Large charge: advanced applications

Domenico Orlando
INFN | Torino

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arXiv:1505.01537, arXiv:1610.04495, arXiv:1707.00711, arXiv:1804.01535, arXiv:1902.09542, arXiv:1905.00026 and more to come…

Who's who

S. Reffert, M. Watanabe (AEC Bern) [O. Loukas];
L. Alvarez Gaumé (CERN and SCGP);
F. Sannino (CP3-Origins)
D. Banerjee (DESY);
S. Chandrasekharan (Duke);
S. Hellerman (IPMU).

Why are we here? Conformal field theories are hard

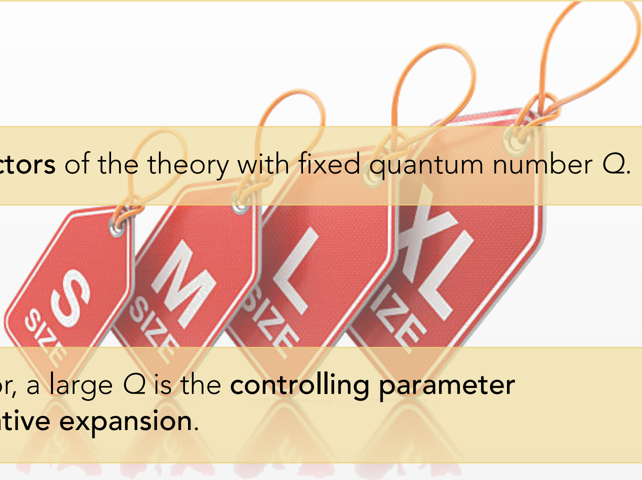
Most conformal field theories (CFTs) lack nice limits where they become simple and solvable.

No parameter of the theory can be dialed to a simplifying limit.

Why are we here? Conformal field theories are hard

In presence of a symmetry there can be sectors of the theory where anomalous dimension and OPE coefficients simplify.

The idea



Study **subsectors** of the theory with fixed quantum number Q.

In each sector, a large Q is the **controlling parameter** in a **perturbative expansion**.

no bootstrap here!

This approach is orthogonal to bootstrap.
We will use an effective action.
We will access sectors that are difficult to reach with bootstrap.
(However, arXiv:1710.11161).

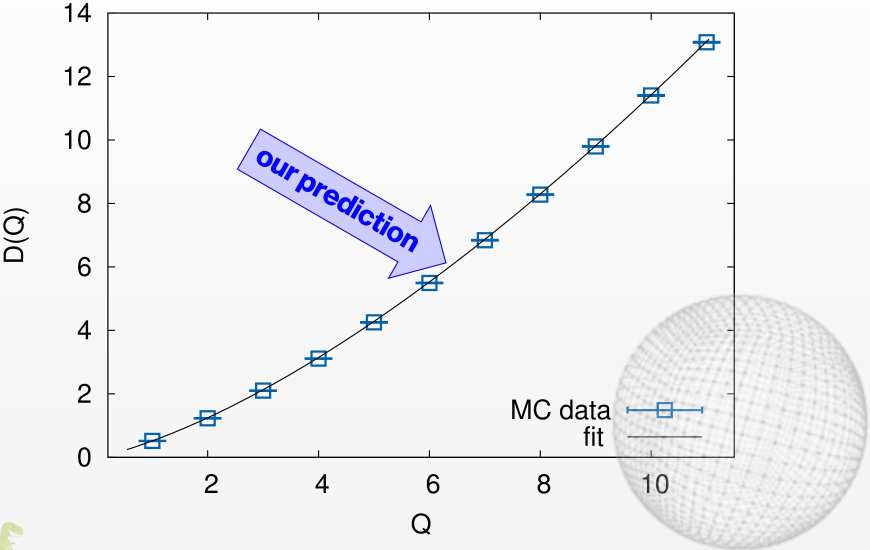
Concrete results

We consider the $O(N)$ vector model in three dimensions. In the IR it flows to a conformal fixed point Wilson & Fisher.

We find an explicit formula for the dimension of the lowest primary at fixed charge:

$$\Delta_Q = \frac{c_{3/2}}{2\sqrt{\pi}} Q^{3/2} + 2\sqrt{\pi}\, c_{1/2} Q^{1/2} - 0.094 + \mathcal{O}\left(Q^{-1/2}\right)$$

Summary of the results: O(2)



Scales

We want to write a Wilsonian effective action.

Choose a cutoff Λ, separate the fields into high and low frequency ϕ_H, ϕ_L and do the path integral over the high-frequency part:

$$e^{iS_\Lambda(\phi_L)} = \int \mathscr{D}\phi_H \, e^{iS(\phi_H,\phi_L)}$$

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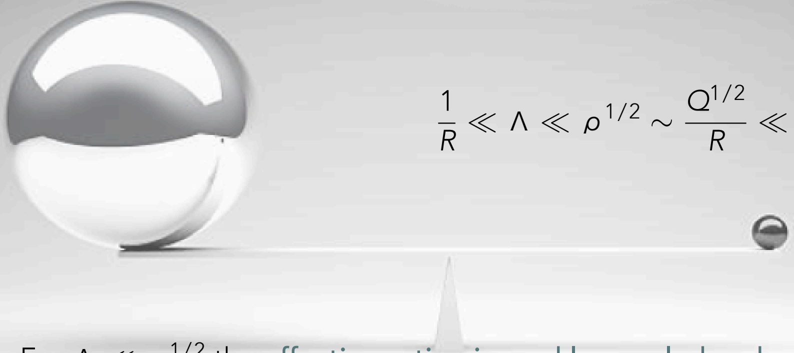
too hard

Scales

- We look at a finite box of typical length R
- The $U(1)$ charge Q fixes a second scale $\rho^{1/2} \sim Q^{1/2}/R$



$$\frac{1}{R} \ll \Lambda \ll \rho^{1/2} \sim \frac{Q^{1/2}}{R} \ll \Lambda_{UV}$$

For $\Lambda \ll \rho^{1/2}$ the effective action is weakly coupled and under perturbative control in powers of ρ^{-1}.

Wilsonian action

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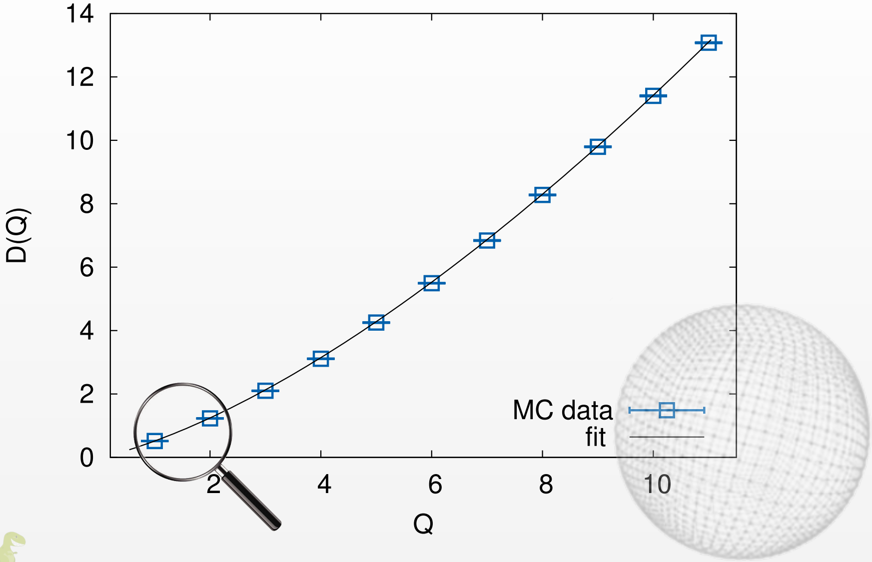
superstition

Too good to be true?

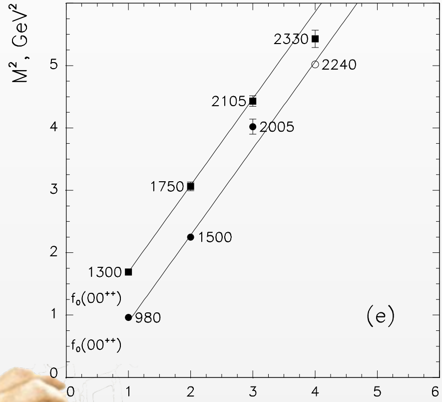


Too good to be true?

Think of Regge trajectories.
The prediction of the theory is

$$m^2 \propto J\left(1 + \mathcal{O}\left(J^{-1}\right)\right)$$

but *experimentally* everything works so well at small J that String Theory was invented.



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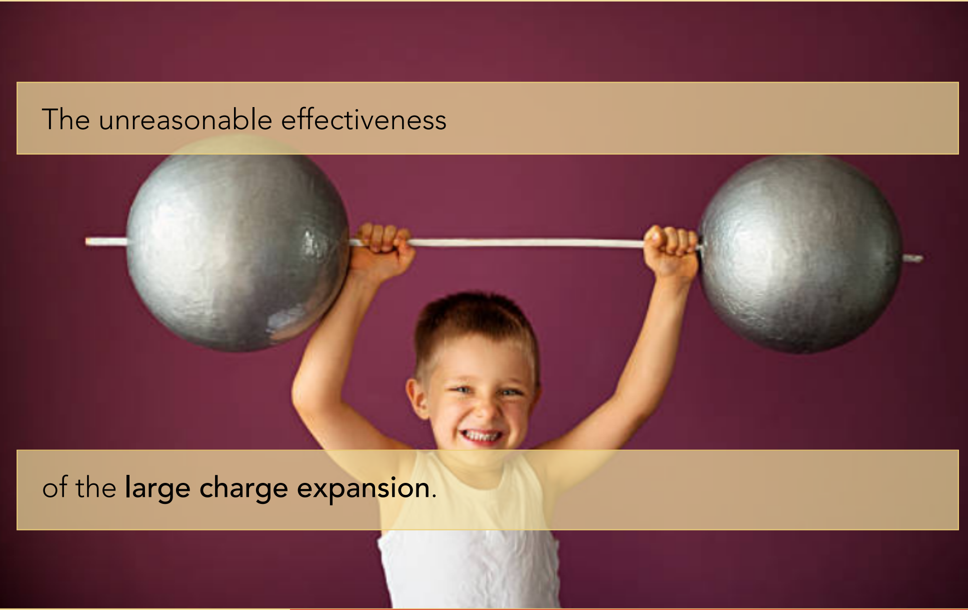

The unreasonable effectiveness

of the **large charge expansion**.

Today's talk

Justify and prove all my claims from first principles, without using effective field theory.

- well-defined asymptotic expansion (in the technical sense)
- justify why the expansion works at small charge
- compute the coefficients in the effective action in large-N

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- compute the coefficients in the effective action in large-N

Use the large-charge expansion together with supersymmetry.

- qualitatively different behavior
- compute three-point functions
- resum the large-charge expansion
- see explicitly the next saddle in the partition function

PARENTAL

ADVISORY

EXPLICIT CONTENT

Large N *vs.* Large Charge

The model

ϕ^4 model on $\mathbb{R} \times \Sigma$ for $2N$ fields

$$S[\varphi] = \frac{1}{2} \sum_{a=1}^{2N} \int \mathrm{d}t\, \mathrm{d}\Sigma \left(\partial_\mu \phi_a \partial^\mu \phi_a + \frac{R}{8} \phi_a \phi_a + u(\phi_a \phi_a)^2 \right)$$

Without quadratic term this flows to the wf fixed point in the ir.
We compute the partition function at fixed charge

$$Z_\Sigma(Q) = \operatorname{Tr}\left[e^{-S[\varphi]} \delta(\hat{Q}[\varphi] - Q) \right]$$

Via the state/operator correspondence we extract the conformal dimensions of operators of fixed charge Q on $\mathbb{R}^3$:

$$\Delta(Q) = -\frac{1}{\beta} \log Z_{S^2}(Q).$$

Fix the charge

We group the ϕ_a into N complex fields φ_i. The charge we fix is the $U(1)$ that rotates all of them together.

$$\hat{Q}[\varphi] = i \sum_{i=1}^{n} \int d\Sigma \, (\varphi_i^* \dot{\varphi}_i - \dot{\varphi}_i^* \varphi_i).$$

So

$$Z_\Sigma(Q) = \mathrm{Tr}\left[e^{-S} \delta(\hat{Q} - Q)\right] = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{-i\theta Q} \mathrm{Tr}\left[e^{-S + i\theta \hat{Q}}\right]$$

Since $\hat{Q}$ depends on the momenta, the integration is not trivial but well understood.

$$\begin{aligned} Z_\Sigma(Q) &= \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{-i\theta Q} \int_{\varphi(2\pi\beta) = e^{i\theta} \varphi(0)} D\varphi_i \, e^{-S[\varphi]} \\ &= \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{-i\theta Q} \int_{\varphi(2\pi\beta) = \varphi(0)} D\varphi_i \, e^{-S^\theta[\varphi]} \end{aligned}$$

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boundary condition

$$Z_\Sigma(Q) = \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta}{2\pi} e^{-i\theta Q} \int_{\varphi(2\pi\beta) = e^{i\theta}\varphi(0)} D\varphi_i \, e^{-S[\varphi]}$$

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covariant derivative

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Effective actions

The covariant derivative approach:

$$S^{\theta}[\varphi] = \sum_{i=1}^{N} \int \mathrm{d}t\,\mathrm{d}\Sigma \left((D_{\mu}\varphi_i)^*(D^{\mu}\varphi_i) + \frac{R}{8}\varphi_i^*\varphi_i + 2u(\varphi_i^*\varphi_i)^2 \right)$$

where

$$\begin{cases} D_0\varphi = \partial_0\varphi + i\frac{\theta}{\beta}\varphi \\ D_i\varphi = \partial_i\varphi \end{cases}$$

Standard construction: introduce a Lagrange multiplier λ and integrate out all the fields but one (which we call σ) [Zinn–Justin].
Effective action

$$S^{\theta}[\sigma, \lambda] = -(N-1)\,\mathrm{Tr}\left[\log\left(-D_{\mu}D^{\mu} + \lambda + \frac{R}{8}\right)\right] + \int \mathrm{d}t\,\mathrm{d}\Sigma \left((D_{\mu}\sigma)^*(D^{\mu}\sigma) + \left(\frac{R}{8} + \lambda\right)\sigma^*\sigma + \frac{\lambda^2}{4u} \right)$$

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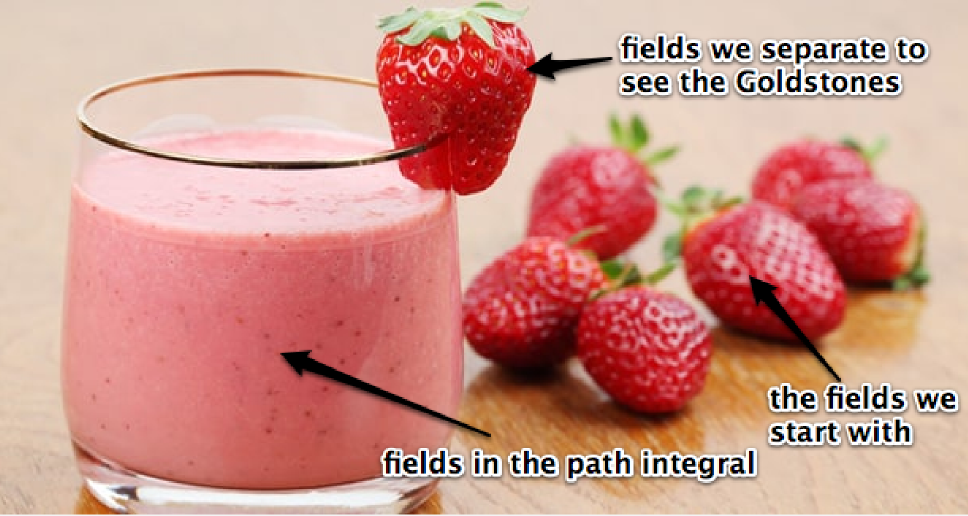
irrelevant

Integrate all but one



Saddle point

We expand the fields λ and σ around a constant vacuum expectation value (vev) to minimize over

$$\lambda = m^2 + \frac{\hat{\lambda}}{(N-1)^{1/2}} \qquad \sigma = v + \hat{\sigma}$$

and we approximate the integral over θ with a saddle point. (Generalized) gap equations:

$$\frac{\mathrm{d}S}{\mathrm{d}m} = 2V\beta\, m v^2 + (N-1)\frac{\partial}{\partial m}\operatorname{Tr}\left[\log\left(-D_\mu D^\mu + m^2\right)\right] = 0$$

$$\frac{\mathrm{d}S}{\mathrm{d}\theta} = iQ + 2V\frac{\theta}{\beta}v^2 + (N-1)\frac{\partial}{\partial\theta}\operatorname{Tr}\left[\log\left(-D_\mu D^\mu + m^2\right)\right] = 0$$

$$\frac{\mathrm{d}S}{\mathrm{d}v} = V\beta\left(m^2 + \frac{\theta^2}{\beta^2}\right)v = 0$$

Zeta functions

In the limit $\beta \to \infty$ (zero temperature), we regularize with a zeta function $\zeta(s|\Sigma, m) = \sum_p (E(p)^2 + m^2)^{-s}$:
The gap equations become

$$Vv^2 + \frac{N-1}{2}\zeta(1/2|\Sigma, m) = 0,$$

$$-iQ + \frac{2V}{\beta}\theta v^2 = 0,$$

$$2V\beta\left(m^2 + \frac{\theta^2}{\beta^2}\right)v = 0,$$

For finite Q we need necessarily $v \neq 0$ and then $\theta = im\beta$. So we get

$$m\,\zeta(1/2|\Sigma, m) = -\frac{Q}{N-1}$$

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natural parameter

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Large N

We have not used at all the large N condition. Now it's the moment.
The action for the fluctuations is an infinite sum of non-local terms

$$
\begin{aligned}
S^\theta[\hat\sigma, \hat\lambda] = {} & -(N-1)\,\mathrm{Tr}\left[\log\left(-D_\mu D^\mu + m^2\right)\right] \\
& + \int \mathrm{d}t\,\mathrm{d}\Sigma \left((D_\mu \hat\sigma)^*(D^\mu \hat\sigma) + (m^2 + \hat\lambda)\,\hat\sigma^* \hat\sigma + \frac{\hat\lambda\, v(\hat\sigma + \hat\sigma^*)}{(N-1)^{1/2}} \right) + \\
& + \frac{1}{2} \int \mathrm{d}x_1\,\mathrm{d}x_2\, \hat\lambda(x_1)\,\hat\lambda(x_2) D(x_1 - x_2)^2 + \\
& + \sum_{n=3}^{\infty} \frac{1}{n(N-1)^{n/2-1}} \int \mathrm{d}x_1 \dots \mathrm{d}x_n \\
& \qquad \hat\lambda(x_1)\,\hat\lambda(x_2)\dots\hat\lambda(x_n) P(x_1, x_2, \dots, x_n)
\end{aligned}
$$

When N is large we have a natural hierarchy

Order N

At leading order in N, the free energy is

$$F(Q) = -\frac{1}{\beta}\left(i\theta\, Q + N \frac{\partial}{\partial s} \frac{\Gamma(s-1/2)}{2\sqrt{\pi}\,\Gamma(s)}\, \beta\, \zeta(s-1/2|\, \Sigma, m)\Big|_{s=0} \right)$$

Using the gap equations

$$F(Q) = mQ + N\zeta(-1/2|\, \Sigma, m)$$

For $\Sigma = S^2$:

$$F(Q) = \frac{N\sqrt{2}}{3}\left(\frac{Q}{N}\right)^{3/2} + \frac{N}{3\sqrt{2}}\left(\frac{Q}{N}\right)^{1/2} - \frac{7N}{180\sqrt{2}}\left(\frac{Q}{N}\right)^{-1/2} + \dots$$

Order N

$$F_{S^2}(Q) = \frac{4N}{3}\left(\frac{Q}{2N}\right)^{3/2} + \frac{N}{3}\left(\frac{Q}{2N}\right)^{1/2} - \frac{7N}{360}\left(\frac{Q}{2N}\right)^{-1/2} - \frac{71N}{90720}\left(\frac{Q}{2N}\right)^{-3/2} + \mathcal{O}\left(e^{-\sqrt{Q/(2N)}}\right)$$

Order N

leading $Q^{3/2}$

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Order N

$1/Q$ **expansion**

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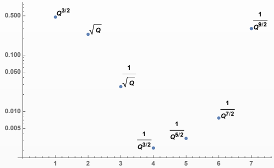

asymptotic expansion

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Order N^0

The order N^0 terms are

$$S^{\theta}[\hat{\sigma}, \hat{\lambda}] = \int \mathrm{d}t\, \mathrm{d}\Sigma \left((D_\mu \hat{\sigma})^* (D^\mu \hat{\sigma}) + (m^2 + \hat{\lambda}) \hat{\sigma}^* \hat{\sigma} + \frac{\hat{\lambda} v(\hat{\sigma} + \hat{\sigma}^*)}{(N-1)^{1/2}} \right) + \frac{1}{2} \int \mathrm{d}x_1\, \mathrm{d}x_2\, \hat{\lambda}(x_1) \hat{\lambda}(x_2) D(x_1 - x_2)^2$$

where $D(x-y)$ is the propagator $(D_\mu D^\mu + m^2)^{-1}$.
At low energies we can approximate the non-local term as

$$\int \mathrm{d}t\, \mathrm{d}\Sigma\, \hat{\lambda}(x)^2 \zeta(2|\theta, \Sigma, m) \approx \frac{V}{2m} \int \mathrm{d}t\, \mathrm{d}\Sigma\, \hat{\lambda}(x)^2$$

and we can integrate $\hat{\lambda}$ out.

Order N^0

The inverse propagator for σ is

$$\begin{pmatrix} 1/2(\omega^2 + p^2 + 4m^2) & m\omega \\ -m\omega & 1/2(\omega^2 + p^2) \end{pmatrix}$$

It describes a massive mode and a massless mode with dispersion

$$\omega^2 + \frac{1}{2}p^2 + \dots = 0 \qquad \omega^2 + 8m^2 + \frac{3}{2}p^2 + \dots = 0$$

This is the conformal Goldstone that we have seen in the EFT.
Its contribution to the partition function is

$$E_G = \frac{1}{2}\frac{1}{\sqrt{2}}\zeta(1/2|S^2) = -0.0937\dots$$

This is universal. Does not depend on N or Q.

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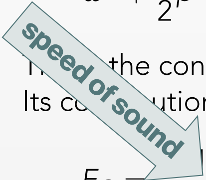

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Higher orders

There are infinite non-local terms

$$S_{\text{nl}} = \sum_{n=3}^{\infty} \frac{1}{n(N-1)^{n/2-1}} \int \mathrm{d}x_1 \ldots \mathrm{d}x_n \, \hat{\lambda}(x_1) \ldots \hat{\lambda}(x_n) P(x_1, \ldots, x_n)$$

At low energy they are approximated by

$$S_{\text{nl}} = \sum_{n=3}^{\infty} \frac{1}{n(N-1)^{n/2-1}} \int \mathrm{d}x \, \hat{\lambda}(x)^n C'_n$$

There is only one scale, the charge density $\rho = Q/V$. We must have

$$C'_n = \rho^{3/2-n} C_n$$

So

$$S_{\text{nl}} = Q^{3/2} \sum_{n=3}^{\infty} \frac{C_n}{n(N-1)^{n/2-1}} \int \mathrm{d}x \, \bar{\lambda}(x)^n$$

Infinite corrections of order $Q^{3/2}$ (and following), controlled by $1/N$.

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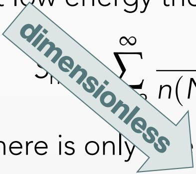

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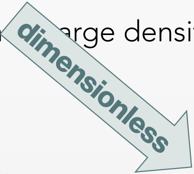


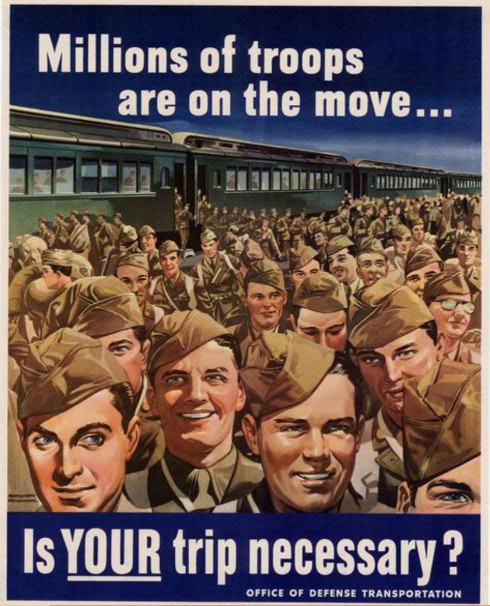
$$C'_n = \rho^{3/2-n} C_n$$

So

$$S_{\text{nl}} = Q^{3/2} \sum_{n=3}^{\infty} \frac{C_n}{n(N-1)^{n/2-1}} \int dx \, \bar{\lambda}(x)^n$$

Infinite corrections of order $Q^{3/2}$ (and following), controlled by $1/N$.

Was it worth it?



Final result

$$
\begin{aligned}
\Delta(Q) = &\left(\frac{4N}{3} + \mathcal{O}(1)\right)\left(\frac{Q}{2N}\right)^{3/2} \\
&+ \left(\frac{N}{3} + \mathcal{O}(1)\right)\left(\frac{Q}{2N}\right)^{1/2} \\
&+ \dots \\
&- 0.0937\dots
\end{aligned}
$$

Final result

$$\Delta(Q) = \left(\frac{4N}{3} + \mathcal{O}(1)\right)\left(\frac{Q}{2N}\right)^{3/2} + \left(\frac{N}{3} + \mathcal{O}(1)\right)\left(\frac{Q}{2N}\right)^{1/2} + \dots - 0.0937\dots$$

no corrections

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& + \dots \\
& - 0.0937\dots
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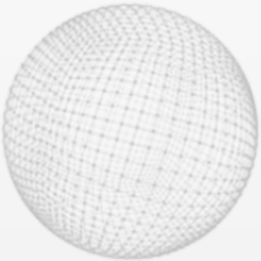
$$

		lattice	leading N	error
$O(2)$	$c_{3/2}$	0.337	0.471	40%
	$c_{1/2}$	0.266	0.236	10%
$O(4)$	$c_{3/2}$	0.301	0.333	10%
	$c_{1/2}$	0.294	0.333	13%



Large charge and supersymmetry

And Now for Something Completely Different

All the models that you have seen have something in common: isolated vacuum. No moduli space.
What happens when there is a flat direction?

Many known examples of (non-Lagrangian) $\mathcal{N} \geq 2$ SCFT in four dimensions.
Coulomb branch with a dimension-one moduli space: all the physics is encoded in a single operator Φ and every chiral operator is just Φ^n.
We will write an effective action for Φ.

Effective action

We have a single vector multiplet. The kinetic term is just

$$L_k = \int \mathrm{d}^4\theta \; \Phi^2 + \text{c.c.} = |\partial\phi|^2 + \text{fermions} + \text{gauge fields}$$

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from anomaly

$$L^{\text{EFT}} = L_K + \alpha L_{WZ}$$

The coefficient α fixes the a-anomaly of the EFT. It has to match the anomaly in the UV.

Claim: at large R-charge this action is all you need for any $\mathcal{N} = 2$ theory (with one-dimensional moduli space).

Observables

Three-point function of the Coulomb branch operators

$$\left\langle \Phi^{n_1}(x_1)\,\Phi^{n_2}(x_2)\,\bar{\Phi}^{n_1+n_2}(x_3) \right\rangle = \frac{C^{n_1,n_2,n_1+n_2}}{|x_1 - x_3|^{2n_1 D}|x_2 - x_3|^{2n_2 D}}$$

The OPE of Φ with itself is regular, so we can set $x_2 = x_1$ and the three-point function is actually a two-point function.

$$C^{n',n-n',n} = |x_1 - x_2|^{2nD} \left\langle \Phi^n(x_1)\,\bar{\Phi}^n(x_2) \right\rangle = e^{q_n - q_0}$$

$Q = nD$ is the controlling parameter (it's the R-charge)

Two-point function

$$\left\langle \Phi^n(x_1)\bar{\Phi}^n(x_2)\right\rangle = \int D\phi \;\phi^n(x_1)\,\bar{\phi}^n(x_2) e^{-S_k}$$

We can just pull the sources in the action and minimize

$$S_k + S_{\text{sources}} \propto k_0 + \int \mathrm{d}^4x \Big[\partial_\mu \phi\, \partial_\mu \bar{\phi} - Q \log \phi\, \delta(x - x_1) - Q \log \bar{\phi}\, \delta(x - x_2)\Big]$$

At the minimum:

$$S = k_0 + k_1 Q - Q \log Q + 2Q \log |x_1 - x_2| + \mathcal{O}\left(Q^0\right)$$

so

$$q_n = k_0 + k_1 Q + \left(Q + \tfrac{1}{2}\right) \log(Q) + \mathcal{O}\left(Q^0\right)$$

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EFT parameters

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Two-point function: tree level

Adding the WZ term gives another contribution

$$q_n = k_1 Q + k_0 + Q \log(Q) + \left(\alpha + \tfrac{1}{2}\right) \log(Q) + \mathcal{O}\left(Q^0\right)$$

This is the tree-level result.

Corrections from quantum fluctuations in the path integral.
No other tree-level terms.

Two-point function: quantum corrections

$1/Q$ is the loop-counting parameter because we are expanding around a VEV that depends on Q.
Sum of a ground state piece and a series in $1/Q$.

$$q_n = k_0 + k_1 Q + Q \log(Q) + \left(\alpha + \tfrac{1}{2}\right) \log(Q) + \sum_{m=1}^{\infty} \frac{k_m(\alpha)}{Q^m}$$

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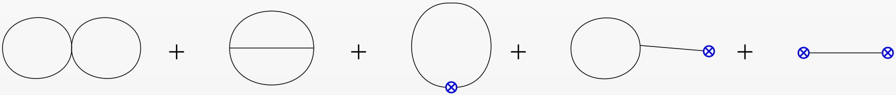
Compute order-by-order



$$k_1(\alpha) = \frac{1}{2}\left(\alpha^2 + \alpha + \frac{1}{6}\right)$$

Supersymmetry to the rescue

There is a better way.
The q_n satisfy a Toda lattice equation arXiv:0910.4963

$$\partial_\tau \partial_{\bar\tau} q_n = e^{q_{n+1}-q_n} - e^{q_n - q_{n-1}}$$

This is integrable, but it's hard to find explicit solutions.
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…we use the form that follows from the existence of the asymptotic expansion

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Recursion relation

We can actually solve the recursion relation, using the value of $k_1(\alpha)$ found at one loop.

$$q_n = k_0(\tau, \bar{\tau}) + Qf(\tau, \bar{\tau}) + \log(\Gamma(2n + \alpha + 1))$$

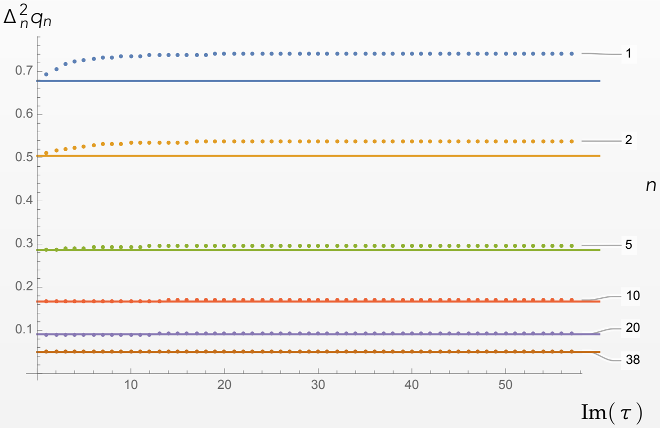
The log term is universal, only depends on α.

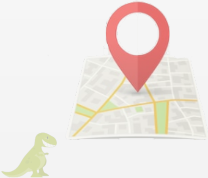
We have completely resummed the $1/Q$ expansion.

Comparison with localization

How well does this work?

For the special case of $SU(2)$ SQCD with $N_f = 4$ we can compare with localization. arXiv:1602.05971

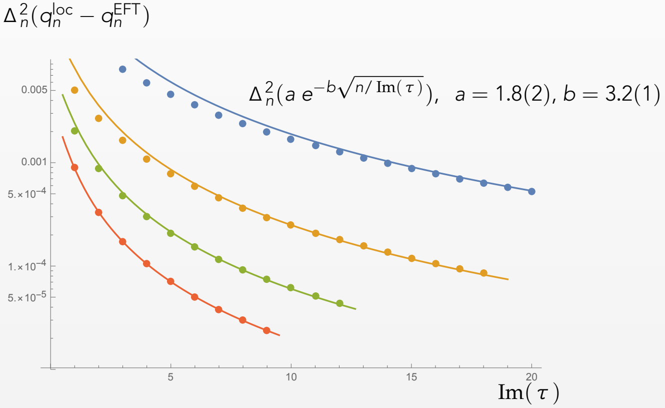




A semi-empirical instanton

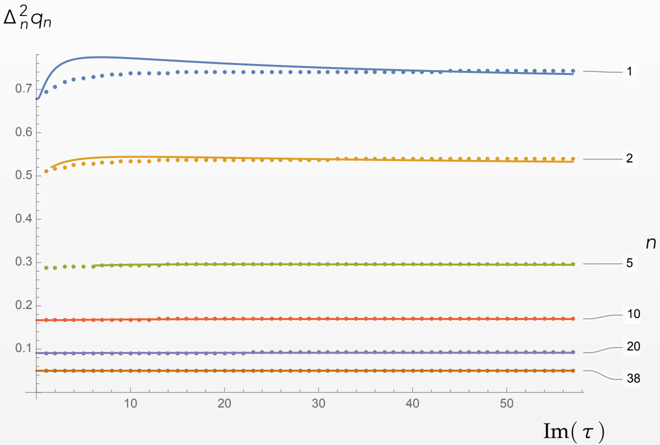
A semi-empirical instanton

We can do better.
We have resummed the $1/Q$ expansion around one vacuum.
Exponential corrections coming from the next saddle in the path integral.



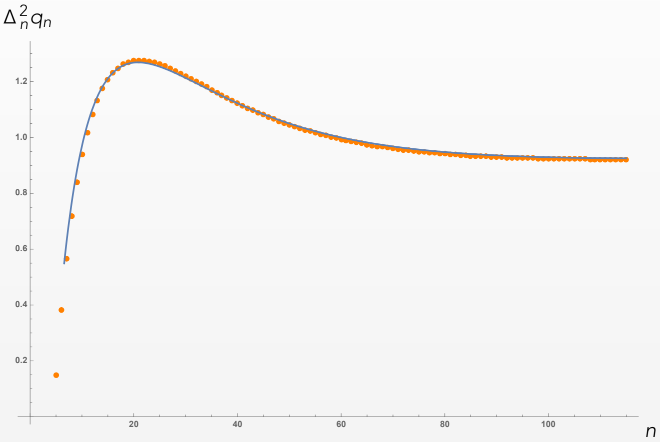
Comparison with localization

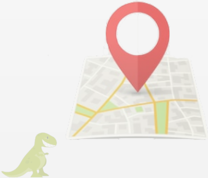
Once we add the first exponential correction



Comparison with localization

Once we add the first exponential correction (fixed $\tau = 6$)

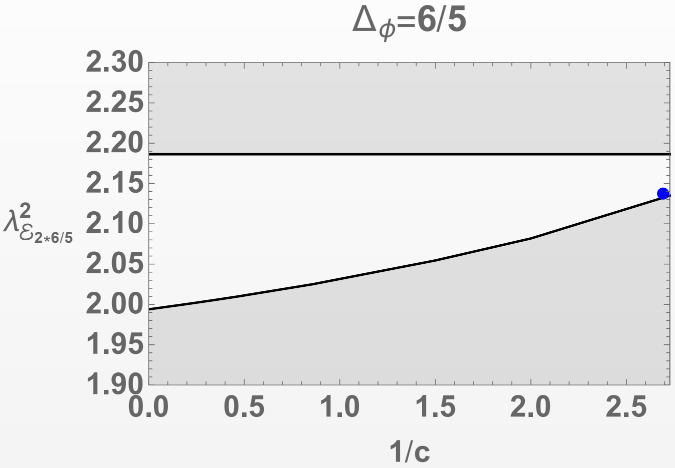




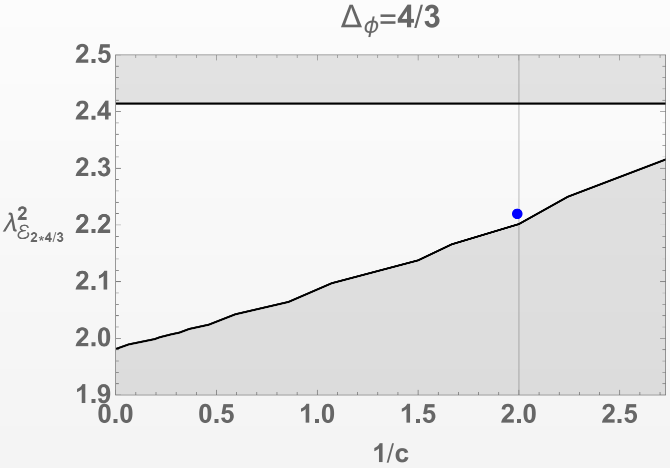
Comparison with bootstrap

For strongly coupled theories one can use bootstrap to place bounds on the three-point coefficients with $n = 1$.
This is the worst possible situation for us. And still...

Comparison with bootstrap



Comparison with bootstrap



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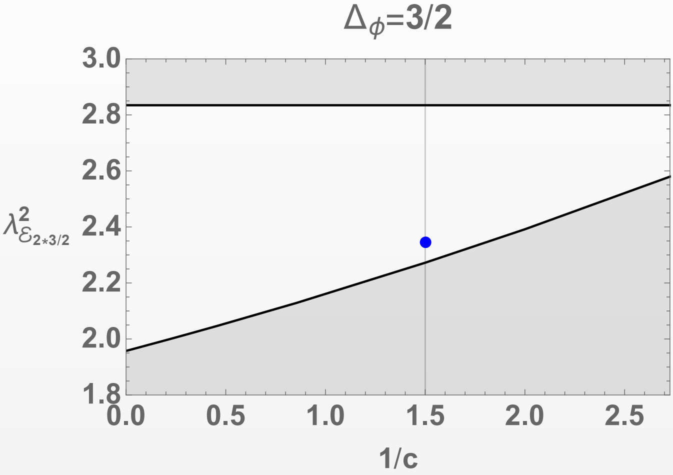

Conclusions

With the large-charge approach we can access strongly coupled theories.

The predictions from the EFT are verified when other methods are available (lattice, large N, localization, bootstrap).

There are different "large-charge universality classes".

For SCFT we can use supersymmetry to resum the large-charge perturbation theory and have a glimpse of what lies beyond the expansion.

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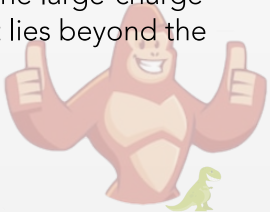

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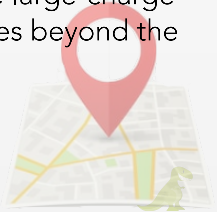

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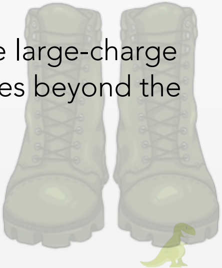

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