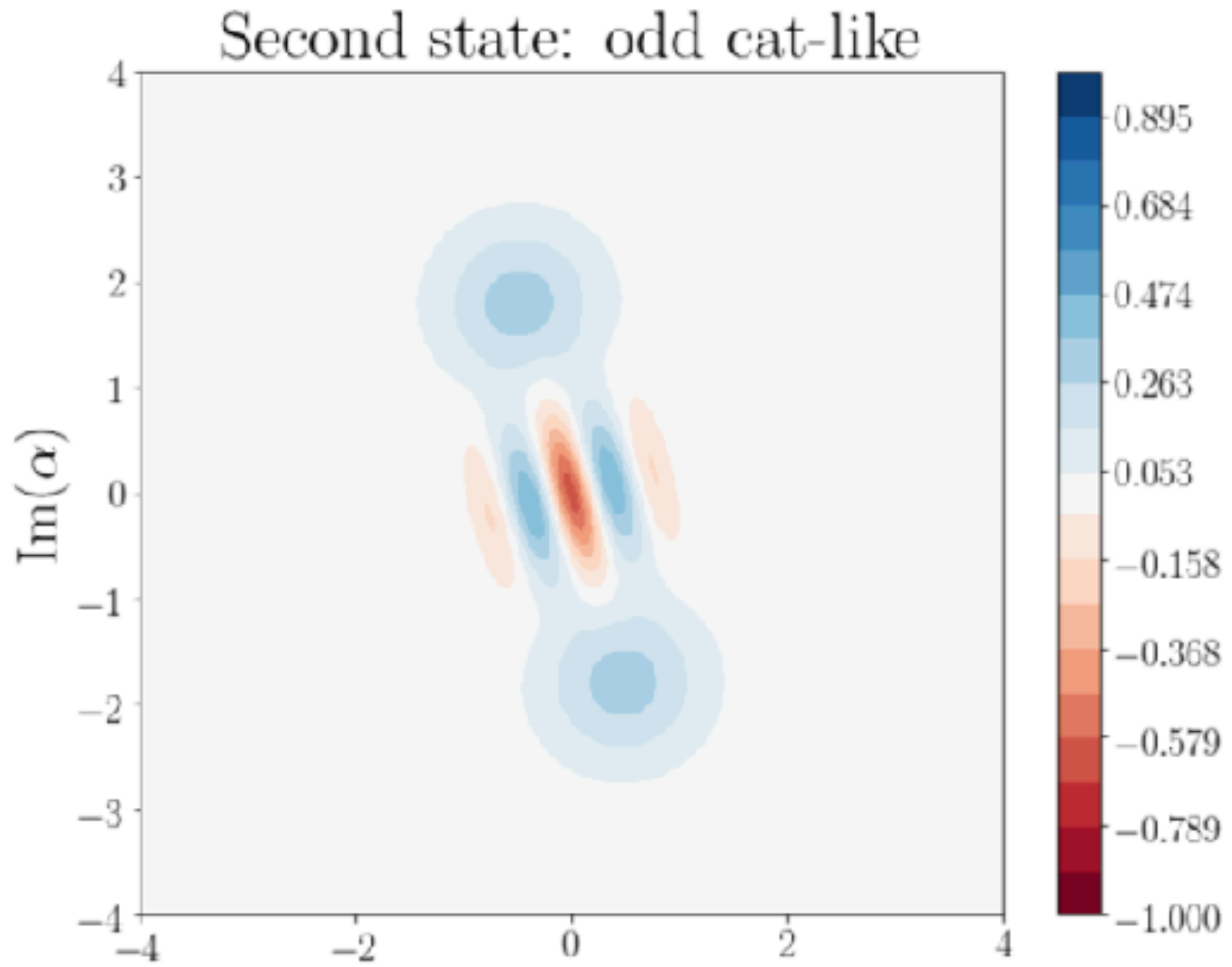
- Overview
- Hadamard
- QFT
- Lindbladian
- Symplectic
- QFT (CRAB)
- state to state (CRAB)
- CNOT
- iSWAP
- Single-qubit rotation
- Toffoli gate

- **Over 60** Jupyter notebook tutorials
- **Over 20** quantum mechanics lectures

QuTiP: Contributing to QuTiP – Tutorials

The **Qu**antum **T**oolbox **i**n **P**ython

Dr. Fabrizio Minganti (RIKEN)



qutip.org/tutorials

Recent addition:
Cat vs coherent states in a Kerr resonator and the role of measurement

Homodyne

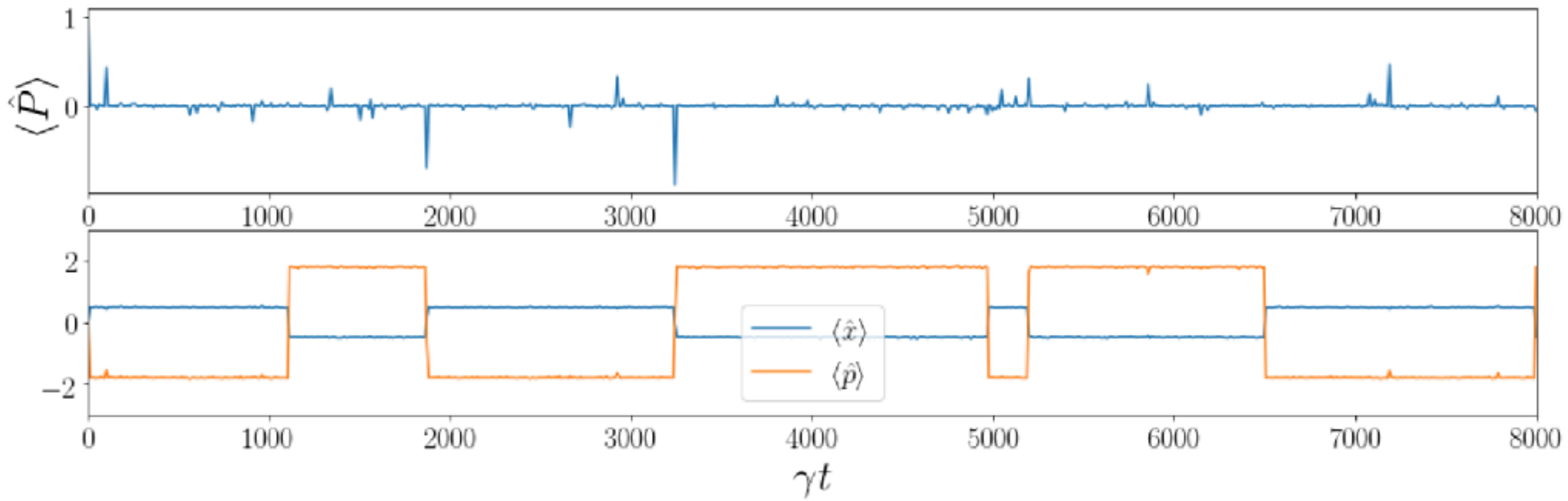
Another possible way to monitor a quantum-optical system is through homodyne detection, a widely-used experimental technique which allows to access the field quadratures [5-6]. To implement this kind of measurement, the cavity output field is mixed to the coherent field of a reference laser through a beam splitter (here assumed of perfect transmittance). Then, the mixed fields are probed via (perfect) photodectors, whose measures are described by new jump operators. We stress that both the coherent and the cavity fields are measured simultaneously.

In the ideal limit $\beta_{1,2} \to \infty$, the system evolves diffusively according to a homodyne stochastic Schr\"odinger equation. Using the ssesolve function with option "method='homodyne'", one can simulate the trajectory.

```
]: tlist=np.linspace(0,8000,800)


sol_hom=ssesolve(H, fock(20,0), tlist, c_ops, [a.dag()*a, (a+a.dag())/2, -1.j*(a-a.dag())/2, parit
```

```
Total run time: 636.04s
```



How to interact with the QuTiP project

Multiple interaction channels

QuTiP
Quantum Toolbox in Python

http://qutip.org/tutorials.html

Website:
Tutorials

http://qutip.org/docs/latest/

Documentation User Guide
Documentation API: docstring comments
Documentation API: source code

https://groups.google.com/forum/#!forum/qutip

Help Group:
Ask questions (installation, physics)
Find *already answered* questions

https://github.com/qutip/qutip

GitHub:
Found a bug?
- Search answer in *already closed* issue
- Search *an open (unsolved)* issue on same topic
- Unknown? Open an Issue

Have a solution? Open a Pull Request (proposal for code modification of the official project source code)

https://gitter.im/qutip

Gitter Chat:
Join the conversation if you have other question

Thank you

@NathanShammah

GitHub: nathanshammah

Extra Slides

QuTiP: The Quantum *Physics* Simulator

The **Qu**antum **T**oolbox **i**n **P**ython: A toolbox to study the **open** quantum dynamics of realistic systems.

QuTiP

Interactive Lectures @ ICTP, Leonardo Building

Tue 25th June - 11:45am, Seminar Room – Driven-dissipative models in quantum physics
Wed 26th June - 11am, Seminar Room – Quantum Open Source & Introduction to QuTiP
Thur 27th June - 9am, Computer Room – Hands-on session on QuTiP's main features
Mon 1st July - 9am, Computer Room – QuTiP stochastic solvers
Tue 2nd July - 9am, Computer Room – PIQS and How to Build your Own Software
(Wed 3rd July - 9am, Computer Room – Extra meeting: SISSA/ICTP projects)

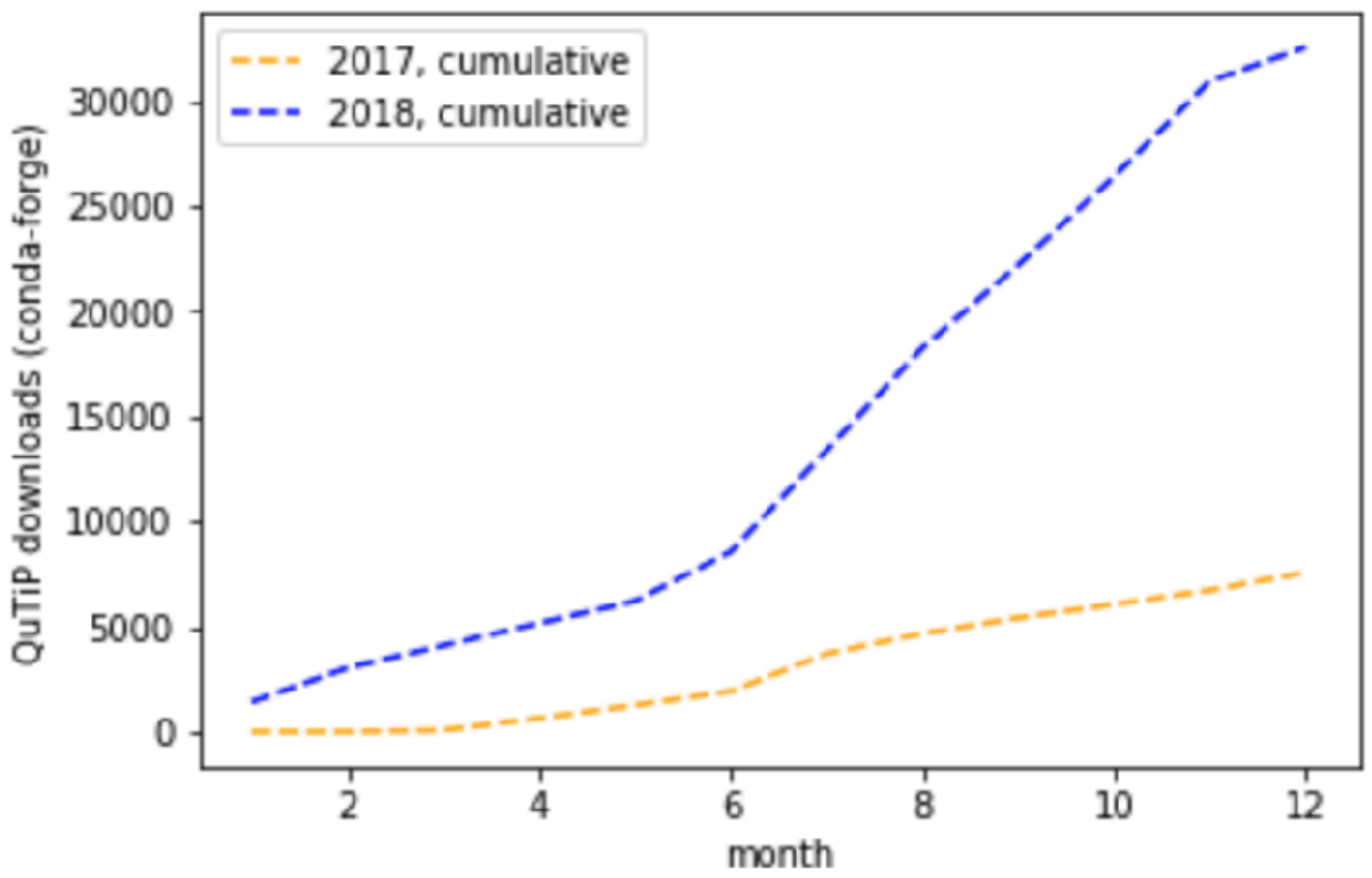
Take a snapshot

https://github.com/nathanshammah/interactive-notebooks

QuTiP: Some Statistics

The **Qu**antum **T**oolbox **i**n **P**ython

conda install qutip



July 2018 downloads 43k

Feb 2019 downloads 79k total

Apr 2019 downloads 112k total

Jun 2019 downloads 130k total

100k downloads/year

GitHub "**Used by**" allows to draw **connection graphs** on the quantum open source ecosystem:

- IBM's qiskit is used by 122 other libraries.

- **qutip by 60**.

- Rigetti's pyquil 53 (grove 8).

- ProjectQ 29.

- Google's OpenFermion 22.

- Dwave's ocean-sdk 14.

QuTiP helps engineer an ecosystem of libraries.

Solving a many-qubit dynamics: Quantum Trajectories

Photon counting statistics: Monte Carlo solver with `mcsolve`

$$\frac{d}{dt}\rho = -\frac{i}{\hbar}[H,\rho] + \gamma\left(\overbrace{L\rho L^\dagger}^{\text{“quantum jump”}} - \underbrace{\frac{1}{2}L^\dagger L\rho - \frac{1}{2}\rho L^\dagger L}_{\text{continuous decay}}\right)$$

A. **Effective Hamiltonian**:
shrinking the state

$$H_{\text{eff}} = H_{\text{sys}} - \frac{i\hbar}{2}L^\dagger L$$

B. **Quantum jump:**

$$|\psi(t+\delta t)\rangle = \frac{L|\psi(t)\rangle}{\left\langle \psi(t) \middle| L^+ L \middle| \psi(t) \right\rangle^{1/2}}$$

1. Generale random number, *r*.
2. Is *r* > norm(psi)?
   I. Yes: keep integrating (A)
   II. No: apply jump and renormalize

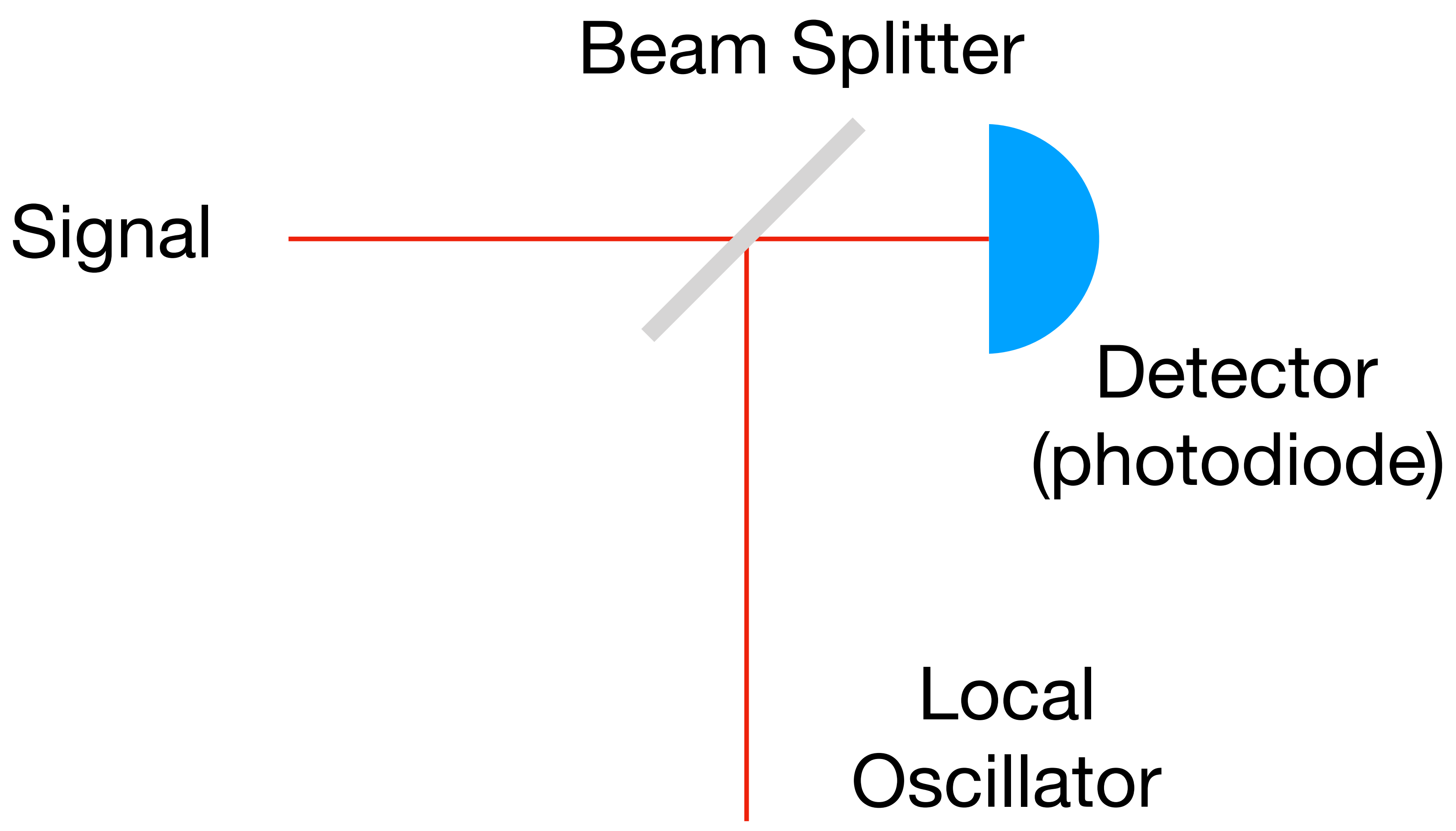
Stochastic solvers

Stochastic master equation: Continuous weak measurements with: \`smesolve\`

- continuous weak measurements

Heterodyne Detection

$$\omega_{LO} \neq \omega_{signal}$$



Homodyne Detection

$$\omega_{LO} = \omega_{signal}$$

```
result = smesolve(H, rho0, times, [], c_ops, e_ops)
```

Stochastic solvers

Stochastic master equation: Continuous weak measurements with: `smesolve`

- continuous weak measurements

Heterodyne Detection

$$d\rho(t) = -i[H,\rho(t)]dt + \gamma\mathcal{D}[a]\rho(t)dt + \frac{1}{\sqrt{2}}\overbrace{dW_1(t)\sqrt{\gamma}\mathcal{H}[a]\rho(t)}^{J_x(t)} + \frac{1}{\sqrt{2}}\overbrace{dW_2(t)\sqrt{\gamma}\mathcal{H}[-ia]\rho(t)}^{J_y(t)}$$

$$\mathcal{D}[A]\rho = A\rho A^\dagger - \frac{1}{2}\left(\rho A^\dagger A + A^\dagger A\rho\right)$$ **Lindblad term**

$$\mathcal{H}[A]\rho = A\rho + \rho A^\dagger - \mathrm{Tr}\left[A\rho + \rho A^\dagger\right]\rho$$ **Nonlinear term**

```
result = smesolve(H, rho0, times, [], c_ops, e_ops, method='heterodyne')
```

```
method = 'heterodyne'
method = 'homodyne'
method = 'photocurrent'
```