Resurgence and Non-Perturbative Physics

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Non-Perturbative Methods in Quantum Field Theory
Abdus Salam ICTP, Trieste, September 3-6, 2019

GD & Mithat Ünsal, review: 1603.04924
A. Ahmed & GD: arXiv:1710.01812
GD, arXiv:1901.02076

O.Costin & GD, 1904.11593, ...

[DOE Division of High Energy Physics]

Physical Motivation

- non-perturbative definition of QFT
- Minkowski vs. Euclidean QFT
- "sign problem" in finite density QFT
- dynamical & non-equilibrium physics in path integrals
- phase transitions (Lee-Yang and Fisher zeroes)
- common thread: analytic continuation of path integrals

- question: does resurgence give (useful) new insight?

Physical Motivation

what does a Minkowski path integral mean, computationally?

$$\int \mathcal{D}A \, \exp\left(\frac{i}{\hbar} S[A]\right) \quad \text{versus} \quad \int \mathcal{D}A \, \exp\left(-\frac{1}{\hbar} S[A]\right)$$

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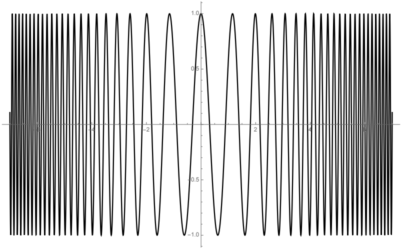
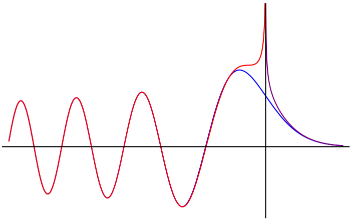




$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\left(\frac{1}{3}\, t^3 + x\, t\right)}\, dt \sim \begin{cases} \dfrac{e^{-\frac{2}{3}\, x^{3/2}}}{2\sqrt{\pi}\, x^{1/4}} & , \quad x \to +\infty \\[2em] \dfrac{\sin\left(\frac{2}{3}\, (-x)^{3/2} + \frac{\pi}{4}\right)}{\sqrt{\pi}\, (-x)^{1/4}} & , \quad x \to -\infty \end{cases}$$

- massive cancellations $\Rightarrow$ $\qquad \mathrm{Ai}(+5) \approx 10^{-4}$

Physical Motivation

- what does a Minkowski space path integral mean?

$$\int \mathcal{D}A \, \exp\left(\frac{i}{\hbar} S[A]\right) \quad \text{versus} \quad \int \mathcal{D}A \, \exp\left(-\frac{1}{\hbar} S[A]\right)$$

- finite dimensions: Stokes/Airy paradigm

- since we need complex analysis and contour deformation to make sense of oscillatory ordinary integrals, it is natural to explore similar methods for path integrals

- Question: can resurgence and Picard-Lefschetz theory be used to tame this long-standing problem?

- phase transition = change of dominant saddle (complex)

Resurgence from Mathematics

Resurgence: 'new' idea in mathematics

(Écalle 1980; Dingle 1960s; Stokes 1850)

resurgence = unification of perturbation theory and non-perturbative physics

resurgence = global complex analysis with asymptotic series

- perturbative series expansion $\longrightarrow$ *trans-series* expansion
- trans-series 'well-defined under analytic continuation'
- non-perturbative saddle expansions are potentially exact
- perturbative and non-perturbative physics entwined
- ODEs, PDEs, difference equations, fluid mechanics, QM, Matrix Models, QFT, Chern-Simons, String Theory, ...
- define the path integral constructively as a trans-series

Resurgence: Implications for QFT

- the physics message from Écalle's resurgence theory: different critical points are related in subtle and powerful ways

The Big Question

- Can we make physical, mathematical and computational sense of a Lefschetz thimble expansion of a path integral?

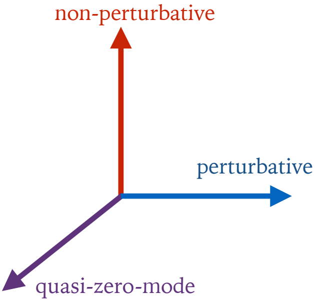
$$Z(\hbar) = \int \mathcal{D}A \, \exp\left(\frac{i}{\hbar} S[A]\right)$$

$$\text{"} = \text{"} \sum_{\text{thimble}} \mathcal{N}_{\text{th}} \, e^{i\,\phi_{\text{th}}} \int_{\text{th}} \mathcal{D}A \times (\mathcal{J}_{\text{th}}) \times \exp\left(\mathcal{R}e\left[\frac{i}{\hbar} S[A]\right]\right)$$

- $Z(\hbar) \to Z(\hbar, \text{masses}, \text{couplings}, \mu, T, B, \ldots)$
- $Z(\hbar) \to Z(\hbar, N)$, and $N \to \infty$ for a phase transition
- resurgence and Stokes transitions:
  metamorphosis/transmutation of trans-series structures across phase transitions

Decoding a Resurgent Trans-series in QFT

$$\int \mathcal{D}A \, e^{-\frac{1}{\hbar}S[A]} = \sum_{\text{all saddles}} e^{-\frac{1}{\hbar}S[A_{\text{saddle}}]} \times (\text{fluctuations}) \times (\text{qzm})$$



- expansions in different directions are <u>quantitatively</u> related
- expansions about different saddles are <u>quantitatively</u> related

Resurgence: Preserving Analytic Continuation Properties

Stirling expansion for $\psi(x) = \frac{d}{dx} \ln \Gamma(x)$ is divergent

$$\psi(1+z) \sim \ln z + \frac{1}{2z} - \frac{1}{12z^2} + \frac{1}{120z^4} - \frac{1}{252z^6} + \dots + \frac{174611}{6600z^{20}} - \dots$$

- functional relation: $\psi(1+z) = \psi(z) + \frac{1}{z}$ ✓

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- functional relation: $\psi(1+z) = \psi(z) + \frac{1}{z}$ ✓
- reflection formula: $\psi(1+z) - \psi(1-z) = \frac{1}{z} - \pi \cot(\pi z)$

$$\Rightarrow \quad \mathrm{Im}\, \psi(1+iy) \sim -\frac{1}{2y} + \frac{\pi}{2} + \pi \sum_{k=1}^{\infty} e^{-2\pi k y}$$

“raw” asymptotics is inconsistent with analytic continuation

- resurgence: add infinite series of non-perturbative terms

"non-perturbative completion"

All-Orders Steepest Descents

Berry/Howls 1991: *hyperasymptotics*

- steepest descent contour integral thru n^{th} saddle point

$$I^{(n)}(g^2) = \int_{C_n} dz\, e^{-\frac{1}{g^2} f(z)} = \frac{1}{\sqrt{1/g^2}}\, e^{-\frac{1}{g^2} f_n}\, T^{(n)}(g^2)$$

- $T^{(n)}(g^2)$: beyond the usual Gaussian approximation

- asymptotic expansion of fluctuations about the saddle n:

$$T^{(n)}(g^2) \sim \sum_{r=0}^{\infty} T_r^{(n)}\, g^{2r}$$

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- $T^{(n)}(g^2)$: beyond the usual Gaussian approximation
- asymptotic expansion of fluctuations about the saddle n:

$$T^{(n)}(g^2) \sim \sum_{r=0}^{\infty} T_r^{(n)}\, g^{2r}$$

- <u>*universal*</u> resurgence relation ($F_{nm} \equiv f_m - f_n$):

$$T_r^{(n)} = \frac{(r-1)!}{2\pi\, i} \sum_m \frac{(-1)^{\gamma_{nm}}}{(F_{nm})^r} \left[T_0^{(m)} + \frac{F_{nm}}{(r-1)}\, T_1^{(m)} + \frac{(F_{nm})^2}{(r-1)(r-2)}\, T_2^{(m)} + \dots \right.$$

  - fluctuations about different saddles are explicitly related !

Resurgence: canonical example = Airy function

- expansions about the two saddles are explicitly related

$$a_n = \frac{\Gamma\left(n+\frac{1}{6}\right)\Gamma\left(n+\frac{5}{6}\right)}{(2\pi)\left(\frac{4}{3}\right)^n n!} = \left\{1, \frac{5}{48}, \frac{385}{4608}, \frac{85085}{663552}, \dots\right\}$$

- large order behavior:

$$a_n \sim \frac{(n-1)!}{(2\pi)\left(\frac{4}{3}\right)^n}\left(1 - \frac{5}{36}\frac{1}{n} + \frac{25}{2592}\frac{1}{n^2} - \dots\right)$$

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- re-express with factors of action difference

$$a_n \sim \frac{(n-1)!}{(2\pi)\left(\frac{4}{3}\right)^n}\left(1 - \left(\frac{4}{3}\right)\frac{5}{48}\frac{1}{(n-1)} + \left(\frac{4}{3}\right)^2\frac{385}{4608}\frac{1}{(n-1)(n-2)} - \dots\right)$$

generic Dingle/Berry/Howls large order/low order relation

- similar behavior in QM, matrix models; leading in QFT

...

Borel summation: extracting physics from asymptotic series

Borel transform of series, where $c_n \sim n! \quad , \quad n \to \infty$

$$f(g) \sim \sum_{n=0}^{\infty} c_n \, g^n \quad \longrightarrow \quad \mathcal{B}[f](t) = \sum_{n=0}^{\infty} \frac{c_n}{n!} t^n$$

new series typically has a finite radius of convergence

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Borel summation of original asymptotic series:

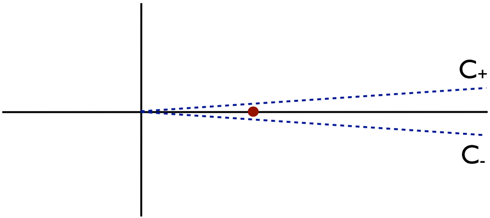
$$\mathcal{S} f(g) = \frac{1}{g} \int_0^{\infty} \mathcal{B}[f](t) e^{-t/g} dt$$

• the singularities of $\mathcal{B}[f](t)$ provide a physical encoding of the global asymptotic behavior of $f(g)$, which is also much more mathematically efficient than the asymptotic series

Borel singularities

Borel transform typically has singularities:
directional Borel sums:

$$\mathcal{S}_\theta f(g) = \frac{1}{g}\int_0^{e^{i\theta}\infty} \mathcal{B}[f](t)e^{-t/g}dt$$



- Borel singularities $\leftrightarrow$ non-perturbative physical objects
- resurgence: isolated poles, algebraic & logarithmic cuts
- "Borel plane is more physical than the physical plane"

Resurgence: canonical example = Airy function

- formal large x solution to ODE $\equiv$ "perturbation theory"

$$y'' = x\, y \;\Rightarrow\; \left\{ \begin{matrix} 2\,\mathrm{Ai}(x) \\ \mathrm{Bi}(x) \end{matrix} \right\} \sim \frac{e^{\mp \frac{2}{3} x^{3/2}}}{2\pi^{3/2}\, x^{1/4}} \sum_{n=0}^{\infty} (\mp 1)^n \frac{\Gamma\left(n+\frac{1}{6}\right)\Gamma\left(n+\frac{5}{6}\right)}{n!\, \left(\frac{4}{3}\, x^{3/2}\right)^n}$$

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- non-perturbative connection formula:

$$\mathrm{Ai}\left(e^{\mp\frac{2\pi i}{3}}\,x\right) = \pm\frac{i}{2}e^{\mp\frac{\pi i}{3}}\mathrm{Bi}\,(x) + \frac{1}{2}e^{\mp\frac{\pi i}{3}}\mathrm{Ai}\,(x)$$

- how do we recover this non-pert. result from the series?

Resurgence: canonical example = Airy function

- Borel sum of the Ai(x) series factor:

$$\sum_{n=0}^{\infty} (-1)^n \frac{\Gamma\left(n+\frac{1}{6}\right)\Gamma\left(n+\frac{5}{6}\right)}{n!}\frac{t^n}{n!} = {}_2F_1\left(\frac{1}{6}, \frac{5}{6}, 1; -t\right)$$

- inverse transform recovers the Ai(x) formal series:

$$Z(x) = \frac{4}{3}x^{3/2}\int_0^\infty dt\, e^{-\frac{4}{3}x^{3/2}t}\; {}_2F_1\left(\frac{1}{6}, \frac{5}{6}, 1; -t\right)$$

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- cut for $t \in (-\infty,-1]$: rotate t contour as x rotates

$${}_2F_1\left(\frac{1}{6},\frac{5}{6},1;t+i\,\epsilon\right) - {}_2F_1\left(\frac{1}{6},\frac{5}{6},1;t-i\,\epsilon\right) = i\ {}_2F_1\left(\frac{1}{6},\frac{5}{6},1;1-t\right)$$

- discontinuity across cut $\Rightarrow$ non-pert. connection formula

$$Z\left(e^{\frac{2\pi i}{3}}\,x\right) - Z\left(e^{-\frac{2\pi i}{3}}\,x\right) = i\,e^{-\frac{4}{3}x^{3/2}}Z\left(x\right)$$

Resurgence: canonical example = Airy function

"path integral"

$$\mathrm{Ai}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, e^{i\left(x\,t+\frac{t^3}{3}\right)} = \frac{\sqrt{r}}{2\pi i}\int_{-i\infty}^{+i\infty} dz\, e^{r^{3/2}\left(e^{i\theta}\,z-\frac{z^3}{3}\right)}$$

- we have written $x \equiv r\, e^{i\theta}$, $t \equiv -i\sqrt{r}z$

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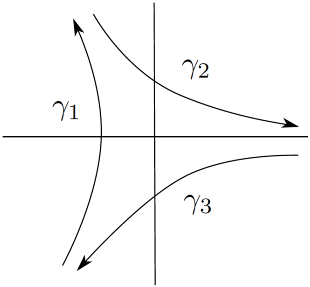
- we have written $x \equiv r\,e^{i\theta}$, $t \equiv -i\sqrt{r}z$
- basis of allowed z-plane contours

$$\mathrm{Ai}(x) = \frac{\sqrt{r}}{2\pi i}\int_{\gamma_k} dz\, e^{r^{3/2}\left(e^{i\theta}\,z-\frac{z^3}{3}\right)}$$



Resurgence: canonical example = Airy function

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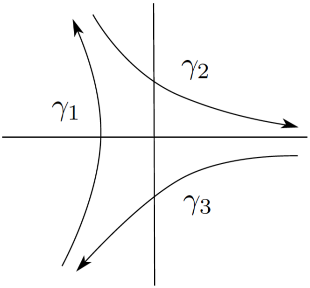
[recall: $x \equiv r\, e^{i\,\theta}$]



- saddles at $z = \pm e^{i\theta/2}$
- saddle exponent ($\equiv$ "action") $= \pm\frac{2}{3} r^{3/2} e^{3i\theta/2}$

$x > 0 \Rightarrow \theta = 0 \Rightarrow$ contour through only 1 saddle $(z = -1)$
$\Rightarrow$ action $= -\frac{2}{3} r^{3/2} = -\frac{2}{3} x^{3/2}$

$x < 0 \Rightarrow \theta = \pm\pi \Rightarrow$ contour through 2 saddles $(z = \pm i)$
$\Rightarrow$ action $= \pm i\frac{2}{3} r^{3/2} = \pm i\frac{2}{3}(-x)^{3/2}$

Resurgence: canonical example = Airy function

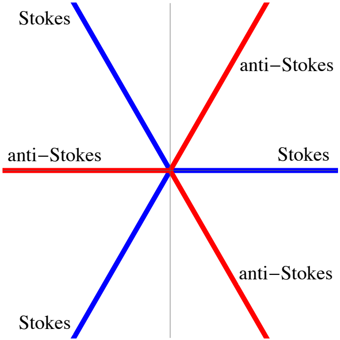
$$\mathrm{Ai}(x) = \frac{\sqrt{r}}{2\pi i} \int_{\gamma_k} dz\, e^{r^{3/2}\left(e^{i\theta}\, z - \frac{z^3}{3}\right)}$$

- saddles at $z = \pm e^{i\theta/2}$, action $= \pm\frac{2}{3} r^{3/2} e^{3i\theta/2}$
- real action when $\theta = 0, \pm\frac{2\pi}{3}$: "Stokes lines"
- imaginary action when $\theta = \pi, \pm\frac{\pi}{3}$: "anti-Stokes lines"

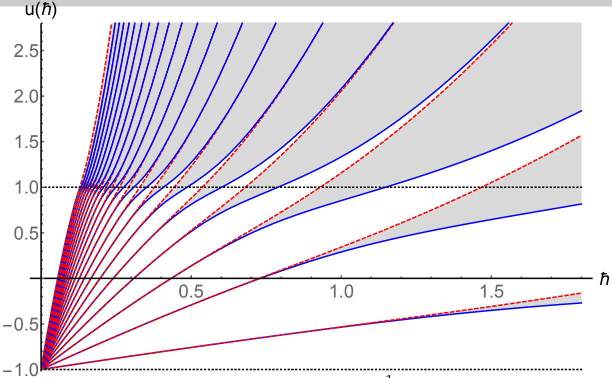
Stokes lines in complex x-plane

$$x = r\, e^{i\,\theta}$$

moral: keep track of **both** saddle contributions as we analytically continue in complex x plane



Mathieu Equation Spectrum: $-\frac{\hbar^2}{2}\frac{d^2\psi}{dx^2}+\cos(x)\,\psi = u\,\psi$



$$u_\pm(\hbar,N) = u_{\rm pert}(\hbar,N) \pm \frac{\hbar}{\sqrt{2\pi}}\frac{1}{N!}\left(\frac{32}{\hbar}\right)^{N+\frac{1}{2}} \exp\left[-\frac{8}{\hbar}\right]\mathcal{P}_{\rm inst}(\hbar,N)+\dots$$

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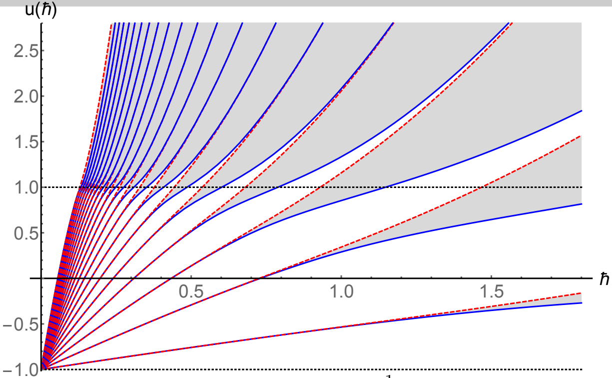


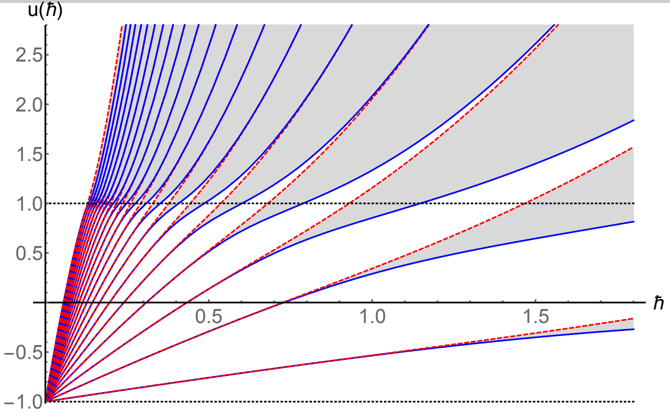
$$u_\pm(\hbar,N)=u_{\rm pert}(\hbar,N)\pm\frac{\hbar}{\sqrt{2\pi}}\frac{1}{N!}\left(\frac{32}{\hbar}\right)^{N+\frac{1}{2}}\exp\left[-\frac{8}{\hbar}\right]\mathcal{P}_{\rm inst}(\hbar,N)+\dots$$

$$\mathcal{P}_{\rm inst}(\hbar,N)=\frac{\partial u_{\rm pert}(\hbar,N)}{\partial N}\exp\left[S\int_0^{\hbar}\frac{d\hbar}{\hbar^3}\left(\frac{\partial u_{\rm pert}(\hbar,N)}{\partial N}-\hbar+\frac{\left(N+\frac{1}{2}\right)\hbar^2}{S}\right)\right]$$

all non-perturbative effects encoded in perturbative expansion

GD & Ünsal (2013); Başar, GD & Ünsal (2017): applies to bands & gaps

Mathieu Equation Spectrum: $-\frac{\hbar^2}{2}\frac{d^2\psi}{dx^2} + \cos(x)\,\psi = u\,\psi$



- phase transition at $\hbar N = \frac{8}{\pi}$: narrow bands vs. narrow gaps
- real vs. complex instantons (Dykhne, 1961; Başar/GD)
- phase transition = "instanton condensation"
- mapping to $\mathcal{N} = 2$ SUSY QFT (Nekrasov et al, Mironov et al)

QM: divergence of perturbation theory is due to factorial growth of number of Feynman diagrams

$$c_n \sim (\pm 1)^n \frac{n!}{(2S)^n}$$

QFT: new physical effects occur, due to running of couplings with the momentum scale

- faster source of divergence: "renormalons"

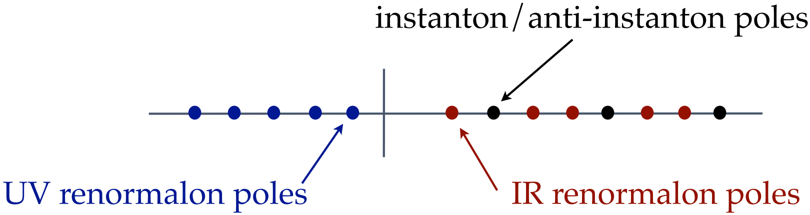
$$c_n \sim (\pm 1)^n \frac{\beta_0^n \, n!}{(2S)^n} = (\pm 1)^n \frac{n!}{(2S/\beta_0)^n}$$

- both positive and negative Borel poles

IR Renormalon Puzzle in Asymptotically Free QFT

Borel sum of perturbation theory: $\rightarrow \quad \pm i \exp\left[-\frac{2S}{\beta_0\, g^2}\right]$

non-perturbative instanton gas: $\rightarrow \quad \pm i \exp\left[-\frac{2S}{g^2}\right]$



appears that Bogomolny/Zinn-Justin cancellation cannot occur

asymptotically free theories remain perturbatively inconsistent

't Hooft, 1980; David, 1981

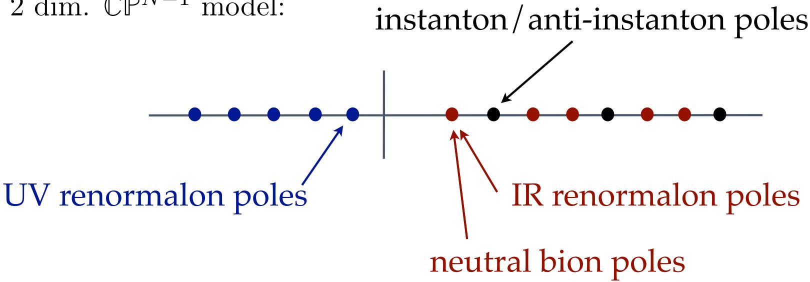
IR Renormalon Puzzle in Asymptotically Free QFT

resolution: there is another problem with the non-perturbative instanton gas analysis (Argyres, Ünsal 1206.1890; GD, Ünsal, 1210.2423)

- scale modulus of instantons

- spatial compactification with $\mathbb{Z}_N$ twisted b.c.'s, & principle of adiabatic continuity

- 2 dim. $\mathbb{CP}^{N-1}$ model:

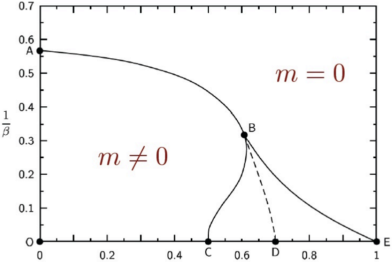


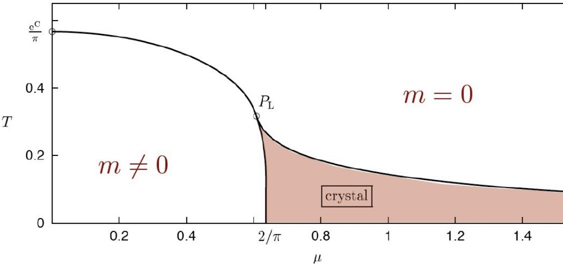
cancellation occurs ! (GD, Ünsal, 1210.2423, 1210.3646)

Phase Transition in 1+1 dim. Gross-Neveu Model

$$\mathcal{L} = \bar{\psi}_a i \partial\!\!\!/ \psi_a + \frac{g^2}{2} \left(\bar{\psi}_a \psi_a\right)^2$$

- asymptotically free; dynamical mass; chiral symmetry
- large N_f chiral symmetry breaking phase transition
- physics = (relativistic) Peierls instability in 1 dimension





saddles from inhomogeneous gap eqn. (Basar, GD, Thies, 2011)

$$\sigma(x;T,\mu) = \frac{\delta}{\delta\sigma(x;T,\mu)} \ln\det\left(i\,\partial\!\!\!/ - \sigma(x;T,\mu)\right)$$

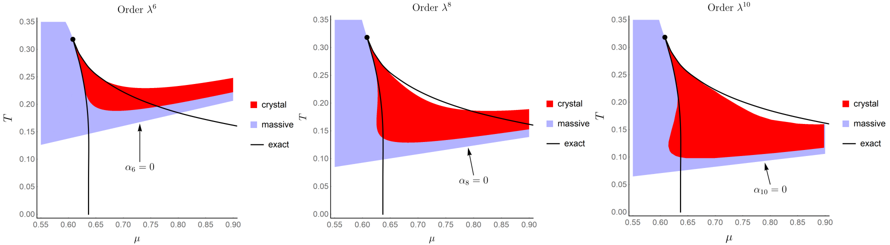
Phase Transition in 1+1 dim. Gross-Neveu Model

- thermodynamic potential

$$\Psi[\sigma;T,\mu] = -T\int dE\,\rho(E)\,\ln\left(1+e^{-(E-\mu)/T}\right)$$
$$= \sum_n \alpha_n(T,\mu) f_n[\sigma(x;T,\mu)]$$

- (divergent) Ginzburg-Landau expansion = mKdV
- saddles: $\sigma(x) = \lambda\,\mathrm{sn}(\lambda\,x;\nu)$
- successive orders of GL expansion reveal the full crystal phase

(Basar, GD, Thies, 2011; Ahmed, 2018)



Phase Transition in 1+1 dim. Gross-Neveu Model

- most difficult point: $\mu_c = \frac{2}{\pi}$, $T = 0$
- high density expansion at $T = 0$: (convergent !)

$$\mathcal{E}(\rho) \sim \frac{\pi}{2}\,\rho^2 \left(1 - \frac{1}{32(\pi\rho)^4} + \frac{3}{8192(\pi\rho)^8} - \dots\right)$$

- low density expansion at $T = 0$: (non-perturbative !)

$$\mathcal{E}(\rho) \sim -\frac{1}{4\pi} + \frac{2\rho}{\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty} \frac{e^{-k/\rho}}{\rho^{k-2}} \mathcal{F}_{k-1}(\rho)$$

(GD, 2018)

- resurgent trans-series
- analogous expansions at fixed T/μ

Phase Transitions and Painlevé VI

- Painlevé I-VI: universal "nonlinear special functions"
- Painlevé VI: Ising diagonal correlators; twistor geometry
- 3 regular points: $0, 1, \infty$; convergent expansions (Jimbo)
- coalescence $\rightarrow$ other Painlevé eqs; irregular points
- scaling limits: $N \rightarrow \infty$ & $T \rightarrow T_c$: PVI $\rightarrow$ PIII (McCoy et al; Jimbo)

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- coalescence $\to$ other Painlevé eqs; irregular points
- scaling limits: $N \to \infty$ & $T \to T_c$: PVI $\to$ PIII (McCoy et al; Jimbo)
- convergent and resurgent (!) conformal block expansions at high and low T (Jimbo; Lisovyy et al; Bonelli et al; GD) (Painlevé I: Eynard et al; Iwaki)

$$\tau(t) \sim \sum_{n=-\infty}^{\infty} s^n \, C(\vec{\theta}, \sigma + n) \, \mathcal{B}(\vec{\theta}, \sigma + n; t)$$

$$\mathcal{B}(\vec{\theta}, \sigma; t) \propto t^{\sigma^2} \sum_{\lambda, \mu \in \mathcal{Y}} \mathcal{B}_{\lambda,\mu}(\vec{\theta}, \sigma) t^{|\lambda| + |\mu|}$$

Other Examples: Phase Transitions

- particle-on-circle (Schulman PhD thesis 1968):
sum over spectrum versus sum over winding (saddles)
- Bose gas (Cristoforetti et al, Alexandru et al)
- Thirring model (Alexandru et al)
- Hubbard model (Tanizaki et al; ...)
- Hydrodynamics: short/late-time (Heller et al; Aniceto et al; Basar/GD)
- Large N matrix models (Mariño, Schiappa, Couso, Russo, ...)
- Painlevé (Jimbo et al; Its et al; Lisovyy et al; Litvinov et al; Costin, GD)
- Gross-Witten-Wadia model (Mariño; Ahmed, GD)
- ...

Gross-Witten-Wadia Unitary Matrix Model

$$Z(g^2, N) = \int_{U(N)} DU \, \exp\left[\frac{1}{g^2} \mathrm{tr}\left(U + U^\dagger\right)\right]$$

- one-plaquette matrix model for 2d lattice Yang-Mills
- two variables: g^2 and N ('t Hooft coupling: $t \equiv g^2 N/2$)
- 3rd order phase transition at $N = \infty$, $t = 1$ (universal!)
- double-scaling limit: Painlevé II
- physics of phase transition = condensation of instantons
- random matrix theory/orthogonal polynomials result:

$$Z(g^2, N) = \det\left(I_{j-k}(x)\right)_{j,k=1,\ldots N} \qquad , \qquad x \equiv \frac{2}{g^2}$$

Gross-Witten-Wadia $N = \infty$ Phase Transition

3rd order transition: kink in the specific heat

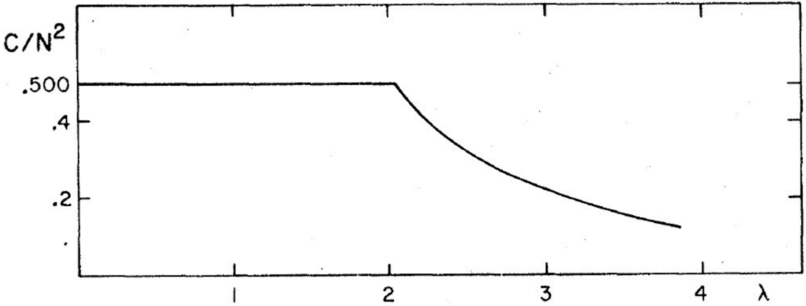

FIG. 2. The specific heat per degree of freedom, C/N^2, as a function of λ (temperature).

D. Gross, E. Witten, 1980

- what about non-perturbative large N effects?

Resurgence in Gross-Witten-Wadia Model: Transmutation of the Trans-series

Ahmed & GD: 1710.01812

- "order parameter":

$$\Delta(x,N) \equiv \langle \det U \rangle = \frac{\det\left[I_{j-k+1}(x)\right]_{j,k=1,\dots,N}}{\det\left[I_{j-k}(x)\right]_{j,k=1,\dots,N}}$$

- for any N, $\Delta(x,N)$ satisfies a Painlevé III equation:

$$\Delta'' + \frac{1}{x}\Delta' + \Delta\left(1-\Delta^2\right) + \frac{\Delta}{(1-\Delta^2)}\left[\left(\Delta'\right)^2 - \frac{N^2}{x^2}\right] = 0$$

- weak-coupling expansion is a divergent series:
  $\rightarrow$ trans-series non-perturbative completion

- strong-coupling expansion is a convergent series:
  but it still has a non-perturbative completion !

- N is a parameter; large N limit by rescaling: $t = \frac{N}{x}$

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- for any N, $\Delta(t, N)$ satisfies a Painlevé III equation:

$$t^2 \Delta'' + t\Delta' + \frac{N^2 \Delta}{t^2}\left(1 - \Delta^2\right) = \frac{\Delta}{1-\Delta^2}\left(N^2 - t^2 \left(\Delta'\right)^2\right)$$

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  → trans-series non-perturbative completion

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  but it still has a non-perturbative completion !

- N is a parameter

Resurgence: Large N 't Hooft limit at Weak Coupling

- large N trans-series at weak-coupling ($t \equiv N/x < 1$)

$$\Delta(t, N) \sim \sqrt{1-t} \sum_{n=0}^{\infty} \frac{d_n^{(0)}(t)}{N^{2n}} - \frac{i}{2\sqrt{2\pi N}} \, \sigma_{\text{weak}} \frac{t \, e^{-N S_{\text{weak}}(t)}}{(1-t)^{1/4}} \sum_{n=0}^{\infty} \frac{d_n^{(1)}(t)}{N^n} + \dots$$

- large N weak-coupling action

$$S_{\text{weak}}(t) = \frac{2\sqrt{1-t}}{t} - 2 \, \text{arctanh}\left(\sqrt{1-t}\right)$$

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- large N weak-coupling action

$$S_{\rm weak}(t) = \frac{2\sqrt{1-t}}{t} - 2\,{\rm arctanh}\left(\sqrt{1-t}\right)$$

- large-order growth of perturbative coefficients ($\forall\, t < 1$):

$$d_n^{(0)}(t) \sim \frac{-1}{\sqrt{2}(1-t)^{3/4}\pi^{3/2}}\,\frac{\Gamma(2n-\frac{5}{2})}{\left(S_{\rm weak}(t)\right)^{2n-\frac{5}{2}}}\left[1 + \frac{(3t^2-12t-8)}{96(1-t)^{3/2}}\,\frac{S_{\rm weak}(t)}{(2n-\frac{7}{2})} + \dots\right.$$

- (parametric) resurgence relations, for all t:

$$\sum_{n=0}^{\infty}\frac{d_n^{(1)}(t)}{N^n} = 1 + \frac{(3t^2-12t-8)}{96(1-t)^{3/2}}\,\frac{1}{N} + \dots$$

Resurgence: Large N 't Hooft limit at Strong Coupling

- large N transseries at strong-coupling: $\Delta(t,N) \approx \sigma J_N\left(\frac{N}{t}\right)$

$$\Delta(t,N) = \sum_{k=1,3,5,\dots}^{\infty} (\sigma_{\text{strong}})^k \Delta_{(k)}(t,N)$$

  - "Debye expansion" for Bessel function: $J_N(N/t)$

$$\Delta(t,N) \sim \frac{\sqrt{t}\, e^{-N S_{\text{strong}}(t)}}{\sqrt{2\pi N}\,(t^2-1)^{1/4}} \sum_{n=0}^{\infty} \frac{U_n(t)}{N^n} + \frac{1}{4(t^2-1)} \left(\frac{\sqrt{t} e^{-N S_{\text{strong}}(t)}}{\sqrt{2\pi N}\,(t^2-1)^{1/4}} \right)^3 \sum_{n=0}^{\infty} \frac{U_n^{(1)}(t)}{N^n} + \dots$$

- large N strong-coupling action: $S_{\text{st}}(t) = \text{arccosh(t)} - \sqrt{1-\frac{1}{t^2}}$

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$$+\frac{1}{4(t^2-1)}\left(\frac{\sqrt{t} e^{-NS_{\text{strong}}(t)}}{\sqrt{2\pi N}\,(t^2-1)^{1/4}}\right)^3 \sum_{n=0}^{\infty} \frac{U_n^{(1)}(t)}{N^n} + \dots$$

- large N strong-coupling action: $S_{\text{st}}(t) = \text{arccosh(t)} - \sqrt{1-\frac{1}{t^2}}$

  - large-order/low-order (parametric) resurgence relations:

$$U_n(t) \sim \frac{(-1)^n (n-1)!}{2\pi(2S_{\text{strong}}(t))^n}\left(1 + U_1(t)\frac{(2S_{\text{strong}}(t))}{(n-1)} + U_2(t)\frac{(2S_{\text{strong}}(t))^2}{(n-1)(n-2)} + \right.$$

- resurgence suggests that local analysis of perturbation theory encodes global information

- Questions:
How much global information can be decoded from a FINITE number of perturbative coefficients ?
How much information is needed to see and to probe phase transitions ?

- resurgent functions have orderly structure in Borel plane
$\Rightarrow$ develop extrapolation and summation methods that take advantage of this!

- high precision test for Painlevé I (but integrability is not important for the method)

Perturbative Expansion of Painlevé I Equation

- Painlevé I equation

$$y''(x) = 6\, y^2(x) - x$$

- large x expansion:

$$y(x) \sim -\sqrt{\frac{x}{6}} \left(1 + \sum_{n=1}^{\infty} a_n \left(\frac{30}{(24\, x)^{5/4}} \right)^{2n} \right) \qquad , \qquad x \to +\infty$$

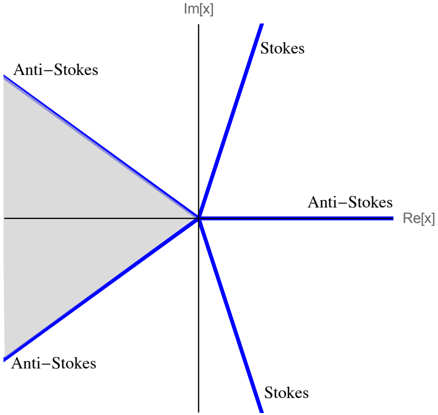
- perturbative input data: $\{a_1, a_2, \ldots, a_N\}$

$$\{\frac{4}{25}, -\frac{392}{625}, \frac{6\,272}{625}, -\frac{141\,196\,832}{390\,625}, \frac{9\,039\,055\,872}{390\,625}, \ldots, a_N\}$$

- this expansion defines the *tritronquée* solution to PI

Reconstruct global behavior from limited $x \to +\infty$ data?

- Painlevé I equation has inherent five-fold symmetry



- do our input coefficients (from $x = +\infty$) "know" this ?
- most interesting/difficult directions: phase transitions

High Precision at the Origin
O.Costin & GD, 1904.11593

- resurgence & Padé-Conformal-Borel transform
- "weak coupling to strong coupling" extrapolation
- $N = 50$ terms and Padé-Conformal-Borel input:

$y(0) \approx -0.18755430834049489383868175759583299323116090976213899693337265167\ldots$

$y'(0) \approx -0.30490556026122885653410412498848967640319991342112833650059344290\ldots$

$y''(0) \approx 0.21105971146248859499298968451861337073253247206264082468899143841\ldots$

$$\left[y''(x) - 6y^2(x) + x\right]_{x=0} = O(10^{-65})$$

- best numerical integration algorithms $\to \quad \approx O(10^{-14})$
- WHY?
- Resurgent extrapolation method encodes global information about the function throughout the entire complex plane, not just along the positive real axis.

Nonlinear Stokes Transition: the Tritronquée Pole Region

- Boutroux (1913): asymptotically, general Painlevé I solution has poles with 5-fold symmetry

- Dubrovin conjecture: *On universality of critical behavior in the focusing nonlinear Schrödinger equation, elliptic umbilic catastrophe and the tritronquée solution to the Painlevé-I equation* (2009): this asymptotic solution to Painlevé I only has poles in a $\frac{2\pi}{5}$ wedge, centered on the negative axis

- proof: Costin-Huang-Tanveer (2012)

Stokes Transition: Mapping the Tritronquée Pole Region

- non-linear Stokes transitions crossing $arg(x) = \pm\frac{4\pi}{5}$

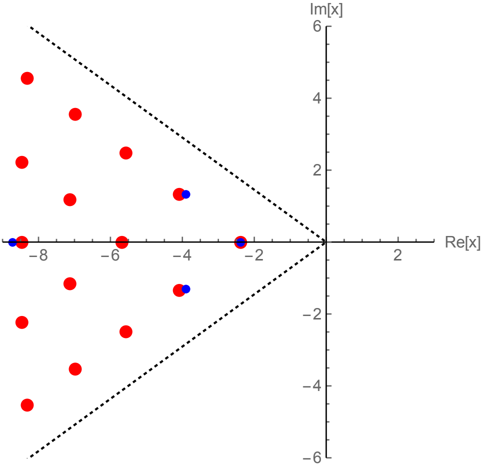


Figure: Complex poles: $N = 10$ (blue); $N = 50$ (red).

Metamorphosis: Asymptotic Series to Meromorphic Function

$$y(x) \approx \frac{1}{(x - {\color{red}x_{\rm pole}})^2} + \frac{x_{\rm pole}}{10}(x - x_{\rm pole})^2 + \frac{1}{6}(x - x_{\rm pole})^3$$
$$+{\color{red}h_{\rm pole}}(x - x_{\rm pole})^4 + \frac{x_{\rm pole}^2}{300}(x - x_{\rm pole})^6 + \ldots$$

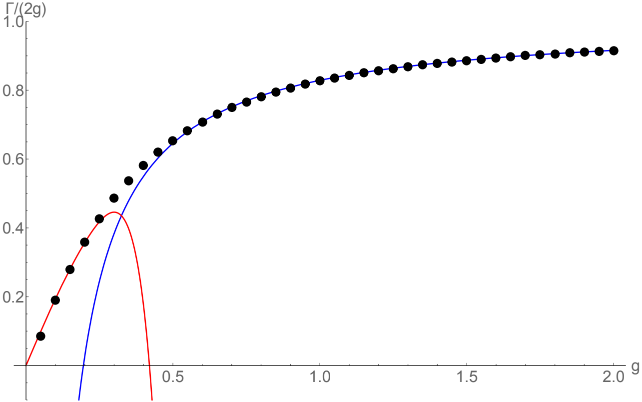
- our extrapolation ($y_N(x)$ with $N = 50$) near 1st pole:

$$y(x) \approx \frac{0.9999999999999999999999999999999999997886}{(x - x_1)^2}$$
$$+3.5 \times 10^{-35} - 2.4 \times 10^{-34}(x - x_1)$$
$${\color{red}-0.238416876956881663929914585244923803}(x - x_1)^2$$
$$+0.166666666666666666666666666666657864(x - x_1)^3$$
$${\color{red}-0.06213573922617764089649014164005140}(x - x_1)^4$$
$$+4 \times 10^{-31}(x - x_1)^5$$
$$+0.0189475357392909503157755851627665(x - x_1)^6 + \ldots$$

- estimate approx 30 digit precision for x_1 and h_1

Other Applications: Cusp-Anomalous Dimension in SYM

(previous: Aniceto; Dorigoni & Hatsuda)



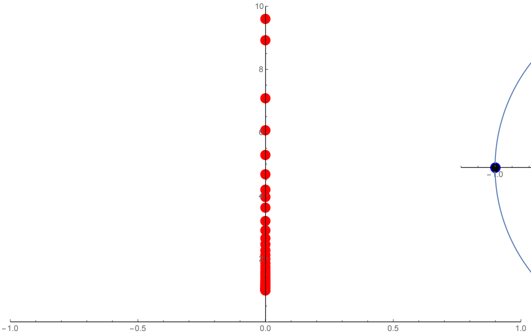
$$\Gamma(g) \approx 4g^2 \left[1 - \frac{\pi^2 g^2}{3} + \frac{11\pi^4 g^4}{45} - 2\left(\frac{73\pi^6}{630} - 4\zeta(3)^2\right) g^6 + \dots \right] \;, \; g \to 0$$

$$\Gamma(g) \sim 2g \left[1 - \frac{3\ln 2}{4\pi g} - \frac{K}{(4\pi g)^2} - \dots \right] \;, \; g \to \infty$$

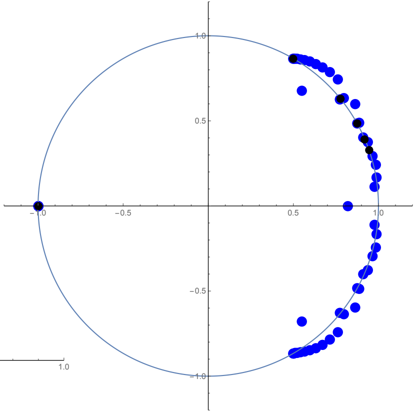
Other Applications: Complex Chern-Simons Theory

(compare: Gukov, Mariño, Putrov)

- Borel structure for Chern-Simons on Seifert $\Sigma(2,3,5)$



Padé-Borel poles



Conformal-Padé-Borel poles

Conclusions

- Resurgence systematically unifies perturbative and non-perturbative analysis, via trans-series, which ‘encode’ analytic continuation information
- QM, matrix models, differential/integral eqns ✓✓✓
- 2d sigma models ✓✓
- integrable/localizable SUSY QFT ✓✓
- 3d Chern-Simons theories ✓+
- numerical Lefschetz thimbles ✓+
- 4d QFT ✓???
- phase transitions ↔ Stokes phenomenon
- non-perturbative effects exist even for convergent series
- resurgent extrapolation: non-perturbative information can be decoded from surprisingly little perturbative data

Applicable resurgent asymptotics: towards a universal theory

Participation in INI programmes is by invitation only. Anyone wishing to apply to participate in the associated workshop(s) should use the relevant workshop application form.

Programme
4th January 2021 to 25th June 2021