Stochastic areas and windings

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Part I. Stochastic areas

The Lévy area formula

Let $Z_t = X_t + iY_t$, $t \geq 0$, be a Brownian motion in the complex plane such that $Z_0 = 0$. Up to a factor $1/2$, the algebraic area swept out by the path of Z up to time t is given by

$$S_t = \int_{Z[0,t]} xdy - yx = \int_0^t X_s dY_s - Y_s dX_s,$$

The Lévy area formula

The Lévy's area formula

$$\mathbb{E}\left(e^{i\lambda S_t} | Z_t = z\right) = \frac{\lambda t}{\sinh \lambda t} e^{-\frac{|z|^2}{2t}(\lambda t \coth \lambda t - 1)}$$

was originally proved by Paul Lévy (1940) by using a series expansion of Z.

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The formula has numerous applications: Rough paths theory, Connections with the Riemann zeta function, Heat kernel on the Heisenberg group,...

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The first observation is that, due to the invariance by rotations of Z, one has for every $\lambda \in \mathbb{R}$,

$$\mathbb{E}\left(e^{i\lambda S_t} | Z_t = z\right) = \mathbb{E}\left(e^{-\frac{\lambda^2}{2}\int_0^t |Z_s|^2 ds} \middle| |Z_t| = |z|\right).$$

The Lévy area formula

One considers then the new probability

$$\mathbb{P}^{\lambda}_{/\mathcal{F}_t} = \exp\left(\frac{\lambda}{2}(|Z_t|^2 - 2t) - \frac{\lambda^2}{2}\int_0^t |Z_s|^2 ds\right)\mathbb{P}_{/\mathcal{F}_t}$$

under which, thanks to Girsanov theorem, $(Z_t)_{t\geq 0}$ is a Gaussian process (an Ornstein-Uhlenbeck process). The Lévy area formula then easily follows from standard computations on Gaussian measures.

The complex projective space $\mathbb{CP}^n$

The complex projective space $\mathbb{CP}^n$ can be defined as the set of complex lines in $\mathbb{C}^{n+1}$. To parametrize points in $\mathbb{CP}^n$, it is convenient to use the local inhomogeneous coordinates given by $w_j = z_j/z_{n+1}$, $1 \le j \le n$, $z \in \mathbb{C}^{n+1}$, $z_{n+1} \neq 0$.

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The map

$$\begin{array}{llcl} \pi: & \mathbb{S}^{2n+1} & \to & \mathbb{CP}^n \\ & (z_1, \cdots, z_{n+1}) & \to & (w_1, \cdots, w_n) \end{array}$$

is a Riemannian submersion with totally geodesic fibers isometric to $\mathbf{U}(1)$.

Brownian motion in $\mathbb{CP}^n$

By using the submersion π, one can construct the Browian motion on $\mathbb{CP}^n$ as

$$w(t) = (w^1(t), \cdots, w^n(t)) = \left(\frac{Z^1(t)}{Z^{n+1}(t)}, \cdots, \frac{Z^n(t)}{Z^{n+1}(t)} \right)$$

where $(Z^1(t), \cdots, Z^{n+1}(t))$ is a Brownian motion on $\mathbb{S}^{2n+1}$.

Stochastic area in $\mathbb{CP}^n$

Let $(w(t))_{t\geq 0}$ be a Brownian motion on $\mathbb{CP}^n$ started at 0[1]. The generalized stochastic area process of $(w(t))_{t\geq 0}$ is defined by

$$\theta(t) = \int_{w[0,t]} \alpha = \frac{i}{2} \sum_{j=1}^{n} \int_0^t \frac{w_j(s) d\overline{w_j}(s) - \overline{w_j}(s) dw_j(s)}{1 + |w(s)|^2},$$

where the above stochastic integrals are understood in the Stratonovitch, or equivalently in the Itô sense.

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where the above stochastic integrals are understood in the Stratonovitch, or equivalently in the Itô sense. The form $d\alpha$ is the Kähler form on $\mathbb{CP}^n$.

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Skew-product decomposition

Theorem

Let $(w(t))_{t\geq 0}$ be a Brownian motion on $\mathbb{CP}^n$ started at 0 and $(\theta(t))_{t\geq 0}$ be its stochastic area process. The $\mathbb{S}^{2n+1}$-valued diffusion process

$$X_t = \frac{e^{-i\theta(t)}}{\sqrt{1+|w(t)|^2}}\,(w(t),1)\,, \quad t \geq 0$$

is the horizontal lift at the north pole of $(w(t))_{t\geq 0}$ by the submersion π.

Skew-product decomposition

Corollary

Let $r(t) = \arctan |w(t)|$. *The process* $(r(t), \theta(t))_{t \geq 0}$ *is a diffusion with generator*

$$L = \frac{1}{2}\left(\frac{\partial^2}{\partial r^2} + ((2n-1)\cot r - \tan r)\frac{\partial}{\partial r} + \tan^2 r \frac{\partial^2}{\partial \theta^2}\right).$$

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As a consequence the following equality in distribution holds

$$(r(t), \theta(t))_{t \geq 0} = \left(r(t), B_{\int_0^t \tan^2 r(s) ds}\right)_{t \geq 0},$$

where $(B_t)_{t \geq 0}$ *is a standard Brownian motion independent from* r.

Consider the Jacobi generator

$$\mathcal{L}^{\alpha,\beta} = \frac{1}{2}\frac{\partial^2}{\partial r^2} + \left(\left(\alpha + \frac{1}{2}\right) \cot r - \left(\beta + \frac{1}{2}\right) \tan r \right) \frac{\partial}{\partial r}, \quad \alpha, \beta > -1$$

We denote by $q_t^{\alpha,\beta}(r_0, r)$ the transition density with respect to the Lebesgue measure of the diffusion with generator $\mathcal{L}^{\alpha,\beta}$.

Theorem

For $\lambda \geq 0$, $r \in [0, \pi/2)$, and $t > 0$ we have

$$\begin{aligned} \mathbb{E}\left(e^{i\lambda\theta(t)} \mid r(t) = r \right) &= \mathbb{E}\left(e^{-\frac{\lambda^2}{2}\int_0^t \tan^2 r(s) ds} \mid r(t) = r \right) \\ &= \frac{e^{-n\lambda t}}{(\cos r)^\lambda} \frac{q_t^{n-1,\lambda}(0, r)}{q_t^{n-1,0}(0, r)}. \end{aligned}$$

Limit distribution

Theorem

When $t \to +\infty$*, the following convergence in distribution takes place*

$$\frac{\theta(t)}{t} \to \mathcal{C}_n,$$

where $\mathcal{C}_n$ *is a Cauchy distribution with parameter* n*.*

The complex hyperbolic space

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be the $2n+1$ dimensional anti-de Sitter space. The map

$$\begin{array}{rccc} \pi : & \mathbb{H}^{2n+1} & \to & \mathbb{CH}^n \\ & (z_1, \cdots, z_{n+1}) & \to & \left(\frac{z_1}{z_{n+1}}, \cdots, \frac{z_n}{z_{n+1}}\right) \end{array}$$

is an indefinite Riemannian submersion whose one-dimensional fibers are definite negative.

Stochastic area in $\mathbb{CH}^n$

To parametrize $\mathbb{CH}^n$, we will use the global inhomogeneous coordinates given by $w_j = z_j / z_{n+1}$ where $(z_1, \dots, z_n) \in \mathbb{M}$ with $\mathbb{M} = \{z \in \mathbb{C}^{n,1}, \sum_{k=1}^{n} |z_k|^2 - |z_{n+1}|^2 < 0\}$.

Definition

Let $(w(t))_{t\geq 0}$ be a Brownian motion on $\mathbb{CH}^n$ started at 0[2]. The generalized stochastic area process of $(w(t))_{t\geq 0}$ is defined by

$$\theta(t) = \int_{w[0,t]} \alpha = \frac{i}{2} \sum_{j=1}^{n} \int_0^t \frac{w_j(s) d\overline{w_j}(s) - \overline{w_j}(s) dw_j(s)}{1 - |w(s)|^2},$$

where the above stochastic integrals are understood in the Stratonovitch sense or equivalently Itô sense.

[2]We call 0 the point with inhomogeneous coordinates $w_1 = 0, \cdots, w_n = 0$

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Theorem

Let $(w(t))_{t\geq 0}$ be a Brownian motion on $\mathbb{CH}^n$ started at 0 and $(\theta(t))_{t\geq 0}$ be its stochastic area process. The $\mathbb{H}^{2n+1}$-valued diffusion process

$$Y_t = \frac{e^{i\theta_t}}{\sqrt{1-|w(t)|^2}}\left(w(t),1\right), \quad t \geq 0$$

is the horizontal lift at $(0,1)$ of $(w(t))_{t\geq 0}$ by the submersion π.

Skew-product decomposition

Theorem

Let $r(t) = \tanh^{-1}|w(t)|$. *The process* $(r(t), \theta(t))_{t\geq 0}$ *is a diffusion with generator*

$$L = \frac{1}{2}\left(\frac{\partial^2}{\partial r^2} + ((2n-1)\coth r + \tanh r)\frac{\partial}{\partial r} + \tanh^2 r \frac{\partial^2}{\partial \theta^2}\right).$$

As a consequence the following equality in distribution holds

$$(r(t), \theta(t))_{t\geq 0} = \left(r(t), B_{\int_0^t \tanh^2 r(s) ds}\right)_{t\geq 0}, \tag{1}$$

where $(B_t)_{t\geq 0}$ *is a standard Brownian motion independent from* r.

Limit law

Theorem

When $t \to +\infty$, the following convergence in distribution takes place

$$\frac{\theta(t)}{\sqrt{t}} \to \mathcal{N}(0,1)$$

where $\mathcal{N}(0,1)$ is a normal distribution with mean 0 and variance 1.

Part II. Stochastic windings

Winding form

In the punctured complex plane $\mathbb{C} \setminus \{0\}$, consider the one-form

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For every smooth path $\gamma : [0, +\infty) \to \mathbb{C} \setminus \{0\}$ one has the representation

$$\gamma(t) = |\gamma(t)| \exp\left(i \int_{\gamma[0,t]} \alpha \right), \quad t \geq 0.$$

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It is therefore natural to call α the winding form around 0 since the integral of a smooth path γ along this form quantifies the angular motion of this path.

Asymptotic Brownian Winding

The integral of the winding form along the paths of a two-dimensional Brownian motion $Z(t) = X(t) + iY(t)$ which is not started from 0 can be defined using Itô's calculus and yields the Brownian winding functional:

$$\zeta(t) = \int_{Z[0,t]} \alpha = \int_0^t \frac{X(s)dY(s) - Y(s)dX(s)}{X(s)^2 + Y(s)^2}.$$

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Theorem (Spitzer, 1958)

When $t \to +\infty$, in distribution

$$\frac{2}{\ln t}\zeta(t) \to C_1$$

where C_1 is a Cauchy distribution with parameter 1.

Winding on $\mathbb{CP}^1$

One has a winding form on $\mathbb{CP}^1 \simeq \mathbb{C} \cup \{\infty\}$. Therefore, if $W(t)$ is a Brownian motion on $\mathbb{CP}^1$ one can consider the winding process

$$\zeta(t) = \int_{W[0,t]} \alpha$$

Theorem (McKean, 1960's)

When $t \to +\infty$, in distribution

$$\frac{1}{t}\zeta(t) \to C_2$$

where C_2 is a Cauchy distribution with parameter 2.

Winding on $\mathbb{CH}^1$

One also has a winding form on $\mathbb{CH}^1 \simeq B_{\mathbb{R}^2}(0,1)$. Therefore, if $W(t)$ is a Brownian motion on $\mathbb{CH}^1$ on can consider the winding process

$$\zeta(t) = \int_{W[0,t]} \alpha$$

Theorem

When $t \to +\infty$, in distribution

$$\zeta(t) \to C_{\ln \coth \|W_0\|}$$

where $C_{\ln \coth \|W_0\|}$ is a Cauchy distribution with parameter $\ln \coth \|W_0\|$.