Problems for the Course: Floquet Engineering.

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1. Let us consider the following Hamiltonian for two identical nonlinear

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{m\omega^2}{2}(x_1^2 + x_2^2) + \frac{\epsilon}{4} x_1^2 x_2^2 \tag{1}$$

Rewrite it in terms of complex amplitudes a_1, a_1^*, a_2, a_2^*:

$$a_j = \frac{1}{\sqrt{2}}\left(\sqrt{m\omega}x_j + i\frac{p_j}{\sqrt{m\omega}}\right), \quad a_j^* = \frac{1}{\sqrt{2}}\left(\sqrt{m\omega}x_j - i\frac{p_j}{\sqrt{m\omega}}\right)$$

Show that within thee rotating frame approximation (RWA) this Hamiltonian reads

$$H_{\rm rot} = \omega(a_1^* a_1 + a_2^* a_2) + \frac{\epsilon}{16 m^2\omega^2}\left(a_1^2 (a_2^*)^2 + (a_1^*)^2 a_2^2 + 4|a_1|^2|a_2|^2\right), \tag{2}$$

Show that unlike the original Hamiltonian the Hamiltonian in the rotating frame has two conservation laws: energy and the number of phonons: $n = a_1^* a_1 + a_2^* a_2$ and thus can not lead to the chaotic motion. In the exercise below we will see that this conservation law persist in the higher (in fact in all) orders of the high frequency expansion. Repeat this exercise for the quantum Hamiltonian, pay attention to the ordering of operators.

2. Consider a system of weakly coupled weakly nonlinear harmonic oscillators on a square lattice of arbitrary dimension such that:

$$H = \sum_j \frac{p_j^2}{2m} + \frac{m\omega^2 x_j^2}{2} + \frac{\epsilon x_j^4}{4} + \frac{m\omega^2\kappa}{2}\sum_{\langle ij\rangle}(x_i - x_j)^2, \tag{3}$$

where $\langle ij\rangle$ stands for the nearest neighbors, κ is the coupling between the oscillators and ϵ is nonlinearity. Both κ and ϵ are assumed to be small.

Show that in the rotating wave approximation with respect to the bare frequency ω the Hamiltonian becomes

$$H_{\rm rot} = \sum_j \omega(1+\zeta\kappa)|a_j|^2 + \frac{3\epsilon}{8m^2\omega^2}|a_j|^4 - \kappa\frac{\omega}{2}\sum_{\langle ij\rangle}(a_i^* a_j + a_j^* a_i), \tag{4}$$

where ζ is the coordination number (e.g. for a two-dimensional square lattice $\zeta = 4$). Show that this Hamiltonian in turn leads to the following (discrete) Gross-Pitaevski equations of motion:

$$i\dot{a}_j = \frac{\partial H}{\partial a_j} = \omega(1+\zeta\kappa)a_j + \frac{3\epsilon}{4m\omega^2}|a_j|^2 a_j - \kappa\omega \sum_{i\in O_j} a_i, \tag{5}$$

where O_j stands for the set of ζ nearest neighbor sites of the site j. These Gross-Pitaveski equations, for example, describe weakly interacting superfluid gases in optical lattices.

Show that sending the discretization step to zero, which is equivalent to considering smooth spatial modulations of the bosonic variables and replacing $a_j \to a(\vec{x})$ leads to the continuous Gross-Pitaveski equations:

$$i\dot{a}(\vec{x}) = \omega a(\vec{x}) - \omega\kappa\nabla^2 a(\vec{x}) + \frac{3\epsilon}{4m\omega^2}|a(\vec{x})|^2 a(\vec{x}), \tag{6}$$

where κ effectively sets the (inverse) of the particle's mass and the nonlinearity $\epsilon > 0$ is equivalent to the repulsive density-density interaction coupling between the particles.

3. Consider the following Hamiltonian describing a one-dimensional nonlinear oscillator:

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} + \frac{\epsilon x^4}{4} = \mathcal{H}_0 + \frac{\epsilon x^4}{4},. \tag{7}$$

Show that in the rotating frame it reads

$$H = \frac{\epsilon}{16m^2\omega^2}\left((\tilde{a}^*)^4 \mathrm{e}^{4i\omega t} + 4(\tilde{a}^*)^3\tilde{a}\mathrm{e}^{2i\omega t} + 6(\tilde{a}^*)^2\tilde{a}^2 + 4\tilde{a}^*\tilde{a}^3\mathrm{e}^{-2i\omega t} + a^4\mathrm{e}^{-4i\omega t}\right) \tag{8}$$

Using the Magnus and Van Vleck expansions find the leading order correction beyond RWA to the Hamiltonian and to the Kick operator.

The same correction can be found using the method introduced by P. Kapitza based on separating fast and slow variables. Namely in the equations of motion for $\tilde{a}$ and $\tilde{a}^*$ assume that the solution can be written in the form

$$\tilde{a}(t) = \tilde{a}_0(t) + \epsilon(b_1(t)\mathrm{e}^{4i\omega t} + b_2(t)\mathrm{e}^{2i\omega t} + b_3\mathrm{e}^{-2i\omega t}), \tag{9}$$

where $\tilde{a}_0(t)$, $b_1(t)$, $b_2(t)$ and $b_3(t)$ are slowly (on the scale of $1/\omega$) varying fields. Plug this expansion into the eqations of motion and separately equate oscillating and non-oscillating parts to the leading order in ϵ. From equating oscillating parts you should get

$$b_1 \approx -\frac{1}{16m^2\omega^3}(\tilde{a}_0^*)^3, \quad b_2 \approx -\frac{3}{8m^2\omega^3}|\tilde{a}_0|^2\tilde{a}_0, \quad b_3 \approx \frac{1}{8m^2\omega^3}\tilde{a}_0^3 \tag{10}$$

Plug this solution to the non-oscillating part of the equation for $\tilde{a}_0$ and collect terms up to the order ϵ^2. You should obtain

$$i\frac{d\tilde{a}_0}{dt} \approx \frac{3\epsilon}{4m^2\omega^2}|\tilde{a}_0|^2\tilde{a}_0 - \frac{45\epsilon^2}{64m^4\omega^5}|\tilde{a}_0|^4\tilde{a}_0.$$

Show that in turn this equation can be interpreted as a Hamiltonian equation of motion

$$i\frac{d\tilde{a}_0}{dt} \approx \frac{\partial \mathcal{H}_{\text{eff}}}{\partial \tilde{a}_0^*}$$

with the Hamiltonian (back in the lab frame)

$$H_{\text{eff}} = \omega|a_0|^2 + \frac{3\epsilon}{8m^2\omega^2}|a_0|^4 - \frac{15\epsilon^2}{64m^4\omega^5}|a_0|^6.$$

Compare your effective Hamiltonian with the one obtained by Magnus and Van Vleck expansions.