Irreversibility of RG in quantum field theories with boundaries

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Overview

1. Boundary RG flows in 2d
   - Boundary entropy from relative entropy
   - Proof of the g-theorem
2. Constraints on higher dimensions
3. Sum rule
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Boundary RG flows in $2d$

Let us consider a BCFT with a boundary at $x_1 = 0$.
We perturb the UV theory $BCFT_{UV}$ by a set of relevant local operators localized at the boundary

$$S = S_{BCFT_{UV}} + \int dx_0 \lambda_i \phi_i(x_0)$$

This perturbation can combine operators from the bulk evaluated at $x_1 = 0$ with quantum mechanical DOF from the impurity. The perturbation trigers a boundary RG flow. We assume it ends on another $BCFT_{IR}$.

There is the intuition that DOF should decrease along this flow

Monotonicity theorems make precise this intuition

They are constructed as constraints on an observable treated as a function over the space of couplings

Friedan and Konechny's g-theorem

The boundary entropy $\log g$ is defined as the size independent term of the thermal entropy

$$s = \frac{c\pi}{3}\frac{L}{\beta} + \log g$$

If μ is the RG parameter.

$$\mu \frac{\partial \log g}{\partial \mu} \leq 0$$

This is equivalent to say that it decreases with the temperature since on dimensional grounds $g = g(\mu\beta)$.

At a fixed point, the thermal entropy can be mapped to an entanglement entropy via a conformal transformation

EE of a segment

The ground state EE of an interval $[0, r)$ attached to the boundary reads

$$S = \frac{c}{6} \log \frac{r}{\epsilon} + c_0 + \log g$$

So we know that the entropic $\log g_{UV} > \log g_{IR}$.

Away from the fixed points the EE can not be mapped to a thermal entropy

Does $\log g(r)$ decrease monotonically along the RG flow?

The relative entropy

The relative entropy between two density matrices is defined as

$$S_{rel}(\rho_1|\rho_0) = \mathrm{Tr}\left(\rho_1 \log \rho_1\right) - \mathrm{Tr}\left(\rho_1 \log \rho_0\right) = \Delta\langle H\rangle - \Delta S$$

with

$$\Delta\langle H\rangle = \mathrm{Tr}\left(\left(\rho_1 - \rho_0\right) H\right) , \qquad \rho_0 = \frac{e^{-H}}{\mathrm{Tr}\, e^{-H}}$$

and

$$\Delta S = S(\rho_1) - S(\rho_0) , \qquad S(\rho) = \mathrm{Tr}\, \rho \log \rho$$

Key property: For a fixed state, it cannot increase when we restrict to a subsystem

In QFT, we consider ρ_V obtained by tracing over the DOF living in the complement $\bar{V}$

S_{rel} increases as we increase the size of the region

Boundary entropy from relative entropy

We want to propose $\log g(r)$ as a g-function

$$S = \frac{c}{6}\log\frac{r}{\epsilon} + c_0 + \log g(r)$$

such that

- $\log g(r) \approx \log g_{UV}$ for $r \ll m^{-1}$
- $\log g(r) \approx \log g_{IR}$ for $r \gg m^{-1}$

We want to show $g'(r) \leq 0$ to get an entropic version of the g-theorem.
Choosing

- $\rho_1 \to \rho$, the BQFT vacua
- $\rho_1 \to$ the BCFT vacua

$$S_{rel}(\rho|\rho_0) = -\log\frac{g(r)}{g(0)} + \mathrm{Tr}\left[\left(\rho - \rho_0\right) H\right]$$

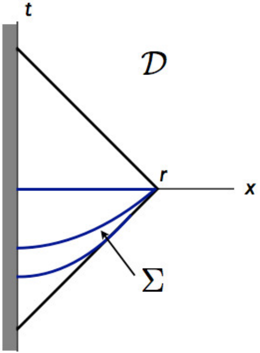
If we manage to kill the modular Hamiltonian contribution we would have an entropic g-theorem

Proof of the g-theorem

In a unitary theory, the EE is the same on any spatial surface.

This is because operators in a given Cauchy surface can be written in any other using causal eoms. Then the full operator algebra in any Cauchy surface will be the same, so the EE will be invariant.

$\Delta\langle H_{BCFT}\rangle$ will depend on the choice of Σ



The conformal vacuum will not change but the the fundamental state of the theory with the relevant perturbation will evolve with an aditional insertion at $x_1 = 0$

H_{BCFT}

The modular Hamiltonian of a BCFT is a flux of a conserved current

$$H_{BCFT} = \int_{\Sigma} ds \eta^{\mu} T_{\mu\nu} \xi^{\nu}$$

where η is the unit vector normal to Σ and

$$\xi^{\mu} = \frac{\pi}{r} \left(r^2 - x_0^2 - x_1^2, -2x_0 x_1 \right)$$

is the Killing vector for a conformal transformation that keeps the interval fixed.

We must now choose Σ such that $\Delta \langle H_{BCFT} \rangle = 0$

Choosing Σ

For a space interval such that $\Sigma \to (t = 0, x)$ we have

$$H_{BCFT} = \pi \int_0^r \frac{r^2 - x_1^2}{r} T_{00}$$

The conservation and tracelessness of $T_{\mu\nu}$ away from the defect constraints

$$\Delta \langle T_{00} \rangle \approx \lambda^2 \epsilon^{1-2\Delta} \delta(x_1)$$

so

$$\Delta \langle H \rangle \approx r$$

Ruining a possible g-theorem

In physical terms, one is placing the impurity at a point with an effective low temperature r^{-1}. As a result the states are highly distinguishable giving a high S_{rel}

Null Σ

On the null boundary of the causal development

$$H_{BCFT} = \pi \int_{-r}^{0} \frac{r^2 - x_+^2}{r} T_{++}$$

with

$$\Delta \langle T_{00} \rangle \approx \delta(x^+ + r)$$

So in the null segment the contribution of $\Delta\langle H \rangle$ vanishes

$$S_{rel}(\rho|\rho_0) = -\log \frac{g(r)}{g(0)}$$

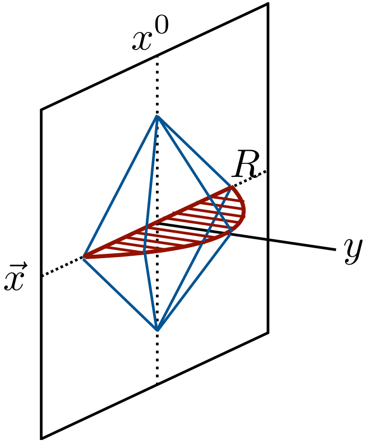
This is because

$$\xi^\alpha \eta_\alpha \Big|_{\Sigma_{null}} \to 0 \,.$$

The proof on higher dimensions

One can extend the previous proof to higher d QFTs with boundaries. This is done by considering a semi-sphere attached to the boundary.

Again we can kill the contribution from the modular hamiltonian by taking the null limit



EE on spheres

The sphere entropy in a CFT without boundary is

$$S(R) = \mu_{d-2}\, R^{d-2} + \mu_{d-4}\, R^{d-4} + \ldots + \begin{cases} (-)^{\frac{d-2}{2}} 4\, A \log(R/\epsilon) & d \text{ even}\,. \\ (-)^{\frac{d-1}{2}} F & d \text{ odd}\,. \end{cases}$$

The coefficients $\mu_k \sim \epsilon^{-k}$ are proportional to inverse powers of the cutoff.

The divergent terms can be written as integrals of local geometric quantities on the entangling surface that respect Lorentz invariance.

The nonlocal contribution, the logarithmic term in even d, comes from a Wess-Zumino action on the surface.

EE on half-spheres

Moving on to the BCFT case, the entangling surface intersects the boundary on a $d-3$ sphere of radius R.

This will give rise to new divergent terms in the EE, which can be written as integrals of local geometric quantities on this S^{d-3} that respect Lorentz invariance.
For d even, we have

$$S(R) = \mu_{d-2}\, R^{d-2} + \tilde{\mu}_{d-3}\, R^{d-3} + \ldots + (-1)^{\frac{d-2}{2}} 4\, A \log(R/\epsilon) + (-1)^{\frac{d-2}{2}}\, \tilde{F}$$

For d odd, we expect

$$S(R) = \mu_{d-2}\, R^{d-2} + \tilde{\mu}_{d-3}\, R^{d-3} + \ldots + (-1)^{\frac{d-3}{2}} 4\, \tilde{A} \log(R/\epsilon) + (-1)^{\frac{d-1}{2}}\, F\,.$$

The logarithmic term, absent for CFTs in odd dimensions, comes from the Wess-Zumino action on S^{d-3}. This should correspond to a Weyl anomaly in the BCFT.

Consequences in $d = 3$

In $d = 3$ we have the fixed point expression

$$S(\rho_0) = \mu_1 \frac{R}{\epsilon} - F + \frac{b_{UV}}{3} \log \frac{R}{\epsilon}$$

where $b_{UV}/12$ is the boundary central charge. The entropy $S(\rho)$ has a more complicated radial dependence along the RG flow, but near the IR fixed point,

$$S(\rho) = \mu_1 \frac{R}{\epsilon} - F + \frac{b_{IR}}{3} \log(mR) - \frac{b_{UV}}{3} \log(m\epsilon)$$

Here $m \sim \lambda^{1/(d-1-\Delta)}$ is a typical mass scale for the RG flow. Then the relative entropy for $mR \gg 1$ becomes

$$S(\rho|\rho_0) \approx \frac{1}{3}(b_{UV} - b_{IR}) \log(mR)$$

The boundary b-anomaly decreases along boundary RG flow

Consequences in higher d

In $d > 3$, for long distances $mR \gg 1$, we have

$$\Delta S = (\tilde{\mu}^{IR}_{d-3} - \tilde{\mu}^{UV}_{d-3})\, R^{d-3} + \dots$$

so that

$$\tilde{\mu}^{UV}_{d-3} \geq \tilde{\mu}^{IR}_{d-3} \,.$$

> The leading area term associated to the boundary can only decrease along boundary RG flows

The flow in the area term in relativistic QFTs is related to the renormalization of Newton's constant. This may be relevant in theories of localized gravity.

The large semi-spheres limit

There is still an alternative derivation that leads to a sum rule.

Consider a half-sphere of radius $R \to \infty$. The entangling region approaches a Rindler wedge along (say) $x^1 \geq 0$. The boundary preserves boosts along x^1, so the modular Hamiltonian is given by

$$H = -2\pi \int_{w^1 \geq 0 \,,\, w^{d-1} \geq 0} d^{d-1}\vec{w}\, w^1\, T_{00}(w)\,,$$

for any boundary QFT. Note that here $w^1 = 0$ is the Rindler edge, while $w^{d-1} = 0$ is the position of the boundary.
Taking $R \to \infty$, $S(\rho)$ is dominated by the IR fixed point, and hence

$$\Delta S = S(\rho) - S(\rho_0) \approx (\tilde{\mu}^{IR}_{d-3} - \tilde{\mu}^{UV}_{d-3})\, R^{d-3}$$

The change $\Delta\tilde{\mu}_{d-3}$ can be obtained by performing a small variation of R,

$$R\,\frac{d\Delta S}{dR} = (d-3)\,\Delta S\,.$$

The sum rule

Under a small change of state $\delta\rho$, the first law allows to relate the variation in the entropy to the change in the modular hamiltonian

As we just did a dilatation

$$\Delta S = \frac{1}{d-3}\int d^d x\,\langle \Theta_\rho(x) H_\rho \rangle - \frac{1}{d-3}\int d^d x\,\langle \Theta_{\rho_0}(x) H_{\rho_0} \rangle$$

Using the conservation and tracelessness of $T_{\mu\nu}$ away from the defect we arrive to a sum rule

$$\Delta\tilde{\mu}_{d-3} = -\frac{2\pi}{(d-1)(d-2)(d-3)}\int_{\partial M} d^{d-1}x\, x^2\,\langle \theta(x)\theta(0) \rangle$$

where θ is the trace of the energy momentum tensor of the boundary theory.

Future directions

The Renyi relative entropies

The relative entropy is part of a family of distance measures defined as

$$S_\alpha(\rho|\sigma) = -\frac{1}{1-\alpha} \log \mathrm{Tr} \left(\sigma^{\frac{1-\alpha}{2\alpha}} \rho \sigma^{\frac{1-\alpha}{2\alpha}}\right)^\alpha$$

- $\alpha = 1$ corresponds to the relative entropy.
- They all satisfy that the increase as we increase the size of the algebra.
- $\alpha = 1/2$ corresponds to the fidelity $F(\rho, \sigma) = \max_{|\psi\rangle, |\phi\rangle} |\langle\psi|\phi\rangle|$

Can we further constraint boundary RG flows using these?

See 1807.03305 for some preliminary results.

Future directions II

Strong subaditivity of the EE

The relative entropy seem to constraint just the leading order terms of the boundary entropy. Can we use the strong subaditivity of the EE to constraint the subleading terms?

In $d > 3$ these terms are not universal. They depoend on the choice of the UV cut-off.

Can we use the strong subaditivity of the EE to constraint the subleading terms?

Thanks!