Floquet Engineering

Hamiltonian dynamics

$$\frac{dx}{dt} = \{x, H\} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = \{p, H\} = -\frac{\partial H}{\partial x}, \quad \{x, p\} = 1$$

Harmonic oscillator

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$$

$$\frac{dx}{dt} = \frac{p}{m} \qquad \frac{dp}{dt} = -m\omega^2 x$$

Another set of canonical variables

$$a = \frac{1}{\sqrt{2}}\left(\sqrt{m\omega}\, x + \frac{ip}{\sqrt{m\omega}}\right) \qquad a^* = \frac{1}{\sqrt{2}}\left(\sqrt{m\omega}\, x - \frac{ip}{\sqrt{m\omega}}\right)$$

$$x = \frac{a + a^*}{\sqrt{2m\omega}} \qquad p = i\sqrt{\frac{m\omega}{2}}\,(a^* - a^{\cancel{*}})$$

$$H = \omega a^* a$$

$$\{A, B\} = \frac{\partial A}{\partial x}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial x} = \left(\frac{\partial A}{\partial a}\frac{\partial a}{\partial x} + \frac{\partial A}{\partial a^*}\frac{\partial a^*}{\partial x}\right)$$

$$\left(\frac{\partial B}{\partial a}\frac{\partial a}{\partial p} + \frac{\partial B}{\partial a^*}\frac{\partial a^*}{\partial p}\right) \cancel{=} \; A \leftrightarrow B = \frac{i}{2}\left(\frac{\partial A}{\partial a} + \frac{\partial A}{\partial a^*}\right)\left(\frac{\partial B}{\partial a} - \frac{\partial B}{\partial a^*}\right) -$$

$$- A \leftrightarrow B = -i\frac{\partial A}{\partial a}\frac{\partial B}{\partial a^*} + i\frac{\partial A}{\partial a^*}\frac{\partial B}{\partial a} = -i\left(\frac{\partial A}{\partial a}\frac{\partial B}{\partial a^*} - \frac{\partial A}{\partial a^*}\frac{\partial B}{\partial a}\right)$$

$\{a, a^*\} = -i \quad \{a, a\}, \{a^*, a^*\} = 0$ } Show that these Brackets are preserved under unitary rotations & Bogoliubov transf.

$\underbrace{i \frac{da}{dt} = i\{a, H\} = \frac{\partial H}{\partial a^*}}_{\text{GP equation}} \qquad i\frac{da^*}{dt} = i\{a^*, H\} = -\frac{\partial H}{\partial a}$

H.O.

$i\frac{da}{dt} = \frac{\partial H}{\partial a^*} = \omega a \Rightarrow a = a_0 e^{-i\omega t} \; ; \quad a^* = a_0^* e^{i\omega t}$

a moves around a circle for h.o., i.e. it rotates

Nonlinear oscillator

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} + \frac{\epsilon x^4}{4} = H_0 + \frac{\epsilon x^4}{4}$$

$$\frac{dp}{dt} = -m\omega^2 x - \epsilon x^3 \qquad \frac{dx}{dt} = \frac{p}{m}$$

Already rather difficult problem

Try new variables

$$H = \omega a^* a + \frac{\epsilon}{16 m^2\omega^2}\left((a^*)^4 + 4(a^*)^3 a + 4a^3 a^* + a^4 + 6(a^*)^2 a^2\right)$$

$$i\frac{da}{dt} = \omega a + \frac{\epsilon}{4m^2\omega^2}\left[(a^*)^3 + 3(a^*)^2 a + a^3 + 3a^* a^2\right]$$

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Solve perturbatively by going to the rotating frame

$$i\frac{da}{dt} = \omega a \Rightarrow \underbrace{a = \tilde{a}\, e^{-i\omega t} \qquad a^* = \tilde{a}^* e^{i\omega t}}_{\text{preserves Poisson brackets.}}$$

Substitute $a = \tilde{a}(t)\, e^{-i\omega t}$

$$i\dot{\tilde{a}} + \cancel{\omega\tilde{a}} = \cancel{\omega\tilde{a}} + \frac{\epsilon}{4m^2\omega^2}\Big[(\tilde{a}^*)\, e^{4i\omega t} + 3(\tilde{a}^*)^2\tilde{a}\, e^{2i\omega t} + 3\tilde{a}^*\tilde{a}^2 + \tilde{a}^2 e^{-2i\omega t}\Big]$$

i.e. a problem with a static Hamilt (Floquet)

Map a static problem to a time dependent problem

Intuition - large ω - oscillating terms average to zero $\Rightarrow$

$$i\frac{d\tilde{a}}{dt} \approx \frac{3\epsilon}{4m^2\omega^2}|\tilde{a}|^2\tilde{a}$$

$\boxed{\text{Show that in this approximation } |\tilde{a}^2(t)| = |a_0^2| = \text{const.}}$

$$\tilde{a} \simeq \tilde{a}_0\, e^{-\frac{3\epsilon|\tilde{a}_0|^2 t}{4m^2\omega^2} i} \Rightarrow a = a_0\, e^{-i\left[\omega t + \frac{3\epsilon}{4m^2\omega^2}|a_0^2|\right]}$$

Frequency shift.

Problem.

Take the quantum Hamiltonian

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} + \frac{\epsilon x^4}{4}$$

Compute energy shifts for n^{th} eigenstates in the leading order in ϵ. Using the relation from WKB approximation that

$$\frac{\partial E_n}{\partial n} \simeq \hbar \Omega = \frac{2\pi\hbar}{T}$$ compute the frequency shift in the h.o. due to ϵ & show it agrees with R.W.A.

Effective hamiltonian in RWA

$$H = \omega a^{+} a + \frac{3\epsilon}{8m^2\omega^2} (a^{+})^2 a^2 .$$

In the rotating frame $a = \tilde{a}\, e^{-i\omega t}$

$$\tilde{H} = \frac{3\epsilon}{8m^2\omega^2} (\tilde{a}^{+})^2 \tilde{a}^2$$ does not acquire any oscillating terms.

In general $H \to \tilde{H}$ is a time dependent canonical transformation (going to a moving frame) - modifies Hamiltonian.

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In xp coordinates H_{RWA} looks a bit unusual

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} + \frac{3\epsilon}{8m^2\omega^4}\left(\frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}\right)^2 \Rightarrow \delta E \simeq \frac{3\epsilon}{8m^2\omega^4} E^2$$

- contains p^2 & p^4 terms, not present in the original Hamiltonian. | H.W. Derive in 2D

Quick detour. General approach to transformations to the rot. frame

$q \to q(\lambda(t), t)$ -
$p \to p(\lambda(t), t)$

$$\frac{\partial q}{\partial \lambda} = -\frac{\partial A_\lambda}{\partial p} = -\{q, A_\lambda\} \qquad \frac{\partial p}{\partial \lambda} = \frac{\partial A_\lambda}{\partial q}$$

preserves Poisson brackets.

Equations of motion

$$\frac{dq}{dt} = \frac{\partial q}{\partial \lambda}\frac{d\lambda}{dt} + \frac{\partial q}{\partial t} = \{q, H\} - \dot{\lambda}\{q, A_\lambda\} = \{q, \underbrace{H - \dot{\lambda} A_\lambda}_{\text{moving frame Hamil}}\}$$

same for p also; Q, Q^*

$$i\frac{\partial Q}{\partial \lambda} = -\frac{\partial A_\lambda}{\partial Q^*}, \quad i\frac{\partial Q^*}{\partial \lambda} = \frac{\partial A_\lambda}{\partial Q}$$

Coordinate shift

$$q = q_0 - \lambda \qquad \frac{\partial q}{\partial \lambda} = -1 = -\frac{\partial A}{\partial p} \qquad \frac{\partial p}{\partial \lambda} = 0 = \frac{\partial A}{\partial q} \Rightarrow A = p$$

Momentum shift

$$p = p_0 - \lambda \qquad \frac{\partial q}{\partial \lambda} = 0 = -\frac{\partial A}{\partial p} \qquad \frac{\partial p}{\partial \lambda} = -1 = \frac{\partial A}{\partial q} \Rightarrow A = -q$$

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Phase shift

$\tilde{a}(\varphi) = a_0 e^{-i\varphi} \qquad \tilde{a}^*(\varphi) = a_0^* e^{i\varphi} \Rightarrow a_0 = \tilde{a} e^{i\varphi}$

$$i \frac{\partial \tilde{a}}{\partial \varphi} = \tilde{a} = -\frac{\partial \mathcal{A}}{\partial \tilde{a}^*} \qquad i\mathcal{A} \Rightarrow \mathcal{A} = -a^* a$$

$i\frac{\partial a^*}{\partial \varphi} = \frac{\partial A}{\partial a} = -a^*$.

Apply to the Hamiltonian

$$H_{eff} = \omega a^* a + \frac{3\epsilon}{8m^2\omega^2} (a^*)^2 a^2$$

$$H_{rot} = \hat{H} = \omega \tilde{a}^* \tilde{a} + \frac{3\epsilon}{8m^2\omega^2} (\tilde{a}^*)^2 \tilde{a}^2 + \dot{\varphi}(-\tilde{a}^* \tilde{a})$$

$$= \omega \tilde{a}^* \tilde{a} + \frac{3\epsilon}{8m^2\omega^2} (\tilde{a}^*)^2 \tilde{a}^2 + \dot{\varphi}\, \tilde{a}^* \tilde{a}$$

Choose $\dot{\varphi} = -\omega \Rightarrow$ Recover previous result.

Quantum Systems

Very similar story

$$\hat{a}, \hat{a}^+ = \frac{1}{\sqrt{2}} \left[\sqrt{\frac{m\omega}{\hbar}}\, \hat{x} \pm \frac{i\hat{p}}{\sqrt{m\omega\hbar}} \right]$$

For the nonlinear osc.

$$\hat{H} = \hbar\omega(\hat{a}^+\hat{a} + 1/2) + \frac{\epsilon\hbar^2}{16m^2\omega^2} \Big[(a^+)^4 + 4(a^+)^3 a + 6(a^+ a)^2 \ldots + a^4 + 6(a^+)^2 + 6\hat{a}^2 + 6a^+a + 3 \Big]$$

exercise

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Rot. frame transformation

$\hat{a} \to \hat{a} e^{-i\omega t} \quad \hat{a}^+ \to \hat{a}^+ e^{i\omega t}$

General theory of transformations

$\hat{a} = \quad |\Psi\rangle = U(\lambda)\, |\tilde{\Psi}\rangle$

$\langle\Psi|\hat{O}|\Psi\rangle = \langle\tilde{\Psi}| U^+ \hat{O} U |\tilde{\Psi}\rangle \Rightarrow \hat{O} \Rightarrow U^+ \hat{O} U = \tilde{O}$

$i\hbar\partial_t |\Psi\rangle = H|\Psi\rangle \quad i\hbar \frac{\partial U}{\partial\lambda}\dot{\lambda}\, |\tilde{\Psi}\rangle + i U \hbar |\dot{\tilde{\Psi}}\rangle = H U |\tilde{\Psi}\rangle$

$i\hbar|\dot{\tilde{\Psi}}\rangle = U^+ H U - i\hbar\dot{\lambda} U^+ \partial_\lambda U = U^+ [H - \dot{\lambda} A_\lambda] U |\tilde{\Psi}\rangle$

$A_\lambda = i\hbar(\partial_\lambda U) U^+$ gauge potential

$\frac{d\hat{O}}{d\lambda} = i\hbar \frac{d\hat{O}}{d\lambda} = i\hbar [-U^+ \partial_\lambda U U^+ \hat{O} U + U^+ \hat{O} U^* U^+ d_\lambda U]$

$= i\hbar [\hat{O}, \tilde{A}_\lambda] = U^+ [O, A_\lambda] U \quad \tilde{A}_\lambda = i\hbar U^+ \partial_\lambda U \Rightarrow i\hbar \frac{d\hat{a}}{d\lambda} = \hat{a}$

$\Rightarrow \hat{a} = e^{-i\lambda}\hat{a}_0$

Our problem $\hat{A}_\lambda = n$, $\underbrace{\dot{\lambda} = \omega \Rightarrow \lambda = \omega t}_{\text{because want to kill}}$

$i\hbar(\partial_\lambda U) U^+ = \hat{n} \Rightarrow U = e^{-i\frac{1}{\hbar}\lambda \hat{n}}$

$\hat{H}_{rot} = \frac{\epsilon\hbar^2}{16 m^2\omega^2} \Big[(\tilde{a}^+)^4 e^{4i\omega t} + 4(\tilde{a}^+)^3 \tilde{a}\, e^{2i\omega t} + 6(\tilde{a}^+)^2 \tilde{a}^2 + 4\tilde{a}^+ \tilde{a}^3 e^{-2i\omega t} + \tilde{a}^4 e^{-4i\omega t} + 6(\tilde{a}^+)^2 e^{2i\omega t} + 6\tilde{a}^2 e^{-2i\omega t} + 6\tilde{a}^+\tilde{a} + 3 \Big]$

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$$\hat{H}_{RWA} = \frac{\epsilon \hbar^2}{16 m^2 \omega^2}\left[6(\tilde{Q}^{+} \tilde{Q})^2 + 6\tilde{Q}^{+}\tilde{Q} + 3 \right]$$

$$\langle n | \hat{H}_{RWA} | n \rangle = \frac{\epsilon \hbar^2}{16 m^2 \omega^2}\left[\overset{6n^2+6n+3}{\cancel{6(n+1)(n-1)+6n+3}} \right] =$$

$$= \frac{\epsilon \hbar^2}{16 m^2 \omega^2}\left[\cancel{6n^2+12n}\ 9 \right] = \frac{\epsilon \hbar^2 (n+1/2)^2 \omega^2}{3 m^2 \omega^4} \left(\frac{n^2+n+1/2}{(n+1/2)^2} \right)$$

$$= \frac{\epsilon E_n^2}{3 m^2 \omega^4} \left(\frac{n^2+n+1/2}{n^2+n+1/4} \right)$$

Agrees with the classical result. At large n

δE computed classically $\Rightarrow$ energy shift within RWA if we adiabatically turn on nonlinearity.

How can we go beyond ~~using~~ avoiding standard language of eigenstates & energy shifts, which quickly becomes incomprehensible, especially for more complex systems?

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Floquet systems. High frequency expansion

$$H = H_0 + \sum_{\ell \neq 0} H_\ell e^{i\omega t \ell} \qquad H_{-\ell} = H_\ell^+$$

General structure of a periodically driven Hamiltonian. Periodicity is not always crucial but stick to it for simplicity

Floquet theorem

Define the evolution operator within the period

$$U_{(T,0)} = \prod_{\Delta t \in [0,T]} e^{-\frac{i}{\hbar} H(t)\Delta t} = T_t\, e^{-\frac{i}{\hbar}\int_0^T H(t)\,dt}$$

Because the Hamiltonian is periodic

$$U(2T,0) = U(2T,T)\,U(T,0) = U^2(T,0)$$

Generalising this we have

$$U(nT+t_0, t_0) = \left(U(T+t_0,t_0)\right)^n$$

Define $e^{-i H_F[t_0] \cdot T \frac{1}{\hbar}} = U(T+t_0, t_0)$

$$\boxed{H_F[t_0] = \frac{i\hbar}{T} \log U(T+t_0,t_0) = \frac{i\hbar}{T} \log T_t\, e^{-\frac{i}{\hbar}\int_{t_0}^{t_0+T} H(t)\,dt}}$$

Stroboscopic evolution

$$U(nT+t_0,t_0) = e^{-\frac{i}{\hbar} H_F[t_0]\, nT}$$

- effectively quench to a static Floquet Hamiltonian

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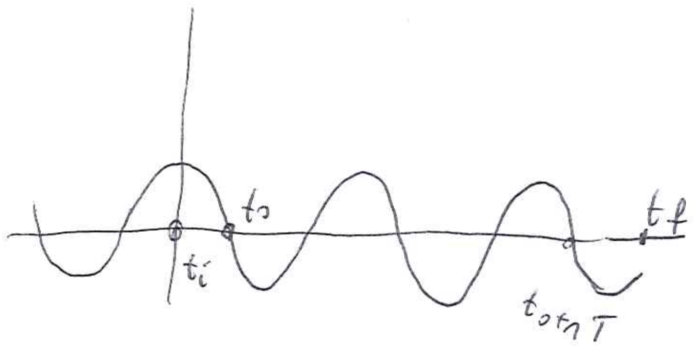
Arbitrary times



$$U(t_f,t_i) = U(t_f, t_0+nT)\, U(t_0+nT, t_0)\, U(t_0,t_i)$$

$$= U(t_f, t_0+nT)\, e^{\frac{i}{\hbar} H_F (t_f - t_0 - nT)}\, e^{-\frac{i}{\hbar} H_F (t_f - t_0 - nT)}\, e^{-\frac{i}{\hbar} H_F nT}$$

$$e^{-\frac{i}{\hbar} H_F (t_0 - t_i)}\, e^{\frac{i}{\hbar} H_F (t_0 - t_i)}\, U(t_0, t_i)$$

$$U(t_f, t_0+nT)\, e^{\frac{i}{\hbar} H_F (t_f - t_0 - nT)} = P(t_f, t_0+nT) \equiv e^{+\frac{i}{\hbar} K(t_f, t_0+nT)}$$

$$e^{\frac{i}{\hbar} H_F (t_0 - t_i)}\, U(t_0, t_i) = \left(U(t_i, t_0)\, e^{\frac{i}{\hbar}(t_i - t_0) H_F} \right)^{+}$$

$$= P_{t_0}(t_f)\, e^{-\frac{i}{\hbar} H_F (t_f - t_i)}\, P_{t_0}(t_i) = e^{\frac{i}{\hbar} K_{t_0}(t_f)}\, e^{-\frac{i}{\hbar} H_F (t_f - t_i)}\, e^{\frac{i}{\hbar} K_{t_0}(t_i)}$$

$K(t_f+T) = K(t_f)$ kick operator.

Proved Floquet theorem

$$U(t_2,t_1) = e^{\frac{i}{\hbar} K(t_f)}\, e^{-\frac{i}{\hbar} H_F (t_f - t_i)}\, e^{-\frac{i}{\hbar} K(t_i)}$$

H_F - Floquet Hamiltonian - defines discrete stroboscopic dynamics

$K(t)$ - Kick operator - defines 1) Basis rotations

$t_f = t_i + nT \Rightarrow K(t_f) = K(t_i) = K \Rightarrow U(t_i, t_f) = e^{\frac{i}{\hbar}K} e^{-\frac{i}{\hbar}H_F(t_f - t_i)} e^{-\frac{i}{\hbar}K}$

2) Defines evolution between periods.

$H_F \leftrightarrow$ Block vector

$K \leftrightarrow$ periodic function in the Bloch theorem $\Psi(\vec{r}) = e^{i\vec{k}\vec{r}} u(\vec{r})$

General (gauge) freedom in choosing Floquet Hamiltonian

And the kick operator

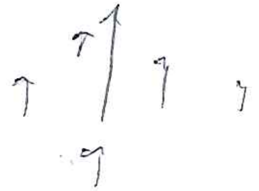
$$e^{\frac{i}{\hbar}K(t_f)} e^{-\frac{i}{\hbar}K'} e^{i/\hbar K'} e^{-i/\hbar H_F(t_f - t_i)} e^{-i/\hbar K'} e^{i/\hbar K'} e^{-i/\hbar K(t_i)}$$

$$e^{\frac{i}{\hbar}\tilde{K}(t)} = e^{\frac{i}{\hbar}K(t)} e^{-\frac{i}{\hbar}K'} \qquad \tilde{H}_F = e^{i/\hbar K'} H_F e^{-i/\hbar K'}$$

K' can depend on t_0 but not on t!

This freedom can be used to modify the kick operator & the Floquet Hamiltonian

Similar to gauge freedom for a particle in a symmetric radial magnetic field



Landau gauge breaks $U(1)$ symmetry

likewise ~~choosing~~ fixing ~~to~~ breaks $U(1)$ symmetry.

Two conventional choices for the gauge

1) Fix t_0 to be some symmetric point ~~in~~ within the period. E.g.

$V(t) = V_0 \cos(\omega t + \varphi)$ $\qquad t \to t_0 + t \Rightarrow \varphi \to \varphi + \omega t_0$

can choose t_0 such that

$V(t) = V_0 \cos[\omega(t - t_0)]$ or $V(t) = V_0 \sin \omega(t - t_0)$

2) Find the gauge $K'(t_0)$ where H_F is independent of t_0 and all dependence on t_0 is in K (similar to symmetric gauge in e/m) $A = \left(\frac{By}{2}, -\frac{Bx}{2}, 0\right)$

High frequency expansion

With a few exceptions it is nearly impossible to find exactly H_F. Similar situation to thermodynamics

$Z = T_\tau e^{-\int_0^\beta H(\tau)d\tau} \qquad F = -T \ln Z$

Need to rely on 1) Exactly solvable (Floquet integrable) systems 2) High frequency ($\cong$ high temperature) expansion 3) perturbative expansions near solvable limits.

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Integrable limits. Most studied are noninteracting limits like h.o. or free models. In such models Algebra closes $\Rightarrow$ $\prod_k e^{-\frac{i}{\hbar}H_k} = e^{-\frac{i}{\hbar}H_F}$

E.g. $e^{iX}e^{iY} = e^{i[X+Y+\frac{i}{2}[X,Y] + \frac{1}{12}[X[X,Y]] + \frac{1}{12}[Y[X,Y]] + \dots}$

If X, Y are finite-dimensional matrices, e.g. 2×2 matrices the commutators start to repeat themselves $\Rightarrow$ $\Rightarrow$ can simply resum the series. Still could be mathematically involved.

2) Perturbative expansion

$$e^{\frac{i}{\hbar}H_F T} = e^{\frac{i}{\hbar}H_0 T} e^{i\epsilon H_1} = e^{\frac{i}{\hbar}X} e^{\frac{i}{\hbar}\epsilon Y} \qquad \epsilon \to 0$$

$\epsilon = 0 \Rightarrow H_F = X$ Try to expand in ϵ

$$\log(e^X e^{\epsilon Y}) = X + \epsilon \frac{Ad_X}{1-e^{-Ad_X}} Y + O(Y^2) = X + \epsilon\left[\frac{1}{2}[X,Y] + \frac{1}{12}[X[X,Y]] + \dots\right]$$

$$Ad_X\, Y = [X,Y]$$

This series can only be resummed in special cases where X does not increase complexity of Y. E.g. X noninteracting, or X is a simple spin rotation or X is a sum of commuting terms $X = \sum_i X_i$ $[X_i, X_j] = 0$ and Y has a finite support. Opposite limit to

FGR - FGR usually applies when there is no perturbative H_F

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3) Most important. High frequency expansion

$$T = \frac{2\pi}{\Omega} \to 0$$

$$e^{-i/\hbar\, H_F T} = T_\tau\, e^{-i/\hbar \int_{t_0}^{T+t_0} H(t')\,dt'}$$ - ~~set~~ use broken

symmetry gauge.

$$e^{-i/\hbar \int_{t_0}^{T+t_0} H(t')dt'} \simeq 1 - \frac{i}{\hbar}\int_{t_0}^{T+t_0} H(t')dt' - \frac{2}{2\hbar^2}\int_{t_0}^{T+t_0} dt_1 \int_{t_0}^{t_1} dt_2\, H(t_1)H(t_2) + \ldots$$

$$H_F = +\frac{\hbar i}{T} \log e^{-i/\hbar \int_{t_0}^{T+t_0} H(t')dt'} \simeq$$

$$= +\frac{i\hbar}{T}\Big[-\frac{i}{\hbar}\int_{t_0}^{T+t_0} H(t')dt' - \frac{2}{2\hbar^2}\iint_{t_1>t_2} dt_1 dt_2\, H(t_1)H(t_2)$$

$$+ \frac{1}{2\hbar^2}\iint dt_1 dt_2\, H(t_1)H(t_2) + \ldots\Big]$$

$$= \frac{1}{T}\int_{t_0}^{T+t_0} H(t')dt' - \frac{i}{T\hbar}\int_{t_0}^{T+t_0} dt_1 \int_{t_0}^{t_1} dt_2\, [H(t_1), H(t_2)] + \ldots$$

$$H_F^0 = \frac{1}{T}\int_{t_0}^{T+t_0} H(t')dt'$$ (recall $H = \sum_\ell H_\ell\, e^{2\pi i \ell\, t/T} = \sum_\ell H_\ell\, e^{i\Omega \ell t}$, $\Omega = \frac{2\pi}{T}$)

$H_F^0 = H_0$ - does not depend on t_0.

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$$H_F^{(1)} = -\frac{i}{\hbar T} \sum_{\ell,\ell'} [H_\ell, H_{\ell'}] \int_{t_0}^{t_0+T} dt_1\, e^{i\Omega \ell t_1} \int_{t_0}^{t_1} dt_2\, e^{i\Omega \ell' t_2}$$

$$\int_{t_0}^{t_1} dt_2\, e^{i\Omega \ell' t_2} = \frac{e^{i\Omega \ell' t_1} - e^{i\Omega \ell' t_0}}{i\Omega \ell'}$$

$$= -\frac{i}{\hbar T\, i\Omega} \sum_{\ell,\ell'} [H_\ell, H_{\ell'}] \frac{1}{\ell'} \Big[\frac{1}{i\Omega(\ell+\ell')} \underbrace{\left(e^{i\Omega(\ell+\ell')(t_0+T)} - e^{i\Omega(\ell+\ell')t_0}\right)}_{0 \text{ if } \ell+\ell' \neq 0,\ T \text{ otherwise}} - \cancel{\frac{1}{i\Omega(\ell+\ell')}} - \frac{e^{-i\Omega \ell' t_0}}{i\Omega \ell} \underbrace{\left(e^{i\Omega \ell(t_0+T)} - e^{i\Omega \ell t_0}\right)}_{T \text{ if } \ell=0,\ 0 \text{ otherwise.}}$$

$$= +\frac{1}{\hbar T\Omega} \sum_{\ell} [H_\ell, H_{-\ell}] \frac{T}{\ell} + \frac{1}{\hbar T\Omega} \sum_{\ell' \neq 0} \frac{t}{\ell'} e^{-i\Omega \ell' t_0} [H_0, H_{\ell'}]$$

$+ h.c.$

$$= \frac{1}{\hbar\Omega} \sum_{\ell > 0} \frac{1}{\ell}\Big([H_\ell, H_{-\ell}] - e^{+i\Omega \ell t_0} [H_\ell, H_0] + e^{-i\Omega \ell t_0} [H_{-\ell}, H_0] \Big)$$

$H_F^{(1)}$ explicitly breaks $U(1)$ invariance

Likewise

$$K_{F\,t_0}^{(0)}(t) = 0$$

$$K_F^{(1)} = \frac{1}{\hbar} \int_{t_0}^{t} dt' \left[H(t') - H_F^{(1)} \right] = \frac{1}{i\hbar\Omega} \sum_{\ell \neq 0} H_\ell \frac{e^{i\ell\Omega t} - e^{-i\Omega \ell t_0}}{\ell}$$

- - -

Can eliminate t_0 dependence by absorbing t_0 dep. terms into K'

$$H_{eff}^{(0)} = H_0 \qquad H_{eff}^{(1)} = \frac{1}{\hbar\Omega} \sum_{\ell\neq 0} [H_\ell, H_{-\ell}]$$

$$K_{eff}^{(0)} = 0 \qquad K_{eff}^{(1)} = \frac{1}{i\hbar\Omega} \sum_{\ell\neq 0} \frac{e^{i\Omega\ell t_0}}{\ell} H_\ell$$

All expansions survive the classical limit despite absence of the FI-theorem

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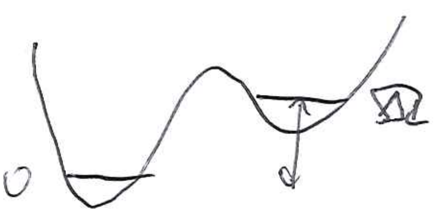
Examples of HFE. Floquet engineering.



2^{nd} quantization

$$\hat{H} = 0 \cdot a_1^+ a_1 + \hbar\Omega\, a_2^+ a_2 - J(a_2^+ a_1 + a_1^+ a_2)$$

$$\hat{H}_{rot} = -J\left(e^{i\Omega t} a_2^+ a_1 + a_1^+ a_2 e^{-i\Omega t}\right)$$

$$H_{rot}^{0} = 0 \qquad H_{rot}^{(1)} = \frac{1}{\hbar\Omega}[a_2^+ a_1, a_1^+ a_2] = \frac{J^2}{\hbar\Omega}(a_2^+ a_2 - a_1^+ a_1)$$

Obtain level repulsion - operator perturbation theory

Apply driving

$$\hat{H} = a_1^+ a_1\ \Delta_0 \cos \omega t\ a_2^+ a_2 - J[a_2^+ a_1 + a_1^+ a_2]$$

Can work in the lab frame $\quad \overline{H}_0 = -J(a_2^+ a_1 + a_1^+ a_2)$

$H_F^{(1)} = 0$ (if we use e.g. Van Vleck expansion)

$H_F^{(2)} \neq 0$

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It is advantageous to go to rot. frame

$$U(t) = e^{\frac{i}{\hbar}\frac{\Delta_0}{\omega}\sin\omega t\, a_2^+ a_2} = e^{-i\varphi(t)\, a_2^+ a_2}$$

$$H_{rot} = H_0 - i\hbar U^+ \partial_t U = -J\left[\tilde{a}_2^+ \tilde{a}_1 e^{i\varphi} + \tilde{a}_1^+ \tilde{a}_2 e^{-i\varphi}\right]$$

$$\overline{H_{rot}} = -J\left[a_2^+ a_1 \overline{e^{+i\frac{\Delta_0}{\omega}\sin\omega t}} + a_1^+ a_2 \overline{e^{-i\frac{\Delta_0}{\omega}\sin\omega t}}\right]$$

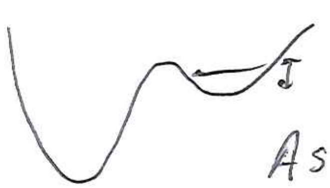
$$= -J\, J_0\!\left(\frac{\Delta}{\omega}\right)\left(a_2^+ a_1 + a_1^+ a_2\right)$$

$\downarrow$
Bessel function

(Exercise - get leading order corr. in the lab. frame)

Get renormalized hopping. Tuning $\frac{\Delta}{\omega} \approx 2.405$

leads to zero hopping - dynamical localization.
Need to work in the limit $\Delta \sim \omega \Rightarrow$ heart of Floquet engineering
Combine the two



$$H = \hbar\Omega\, a_2^+ a_2 - J\left(a_1^+ a_2 + a_2^+ a_1\right) - \Delta\cos(\omega t + \varphi)\, a_2^+ a_2$$

Assume $\Omega = n\omega + \varepsilon$

Go to the joint rotating frame w.r.t. to $n\omega$ & $\Delta\cos\omega t$

$$H_{rot} = -J\left[a_1^+ a_2 e^{-i\varphi} + a_2^+ a_1 e^{i\varphi}\right] + \varepsilon\, a_2^+ a_2$$

$$\varphi(t) = \omega n t - \frac{\Delta}{\omega}\sin[\omega t + \varphi]$$

$$\overline{H_{rot}} = \overline{e^{i\omega n t - i\frac{\Delta}{\omega}\sin\omega(t-t_0)}} \qquad + \omega t_0 = \varphi$$

$$= e^{i\omega n t_0}\, \overline{e^{i\omega n(t-t_0)}\, e^{-i\frac{\Delta}{\omega}\sin\omega(t-t_0)}} = e^{\pm i n \varphi_0} J_n\left(\tfrac{\Delta}{\omega}\right)$$

$$\overline{H_{rot}} = -J\, J_n\left(\frac{\Delta}{\omega}\right)\left[a_1^+ a_2\, e^{i n \varphi_0} + a_2^+ a_1\, e^{-i n \varphi_0}\right] +$$

$$+ \varepsilon\, a_2^+ a_2$$

Get renormalized complex hopping & detuning

Higher order get correction to detuning $\sim \frac{J^2}{\omega}$

SW transformation = van Vleck expansion

Can get a similar expansion for the kick operator

$$K_{eff}^{(1)} = \frac{1}{i\omega}\sum_{\ell=1}^{\infty}\frac{1}{\ell}\left(e^{i\ell\omega t} H_\ell - e^{-i\omega\ell t} H_{-\ell}\right)$$

$$K_F^{(1)} = \frac{1}{i\omega}\sum_{\ell=1}^{\infty}\frac{1}{\ell}\left[\left(e^{i\ell\omega t} - 1\right)H_\ell - \left(e^{-i\omega\ell t} - 1\right)H_{-\ell}\right]$$

LFE

The Dicke model - Hydrogen atom for quantum optics

work with $\hbar \to 1$

$$H = \hbar\omega\, a^{+}a + \Delta \sum_j \sigma_z^j + \underbrace{\frac{g}{\sqrt{N}}\,(a^{+}+a)}_{} \sum_j \sigma_x^j$$

spin flip operator

like photon electric field

- analogue of a dipole coupling if we focus on 2 relevant levels

let us assume $\Delta = \frac{\omega}{2} + \varepsilon$ ← detuning

Go to the rotating frame w.r.t. large fields

$\omega a^{+}a$ & $\omega \sum_j \sigma_z^j$

1. $a \to \tilde{a}\, e^{-i\omega t}$ $\quad a^{+} \to \tilde{a}^{+} e^{i\omega t}$

2. $\underbrace{\sigma_z \to \sigma_z \quad \sigma^{+} \to \sigma^{+} e^{i\omega t} \quad \sigma^{-} \to \sigma^{-} e^{-i\omega t}}_{}$

either comes from Schw. Boson rep. of spins

or directly from $A_\lambda = \sigma_z/2 \quad \dot{\lambda} = \omega$

$$i\frac{d\sigma^{+}}{d\lambda} = [\sigma^{+}, A_\lambda] = -i\sigma_y - \sigma_x = -\sigma^{+} \Rightarrow \sigma^{+} = \sigma_0^{+} e^{i\lambda}$$

$$H_{rot} = \varepsilon \sum_j \sigma_z^j + \frac{g}{\sqrt{N}}\left(a^{+}\sum_j \sigma_-^j + a \sum_j \sigma_+^j\right) +$$

$$+ \frac{g'}{\sqrt{N}}\left(\sum a^{+}\sigma_+^j e^{2i\omega t} + \sum a\, \sigma_-^j e^{-2i\omega t}\right)$$

generally one can assume $g \neq g'$

FE

RWA

$$H_{rot} = \varepsilon \sum_j \sigma_z^j + \frac{g}{\sqrt{N}} \left(a^\dagger \sum_j \sigma_-^j + a \sum_j \sigma_+^j \right)$$

Jaynes-Cummings model.

Leading correction beyond RWA - use van Vleck.

$$[H^+, H^-] = \frac{(g')^2}{N \cdot 2\omega} \left[a^\dagger \sum_j \sigma_+^j , a \sum_j \sigma_-^j \right]$$

$$a^\dagger a \sum_j \sum_k \sigma_+^j \sigma_-^k - a a^\dagger \sum_{jk} \sigma_-^k \sigma_+^j =$$

$$= \cancel{a^\dagger a \left(\sum_{jk} \sigma_+^j \sigma_-^k - \sigma_-^k \sigma_+^j \right)} - \sum_{jk} \sigma_-^k \sigma_+^j$$

$$= \frac{a^\dagger a + a a^\dagger}{2} \left(\sum_{j,k} \sigma_+^j \sigma_-^k - \sigma_-^k \sigma_+^j \right) + \frac{a^\dagger a - a a^\dagger}{2} \sum \sum \sigma_+^j \sigma_-^k + \sigma_-^k \sigma_+^j$$

only $j = k$ contribute

$$[\sigma_+, \sigma_-] = -i [\sigma_x, \sigma_y] + i [\sigma_y, \sigma_x] = 2\sigma_z$$

$$H_{eff}^{(1)} = \varepsilon \sum_j \sigma_z^j + (g')^2 \frac{g}{\sqrt{N}} \left[a^\dagger \sum_j \sigma_-^j + a \sum_j \sigma_+^j \right]$$

$$+ \frac{(g')^2}{2N\omega} (a^\dagger a + a a^\dagger) \sum_j \sigma_z^j + \frac{(g')^2}{4N\omega} \{ S_+, S_- \}_+$$

Floquet limit $a = \sqrt{n} \, e^{i\phi}$ $\Rightarrow$ second term disappears.
very large $N \to \infty$

FE

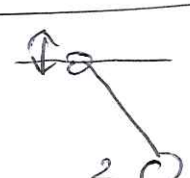
$$H_{eff}^{(1)} \simeq \varepsilon \sum_j \sigma_z^j + g\,[\,\cancel{a}^{+} \sum_j \sigma_-^j e^{-i\varphi} + \sum_j \sigma_+^j e^{i\varphi}\,]$$

$$+ \frac{(g')^2}{\omega} 2 \sum_j \sigma_z^j$$

Quantum photon cavity get extra all to all interactions. Very important for fate of MBL

externally driven disordered spins ⇒ MBL

spins coupled to few photons — ~~no~~ no (M. Kolodrubetz 2018)



Kapitza pendulum

$$H = \frac{1}{2m\ell^2}\left[p - m\ell^2 \frac{a\Omega}{\ell} \sin\theta \sin \Omega t\right]^2 - m\ell^2 \omega_0^2 \cos\theta$$

Momentum shift ~~A=~~ $p \to p - \lambda \sin\theta \qquad \frac{\partial p}{\partial \lambda} = -\sin\theta = -\frac{\partial A}{\partial \theta}$

$\Rightarrow A = \cos\theta$

$$H \to H - \dot{\lambda} A = \frac{p^2}{2m\ell^2} - m\ell^2 \omega_0^2 \cos\theta - m\ell a \Omega^2 \cos\Omega t \cos\theta$$

$$= \frac{p^2}{2m\ell^2} - m\ell^2 \cos\theta \left[\omega_0^2 + \overbrace{\frac{a\Omega}{\ell}}^{\lambda} \Omega \cos \Omega t\right] \quad \text{desired form}$$

Go to rot. frame w.r.t. to driving = go back

$$U = e^{i m \lambda \sin \Omega t}$$

FE

$$H = \underbrace{U^{+} \frac{p^2}{2m} U}_{\text{recall } p \text{ is second derivative}} - m e^2 \cos\theta$$

$$H_{rot}^{r} = \frac{p^2}{2m} - m\omega_0^2 \cos\theta + \frac{m^2\lambda^2}{2m} \sin^2\theta \sin^2\Omega t$$

$$- \frac{m\lambda}{2m} \sin\Omega t \; \{\sin\theta, p\}_{+}$$

$$\overline{H_{rot}} = \frac{p^2}{2m} - m\omega_0^2 \cos\theta + \frac{m\lambda^2}{4} \sin^2\theta$$

$\frac{m\omega_0^2}{2} \cancel{<} \frac{m\lambda^2}{4} \Leftrightarrow \lambda \gtrsim \sqrt{2}\,\omega_0$ get a new inverted minimum - engineer new stable equilibrium at π

Can analyze $\frac{1}{\omega}$ corrections

$$H_{eff}^{(1)} = 0$$

$$\underbrace{K_{eff}^{(1)} = \frac{1}{\Omega}\left[\frac{\lambda}{2} \cos\Omega t \; \{\sin\theta, p\}_{+} - \frac{m\lambda^2}{8} \sin(2\Omega t) \sin^2\theta\right]}_{}$$

describes fast dynamics & dressing of x, p to $\frac{1}{\Omega}$ order.

LFE

Artificial mag. fields (I. Block, W. Ketterle)

$$H_0 = -\sum_{m,n} J_x \left[a^{\dagger}_{m+1,n} a_{mn} + h.c. \right] - J_y \left(a^{\dagger}_{m\,n+1} a_{mn} + h.c. \right) \Big]$$

$$+ \frac{U}{2} \sum n_{mn} (n_{mn} - 1) + \Omega \sum_{m,n} \underbrace{\left[\frac{\lambda}{2} \sin\left(\Omega t - \varphi_{mn} + \frac{\Phi_\Box}{2}\right) + \Omega m \right] n_{mn}}_{\text{shaking with a site dep. phase}}$$

$$+ \underbrace{\Omega m \sum n_{mn}}_{\text{field along } x}$$

$\varphi_{mn} = \Phi_\Box (n+m)$ phase gradient (for simplicity same along x, y)

$$H_{rot} = H_{int} + W + W^{\dagger}$$

$$W = -\sum_{m,n} \Big[J_x\, e^{-i\rho \sin(\Omega t - \varphi_{mn}) + i\Omega t}\, a^{\dagger}_{m+1,n} a_{mn}$$

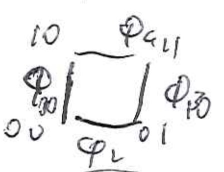
$$+ J_y\, e^{-i\rho \sin(\Omega t - \varphi_{mn})}\, a^{\dagger}_{m\,n+1} a_{mn} \Big]$$

$$\rho = \lambda \sin \Phi_\Box / 2$$

$$\overline{H_{rot}} = \frac{U}{2} \sum_{nm} n_{nm}(n_{mn} - 1) - K \sum e^{i\varphi_{mn}} a^{\dagger}_{m+1} a_n + h.c.$$

$$- J \sum \left(a^{\dagger}_{m\,n+1} a_{mn} + h.c. \right)$$

$$K = J_x\, J_1(\rho) \qquad J = J_y\, J_0(\rho)$$



$\Delta\varphi = \Phi_\Box$

$\cancel{\Delta\varphi_3 - \varphi_1 - \Phi_\Box}$

realize flux per plaquette = magnetic field

Kick operators are important