MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

CSC COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

# Stabilization of Nonlinear Systems by Oscillating Controls with Application to Nonholonomic and Fluid Dynamics

**Alexander Zuyev**

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## Outline

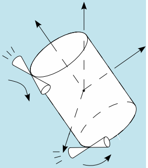
# Systems with Uncontrollable Linearization

## Unicycle

$$\dot x_1 = u_1 \cos x_3,$$
$$\dot x_2 = u_1 \sin x_3,$$
$$\dot x_3 = u_2.$$

## Euler's equations in rigid body dynamics

$$J_1 \dot x_1 = (J_2 - J_3) x_2 x_3 + \mu_{11} u_1 + \mu_{21} u_2,$$
$$J_2 \dot x_2 = (J_3 - J_1) x_1 x_3 + \mu_{12} u_1 + \mu_{22} u_2,$$
$$J_3 \dot x_3 = (J_1 - J_2) x_1 x_2 + \mu_{13} u_1 + \mu_{23} u_2.$$



## The Navier–Stokes and Euler equations on $\mathbb{T}^2$ (incompressible case)

$$\frac{\partial v}{\partial t} + (v \cdot \nabla) v + \frac{1}{\rho} \nabla p - \nu \Delta v = \sum_{j=1}^{m} u_j F_j(y),$$

$$\nabla \cdot v = 0, \quad y = (y_1, y_2) \in \mathbb{T}^2,$$

$v = (v_1(t,y), v_2(t,y))$ – velocity, $p = p(t,y)$ – pressure.

The Euler equations: $\nu = 0$.

Consider

$$\dot{x} = f_0(x) + \sum_{j=1}^{m} u_j f_j(x) \equiv f(x,u), \quad x \in D \subset \mathbb{R}^n,\ u \in \mathbb{R}^m,\ 0 \in D, \quad (\Sigma)$$

where $f_0$, $f_1$, ..., $f_m$ are smooth, $f_0(0) = 0$, and $m < n$ .

## Motivation: Controllability $\Rightarrow$ Stabilizability ?

Consider

$$\dot{x} = f_0(x) + \sum_{j=1}^{m} u_j f_j(x) \equiv f(x,u), \quad x \in D \subset \mathbb{R}^n,\ u \in \mathbb{R}^m,\ 0 \in D, \quad (\Sigma)$$

where $f_0$, $f_1$, ..., $f_m$ are smooth, $f_0(0) = 0$, and $m < n$ .

### Stabilization by a time-invariant feedback

Find a continuous $u = k(x)$, $k(0) = 0$ s.t. the solution $x = 0$ of $\dot{x} = f(x, k(x)) \equiv F(x)$ is asymptotically stable in the sense of Lyapunov.

### References

R.E. Kalman (1961), N.N. Krasovskii (1966), G.V. Kamenkov (1972), V.I. Korobov (1973), Z. Artstein (1983), R.W. Brockett (1983), V.G. Veretennikov (1984), M. Kawski (1989), J.-M. Coron, L. Praly, A. Teel (1995), F.H. Clarke, Yu. S. Ledyaev, E.D. Sontag, A.I. Subbotin (1997), S. Celikovsky, E. Aranda-Bricaire (1999), P. Morin, J.-B. Pomet, C. Samson (1999), ... , F. Gao, Y. Wu, H. Li, Y. Liu (2018), ...

**Krasnoselskii–Zabreiko theorem (1974)**

If $x = 0$ is asymptotically stable for $\dot{x} = f(x, k(x)) \equiv F(x),\ x \in \mathbb{R}^n$, then $\gamma[F, S_\varepsilon] = (-1)^n$ for any small enough $\varepsilon > 0$.

**Rotation (degree) of a continuous vector field $F : S_\varepsilon \to \mathbb{R}^n$**

If $F(x) \neq 0$ on a sphere $S_\varepsilon = \varepsilon S^{n-1}\mathbb{R}^n$ then $\gamma[F, S_\varepsilon] \in \mathbb{Z}$ is well-defined.

## Motivation: Obstacles for asymptotic stability

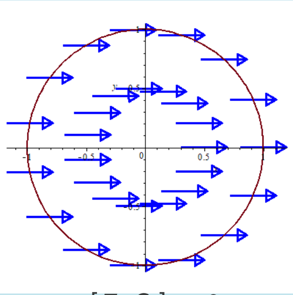
### Krasnoselskii–Zabreiko theorem (1974)

If $x = 0$ is asymptotically stable for $\dot{x} = f(x, k(x)) \equiv F(x)$, $x \in \mathbb{R}^n$, then $\gamma[F, S_\varepsilon] = (-1)^n$ for any small enough $\varepsilon > 0$.

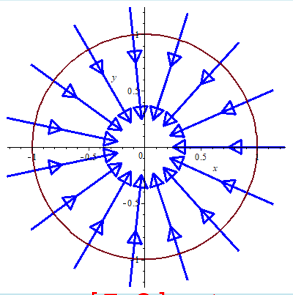
### Rotation (degree) of a continuous vector field $F : S_\varepsilon \to \mathbb{R}^n$

If $F(x) \neq 0$ on a sphere $S_\varepsilon = \varepsilon S^{n-1} \mathbb{R}^n$ then $\gamma[F, S_\varepsilon] \in \mathbb{Z}$ is well-defined.

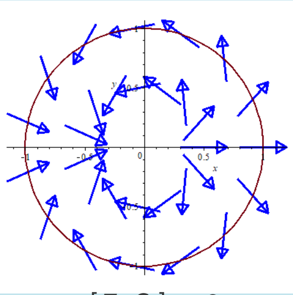
### Topological constraints for asymptotic stability



$\gamma[F, S_\varepsilon] = 0$



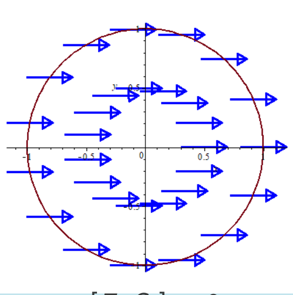
$\gamma[F, S_\varepsilon] = 1$



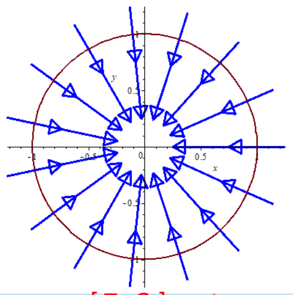
$\gamma[F, S_\varepsilon] = 2$

# Motivation: Obstacles for asymptotic stability

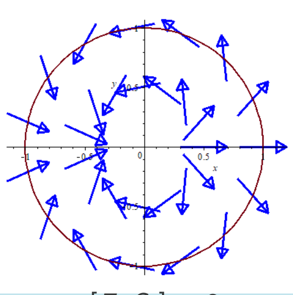
## Topological constraints for asymptotic stability



$\gamma[F, S_\varepsilon] = 0$



$\gamma[F, S_\varepsilon] = 1$

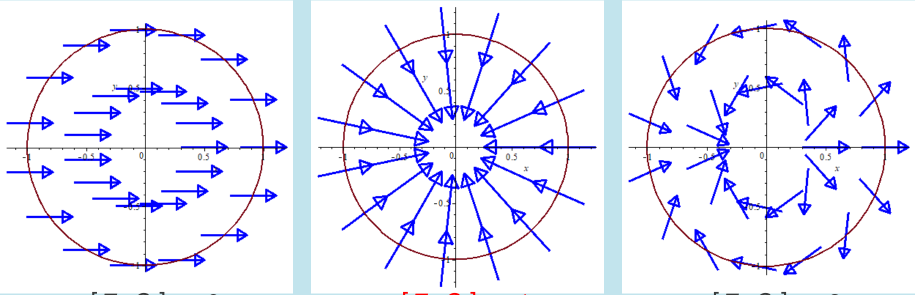


$\gamma[F, S_\varepsilon] = 2$

## Principle of nonzero rotation

If $F \in C(\bar{B})$, $\bar{B}$ - closed ball, $\gamma[F, \partial B] \neq 0 \Rightarrow \exists \tilde{x} \in B : F(\tilde{x}) = 0$.

## Topological constraints for asymptotic stability



$\gamma[F, S_\varepsilon] = 0$ $\quad$ $\gamma[F, S_\varepsilon] = 1$ $\quad$ $\gamma[F, S_\varepsilon] = 2$

## Brockett's necessary stabilizability condition (1983)

If $x = 0$ is stabilizable for $\dot{x} = f(x, u)$ by a continuous feedback law $u = k(x)$, $k(0) = 0$, then $\forall \varepsilon > 0\ \exists \delta > 0$ s.t.

$$B_\delta(0) \subset f\big(B_\varepsilon(0), B_\varepsilon(0)\big), \quad B_\varepsilon(x^*) := \{x : \|x - x^*\| < \varepsilon\}.$$

# Motivation: Obstacles for asymptotic stability

## Examples of non-stabilizable systems

$$\dot{x}_1 = u_1,\ \dot{x}_2 = u_2,\ \dot{x}_3 = x_2 u_1 - x_1 u_2. \qquad (R.W.\ Brockett'83)$$

$$\dot{x}_1 = x_3,\ \dot{x}_2 = x_1^2 - 2x_1 x_3^2,\ \dot{x}_3 = u. \qquad (J.-M.\ Coron\ \&\ L.\ Rosier'92)$$

$$\dot{z} = f_0 z^s + u g_0 z^q,\ z = x_1 + i x_2,\ 2q-1 > s > 1.\ (B.\ Jakubczyk\ \&\ A.Z.'05)$$

## An academic example (Brockett's example)

$$\dot{x}_1 = u_1,\ \dot{x}_2 = u_2,\ \dot{x}_3 = x_2 u_1 - x_1 u_2.$$

Brockett's condition fails: the system of algebraic equations

$$\begin{pmatrix} u_1 \\ u_2 \\ x_2 u_1 - x_1 u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ p_3 \end{pmatrix} \quad \text{has no solutions if } p_3 \neq 0.$$

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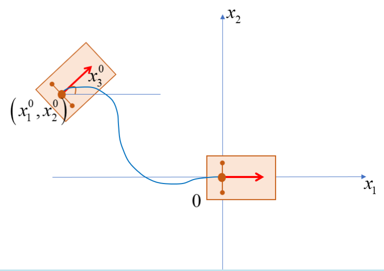
$$\dot{z} = f_0 z^s + u g_0 z^q,\ z = x_1 + i x_2,\ 2q-1 > s > 1.\ (B.\ Jakubczyk\ \&\ A.Z.'05)$$

## A practical motivation: stabilization of nonholonomic systems




Unicycle

$$\dot{x}_1 = u_1 \cos x_3,$$
$$\dot{x}_2 = u_1 \sin x_3,$$
$$\dot{x}_3 = u_2, \quad x \in \mathbb{R}^3,\ u \in \mathbb{R}^2.$$

Control Lyapunov functions do not exist for underactuated ($m < n$) driftless ($f_0(x) \equiv 0$) systems!

## Dynamic extension of Euler's equations with $dim(u) = 2$

$$\begin{aligned}
\dot\omega &= A\omega \times \omega + \mu_1 u_1 + \mu_2 u_2,\\
\dot\phi &= \omega_1 \cos\theta + \omega_3 \sin\theta,\\
\dot\theta &= \omega_1 \sin\theta \tan\phi + \omega_2 - \omega_3 \cos\theta \tan\phi,\\
\dot\psi &= -\omega_1 \sin\theta \sec\phi + \omega_3 \cos\theta \sec\phi.
\end{aligned}$$

## C. Byrnes (2008): Brockett's condition is violated!

The algebraic equation

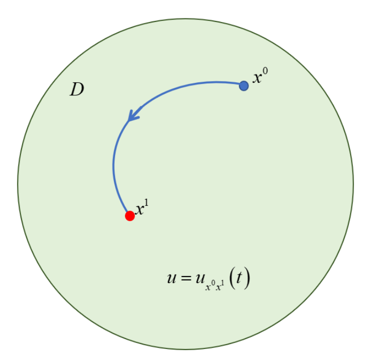
$$f(x, \phi, \theta, \psi, u_1, u_2) = (y_1, y_2, y_3, 0, 0, 0)^T$$

has no solutions generically for small $|y|$.

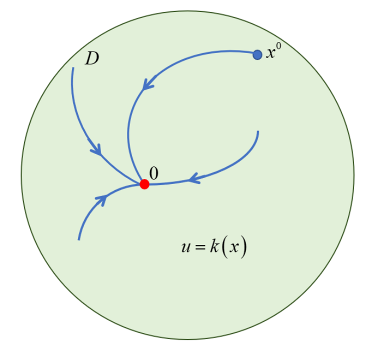
# Motivation: Controllability $\Rightarrow$ Stabilizability ?

$$\dot{x} = f_0(x) + \sum_{j=1}^{m} u_j f_j(x) \equiv f(x,u), \quad x \in D \subset \mathbb{R}^n,\ u \in \mathbb{R}^m,\ m < n. \qquad (\Sigma)$$

**General question: Controllability $\Rightarrow$ Stabilizability ?**




? $\Rightarrow$




$\forall x^0, x^1 \in D\ \exists u_{x^0 x^1} \in L^\infty[0,T]$

$\exists k \in C(D) :\ k(0) = 0$

$$\dot{x} = f_0(x) + \sum_{j=1}^{m} u_j f_j(x) \equiv f(x,u), \quad x \in D \subset \mathbb{R}^n,\ u \in \mathbb{R}^m,\ m < n. \qquad (\Sigma)$$

## General question: Controllability $\Rightarrow$ Stabilizability ?

### Linear and Linearizable Systems

$$\dot{x} = Ax + Bu, \quad \operatorname{rank}(B, AB, ..., A^{n-1}B) = n \quad \Rightarrow \quad \exists u = Kx : \quad x = 0 \text{ - exponentially stable}$$

### General Systems of the Form $(\Sigma)$

$$\mathrm{Lie}_{x=0}\{f_0, f_1, ..., f_m\} = \mathbb{R}^n \quad \not\Rightarrow \quad \exists u = k(x) : k \in C(D),\ k(0) = 0 :$$

(Lie algebra rank condition) $\qquad x = 0$ - asymptotically stable

Existence Results

### J.-M. Coron (1995)

Assume that $x = 0$ is locally continuously reachable in small time for the control system

$$\dot{x} = f(x,u),\ (x,u) \in \mathcal{O} \subset \mathbb{R}^n \times \mathbb{R}^m,\ (0,0) \in \mathcal{O},\ f(0,0) = 0, \qquad (\Sigma)$$

that $(\Sigma)$ satisfies the Lie algebra rank condition at $(0,0) \in \mathbb{R}^n \times \mathbb{R}^m$ and that $n \notin \{2,3\}$. Then $(\Sigma)$ is locally stabilizable in small time by means of almost smooth periodic time-varying feedback laws $u = k(x,t)$.

### F.H. Clarke, Yu.S. Ledyaev, E.D. Sontag, A.I. Subbotin (1997)

System $(\Sigma)$ is asymptotically controllable if and only if it admits an $s$-stabilizing feedback $u = k(x)$. (Solutions are defined in the sense of sampling – "$\pi$-trajectories" or "$\pi_\varepsilon$-solutions").

## Partition of $t \in [0, +\infty)$

For a given $\varepsilon > 0$, we denote by $\pi_\varepsilon$ the partition of $[0, +\infty)$ into intervals

$$I_j = [t_j, t_{j+1}),\ t_j = \varepsilon j, \quad j = 0, 1, 2, \dots\ .$$

## $\pi_\varepsilon$-solutions

Assume given a feedback $u = h(t, x)$, $h : [0, +\infty) \times D \to \mathbb{R}^m$, $\varepsilon > 0$, and $x^0 \in \mathbb{R}^n$. A $\pi_\varepsilon$-solution of system $(\Sigma)$ corresponding to $x^0 \in D$ and $h(t, x)$ is an absolutely continuous function $x(t) \in D$, defined for $t \in [0, +\infty)$, which satisfies the initial condition $x(0) = x^0$ and the following differential equations

$$\dot{x}(t) = f(x(t), h(t, x(t_j))), \quad t \in I_j = [t_j, t_{j+1}),$$

for each $j = 0, 1, 2, \dots\ .$

## General formulation

Let the assumptions of Coron's theorem be satisfied for the control system

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} u_i f_i(x), \quad x \in D \subset \mathbb{R}^n,\ u \in \mathbb{R}^m,\ 0 \in D. \qquad (\Sigma)$$

Is it possible to construct a time-varying feedback law

$$u_j = \sum_{k=-N}^{N} a_{jk}(x) \exp\left\{ i \frac{2\pi k t}{\varepsilon} \right\} \in \mathbb{R}, \quad j = 1, 2, ..., m, \qquad (C)$$

such that the solution $x = 0$ of $(\Sigma)$, $(C)$ is asymptotically (exponentially) stable? Here $a_{jk}(x)$ are piecewise smooth functions, $a_{jk}(x) \to 0$ as $x \to 0$.

## Why trigonometric polynomials?

### Sine and cosine controls

Let

$$\dot{x} = u_1(t)f_1(x) + u_2(t)f_2(x),\ x(0) = x^0,$$

$$u_1(t) = a\cos\left(\frac{2\pi kt}{\varepsilon}\right),\ u_2(t) = a\sin\left(\frac{2\pi kt}{\varepsilon}\right),\ k \in \mathbb{Z}\setminus\{0\},\ t \in [0,\varepsilon].$$

Then

$$x(\varepsilon) = x^0 + \frac{\varepsilon^2 a^2}{4\pi k}[f_1, f_2](x^0) + O(|a|^3\varepsilon^3), \quad [f_1, f_2](x) := \frac{\partial f_2}{\partial x}f_1(x) - \frac{\partial f_1}{\partial x}f_2(x).$$

### Applications to optimal control, motion planning, stabilization, ...

R.W. Brockett (1981), H.J. Sussmann and W. Liu (1991), R.M. Murray and S.S. Sastry (1993), W. Liu (1997), P. Morin, J.-B. Pomet, and C. Samson (1999), A. Agrachev and A. Sarychev (2005), J.-P. Gauthier, B. Jakubczyk, and V. Zakalyukin (2010), Y. Chitour, F. Jean, and R. Long (2013), F. Jean (2014), ... .

## Nonholonomic system

$$\dot{x} = \sum_{i=1}^{m} u_i f_i(x), \quad x \in D \subset \mathbb{R}^n,\ 0 \in D,\ \|f_i\|_{C^2(D)} < \infty,\ m < n. \qquad (\Sigma_0)$$

Assume the following step-2 bracket generating property at $x = 0$:

$$\mathrm{span}\,\{f_i(x), [f_j, f_l](x)\,|\, i = 1, 2, ..., m, (j, l) \in S\} = \mathbb{R}^n, \qquad (B)$$

where $S \subseteq \{1, \dots, m\}^2$, $m + |S| = n$.

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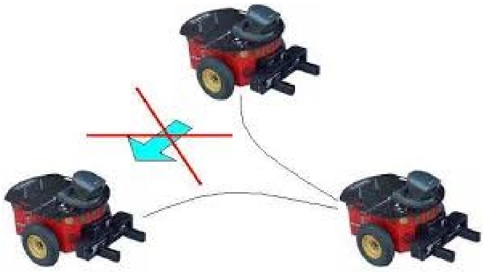
where $S \subseteq \{1, \dots, m\}^2,\ m + |S| = n$.

## Time-varying feedback controls $u_i = u_i^{\varepsilon}(t, x),\ i = 1, 2, ..., m$:

$$u_i^{\varepsilon}(t, x) = v_i + \sum_{(j,l)\in S} a_{jl} \left\{ \delta_{ij} \cos\left(\frac{2\pi k_{jl} t}{\varepsilon}\right) + \delta_{il}\, \mathrm{sign}(a_{jl}) \sin\left(\frac{2\pi k_{jl} t}{\varepsilon}\right) \right\},$$

$$v_i = v_i(x),\ a_{jl} = a_{jl}(x),\ k_{jl} \in \mathbb{Z},\ \varepsilon > 0. \qquad (C)$$



Nonholonomic system $(\Sigma)$

$$\dot{x} = \sum_{i=1}^{m} u_i f_i(x),$$

$$x \in D \subset \mathbb{R}^n,\ u \in \mathbb{R}^m,\ m < n.$$

Extended system $(\Sigma_e)$

$$\dot{x} = \sum_{i=1}^{m} \bar{u}_i f_i(x) + \sum_{(j,l)\in S} \bar{u}_{jl}[f_j, f_l](x),$$

$$\bar{u} = (\bar{u}_1, \ldots, \bar{u}_m, \bar{u}_{jl})_{(j,l)\in S} \in \mathbb{R}^n.$$

## Control Design Scheme

### Main idea: Consider a positive definite function $V(x)$

Define controls of the form $(C)$ to approximate the flow of $\dot{\tilde{x}} = -\nabla V(\tilde{x})$ by trajectories of $(\Sigma_0)$.

### Algebraic equations w.r.t. $v_i$ and $a_{jl}$:

$$\sum_{i=1}^{m} v_i f_i(x) + \frac{\varepsilon}{4\pi} \sum_{(i,j)\in S} \frac{a_{ij}|a_{ij}|}{k_{ij}} [f_i, f_j](x) + \frac{\varepsilon}{2} \sum_{i,j=1}^{m} v_i v_j \frac{\partial f_j(x)}{\partial x} f_i(x) +$$

$$+ \frac{\varepsilon}{2\pi} \sum_{i<j} \left( v_j \sum_{(q,i)\in S} \left| \frac{a_{qi}}{k_{qi}} \right| - v_i \sum_{(q,j)\in S} \left| \frac{a_{qj}}{k_{qj}} \right| \right) [f_i, f_j](x) = -\nabla V(x). \quad (\Sigma_A)$$

### Non-resonance assumption w.r.t. $k_{jl} \in \mathbb{Z} \setminus \{0\}$:

$$|k_{ql}| \neq |k_{jr}| \quad \text{for all} \quad (q,l) \in S, (j,r) \in S, (q,l) \neq (j,r). \qquad (NR)$$

**Theorem 1.** Let $V(x)$ be a function of class $C^2(D)$ such that

$$V(0)=0,\ \|\nabla V(x)\|^2 \geq \alpha_1 V(x),\ V(x)\geq \beta_1\|x\|^2,\ \alpha_1>0,\ \beta_1>0, \tag{1}$$

$$\left\|\frac{\partial f_i(x)}{\partial x}\right\| \leq L, \quad \forall x\in D,\ i\in\{1,\dots,m\}, \tag{2}$$

and let $v_i = v_i^\varepsilon(x)$, $a_{jl} = a_{jl}^\varepsilon(x)$ ($\|x\|\leq\rho_0$, $\varepsilon\leq\varepsilon_0$) be a solution of $(\Sigma_A)$ such that

$$\lim_{\varepsilon\to 0}\left(\sup_{0<\|x\|\leq\rho_0}\frac{\|v^\varepsilon(x)\|+\|a^\varepsilon(x)\|}{\|x\|^{1/3}}\varepsilon^{2/3}\right)=0. \tag{3}$$

Then there exist $\rho\in(0,\rho_0]$, $\bar\varepsilon\in(0,\varepsilon_0]$, and $\lambda>0$:

$$\|x^0\|\leq\rho, \varepsilon\in(0,\bar\varepsilon) \Rightarrow \|x(t)\| = O(e^{-\lambda t}), \|u^\varepsilon(t,x(t))\| = O(e^{-\frac{\lambda t}{3}}) \text{ as } t\to+\infty, \tag{4}$$

for the $\pi_\varepsilon$-solutions of system $(\Sigma_0)$ with controls $(C)$.

- A. Z. "Exponential stabilization of nonholonomic systems by means of oscillating controls", SIAM J. Control Optim., 2016, Vol. 54, P. 1678-1696.

**Theorem 2 (Local Solvability of the System of Algebraic Equations)**

Assume that $f_1(x)$, $f_2(x)$, ..., $f_m(x)$ satisfy the condition $(B)$ at $x = 0$, $|S| = n - m$, and let $V \in C^2(D)$ be a positive definite function. Then, for any small enough $\varepsilon > 0$, there exists a $\Delta > 0$ such that $(\Sigma_A)$ has a solution

$$v^\varepsilon(x) = (v_1^\varepsilon(x), ..., v_m^\varepsilon(x))',\ a^\varepsilon(x) = (a_{jl}^\varepsilon(x)_{(j,l)\in S})',$$

$$k^\varepsilon(x) = (k_{jl}^\varepsilon(x)_{(j,l)\in S})',\ x \in B_\Delta(0).$$

The above solution satisfies

$$\|v^\varepsilon(x)\| \le M_v\|x\|,\ \|a^\varepsilon(x)\| \le M_a\sqrt{\frac{\|x\|}{\varepsilon}}, \quad x \in B_\Delta(0), \tag{5}$$

where the positive constants $M_v$ and $M_a$ do not depend on $\varepsilon$.

## Corollary of Theorems 1 and 2

Assume that $f_1(x)$, $f_2(x)$, ..., $f_m(x)$ satisfy the condition $(B)$ with $|S| = n - m$ at $x = 0$. Then, for any positive definite quadratic form $V(x)$, there exist constants $\rho_0 \geq \rho > 0$ and $\varepsilon_0 \geq \bar{\varepsilon} > 0$ such that the algebraic system $(\Sigma_A)$ admits a solution

$$v_i = v_i^\varepsilon(x),\ a_{jl} = a_{jl}^\varepsilon(x), \quad x \in \overline{B_{\rho_0}(0)} \subset D,\ \varepsilon \in (0, \varepsilon_0],$$

$$i \in \{1, ..., m\},\ (j, l) \in S,$$

and, for any $\varepsilon \in (0, \bar{\varepsilon}]$, there is a $\lambda = \lambda(\varepsilon) > 0$:

$$x^0 \in \overline{B_\rho(0)} \Rightarrow \|x(t)\| = O(e^{-\lambda t}),\ \|u^\varepsilon(t, x(t))\| = O(e^{-\lambda t/3}) \text{ as } t \to +\infty, \quad (6)$$

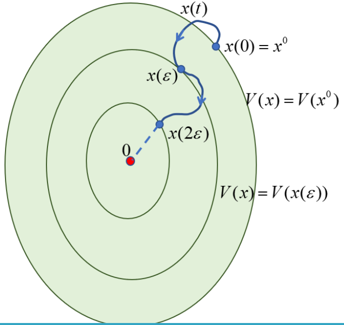
for each $\pi_\varepsilon$-solution $x(t)$ of system $(\Sigma_0)$ with $(C)$.

**Technical Lemma.** Assume that $V \in C^2(D)$,

$$\beta\|x\|^2 \le V(x) \le \gamma_1\|x\|^2,\ \alpha V(x) \le \|\nabla V(x)\|^2 \le \gamma_2 V(x),\ \left\|\frac{\partial^2 V(x)}{\partial x^2}\right\| \le \mu.$$

If $x : [0,\varepsilon] \to D$ is a function s.t. $x(\varepsilon) = x(0) - \varepsilon \nabla V(x(0)) + r_\varepsilon$, $x(0) \ne 0$, then

$$V(x(\varepsilon)) \le V(x(0)) \left\{1 - \alpha\varepsilon + \frac{\gamma_2 \varepsilon^2 \mu}{2} + \frac{\mu \|r_\varepsilon\|^2}{2\beta\|x(0)\|^2} + \frac{\sqrt{\gamma_2}(1+\varepsilon\mu)\|r_\varepsilon\|}{\sqrt{\beta}\|x(0)\|}\right\}.$$

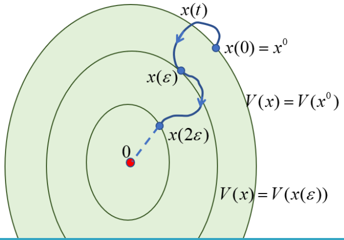


## Volterra (Chen–Fliess) expansion of the solutions of $(\Sigma_0)$

$$x(t)=x^0+\sum_{i=1}^{m} f_i(x^0)\cdot\int_0^t u_i(s)ds+\sum_{i,j=1}^{m}\frac{\partial f_i}{\partial x}f_j\Big|_{x=x^0}\cdot\int_0^t\int_0^s u_i(s)u_j(v)dv\,ds$$

$$+\sum_{i,j,l=1}^{m}\frac{\partial}{\partial x}\left(\frac{\partial f_i}{\partial x}f_j\right)f_l\Big|_{x=x^0}\cdot\int_0^t\int_0^s\int_0^v u_i(s)u_j(v)u_l(p)dp\,dv\,ds+R(t),$$

$$\|R(t)\|=O\left(t^4\|u\|^4_{L^\infty[0,\varepsilon]}\right),\ 0\le t\le\varepsilon.$$



Consider the control system

$$\dot{x}_1 = u_1,\ \dot{x}_2 = u_2,\ \dot{x}_3 = u_1 x_2 - u_2 x_1, \tag{7}$$

where $x = (x_1, x_2, x_3)^* \in \mathbb{R}^3$ is the state and $u = (u_1, u_2)^* \in \mathbb{R}^2$ is the control.

**A. Astolfi (1999):**

System (7) can be exponentially stabilized by a time-invariant feedback law for the initial values in some open and dense set $\Omega \neq \mathbb{R}^3$, $0 \notin \operatorname{int} \Omega$.

# Example 1: Brockett's Example

System (7) satisfies the rank condition $(B1)$ with $S = \{(1,2)\}$:

$$\mathrm{span}\{f_1(x), f_2(x), [f_1, f_2](x)\} = \mathbb{R}^3 \quad \text{for each} \;\; x \in \mathbb{R}^3,$$

where

$$f_1(x) = \begin{pmatrix} 1 \\ 0 \\ x_2 \end{pmatrix}, \; f_2(x) = \begin{pmatrix} 0 \\ 1 \\ -x_1 \end{pmatrix},$$

$$[f_1, f_2](x) = \frac{\partial f_2(x)}{\partial x} f_1(x) - \frac{\partial f_1(x)}{\partial x} f_2(x) = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix}.$$

## Example 1: Brockett's Example

$$V(x) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2).$$

$$u_1^\varepsilon(t,x) = v_1(x) + a(x)\cos\left(\frac{2\pi k}{\varepsilon}t\right), \tag{8}$$

$$u_2^\varepsilon(t,x) = v_2(x) + |a(x)|\sin\left(\frac{2\pi k}{\varepsilon}t\right), \tag{9}$$

where

$$v_1(x) = -x_1,\ v_2(x) = -x_2,\ k \in \mathbb{N},$$

$$a(x) = \begin{cases} -x_1 \pm \sqrt{x_1^2 + \frac{2\pi|x_3|}{\varepsilon}}, & x_3 \neq 0, \\ 0, & x_3 = 0. \end{cases}$$

By Theorem 1, the feedback control (8)–(9) ensures exponential stability of the equilibrium $x = 0$, provided that $\varepsilon > 0$ is small enough.

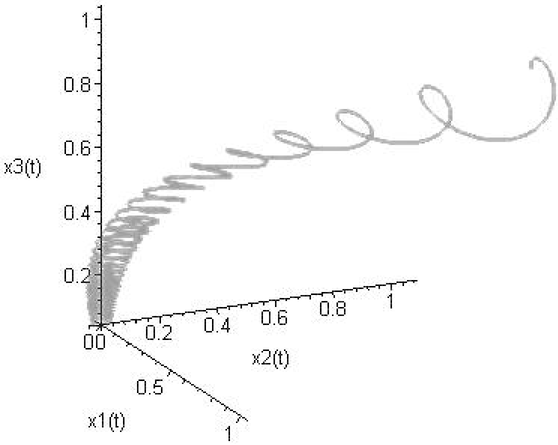



Figure: Trajectory of the closed-loop system (7)–(9) with $x_1^0 = x_2^0 = x_3^0 = 1$ and $\varepsilon = 1$.

## Rolling without slipping

$$\dot{x}_1 = u_1 \cos x_3,\ \dot{x}_2 = u_1 \sin x_3,\ \dot{x}_3 = u_2. \tag{10}$$

Time-varying feedback:

$$u_1^{\varepsilon}(t,x) = v_1(x) + a(x)\cos\left(\frac{2\pi}{\varepsilon}t\right), \tag{11}$$

$$u_2^{\varepsilon}(t,x) = v_2(x) + |a(x)|\sin\left(\frac{2\pi}{\varepsilon}t\right), \tag{12}$$

$$v_1(x) = -x_1\cos x_3 - x_2 \sin x_3,\ v_2(x) = -\varkappa x_3,\ \varkappa > 0,$$

$$a(x) = v_1 \pm \sqrt{v_1^2 - 2\pi v_1 x_3 + \frac{4\pi}{\varepsilon}(x_2\cos x_3 - x_1 \sin x_3)}.$$

## Unit disc rolling on the plane

$$\begin{aligned}\dot{x}_1 &= u_1 \cos x_3,\\ \dot{x}_2 &= u_1 \sin x_3,\\ \dot{x}_3 &= u_2,\\ \dot{x}_4 &= u_1, \qquad x=(x_1,x_2,x_3,x_4)^T \in \mathbb{R}^4,\ u=(u_1,u_2)^T \in \mathbb{R}^2.\end{aligned} \tag{13}$$

Z. Li, J. Canny "Motion of two rigid bodies with rolling constraint", IEEE Tr. Robotics and Autom., 1990, Vol. 6, P. 62-71.

## Bracket generating condition

$$\mathrm{span}\{f_1(x), f_2(x), [f_1,f_2](x), \big[[f_1,f_2],f_2\big](x)\} = \mathbb{R}^4,$$
$$f_1(x) = (\cos x_3, \sin x_3, 0, 1)^T,\ f_2(x) = (0,0,1,0)^T.$$

## Unit disc rolling on the plane

$$
\begin{aligned}
\dot{x}_1 &= u_1 \cos x_3, \\
\dot{x}_2 &= u_1 \sin x_3, \\
\dot{x}_3 &= u_2, \\
\dot{x}_4 &= u_1, \qquad x=(x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4,\ u=(u_1, u_2)^T \in \mathbb{R}^2.
\end{aligned}
\tag{13}
$$

Z. Li, J. Canny "Motion of two rigid bodies with rolling constraint", IEEE Tr. Robotics and Autom., 1990, Vol. 6, P. 62-71.

## Stabilizing controls

$$
\begin{aligned}
u_1^\varepsilon(t,x) &= a_1(x) + a_{12}(x) \cos \frac{2\pi k_{12}}{\varepsilon} t + a_{122}(x) \cos \frac{2\pi k_{1122}}{\varepsilon} t, \\
u_2^\varepsilon(t,x) &= a_2(x) + a_{12}(x) \sin \frac{2\pi k_{12}}{\varepsilon} t + a_{122}(x) \Big( \sin \frac{2\pi k_{2122}}{\varepsilon} t + \sin \frac{2\pi k_{3122}}{\varepsilon} t \Big).
\end{aligned}
$$

## Stabilizing controls

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u_1^\varepsilon(t,x) &= a_1(x) + a_{12}(x)\cos\frac{2\pi k_{12}}{\varepsilon}t + a_{122}(x)\cos\frac{2\pi k_{1122}}{\varepsilon}t,\\
u_2^\varepsilon(t,x) &= a_2(x) + a_{12}(x)\sin\frac{2\pi k_{12}}{\varepsilon}t + a_{122}(x)\Big(\sin\frac{2\pi k_{2122}}{\varepsilon}t + \sin\frac{2\pi k_{3122}}{\varepsilon}t\Big).
\end{aligned}
\tag{14}
$$

Here

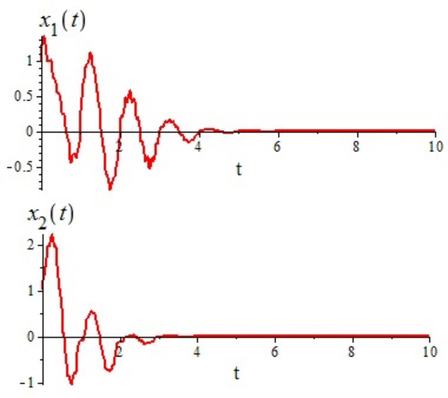
$$
a_1(x) = -\frac{1}{\varepsilon}\frac{\partial V(x)}{\partial x_4},\; a_2(x) = -\frac{1}{\varepsilon}\frac{\partial V(x)}{\partial x_3},\; a_{12}(x) = a_1(x) + \sqrt{a_1(x)^2 + 2k_{12}h(x)},
$$

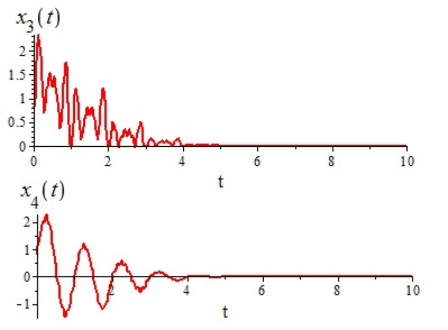
$$
h(x) = \pi a_1(x)a_2(x) + a_1(x)a_{122}(x)\Big(\frac{1}{k_{2122}} + \frac{1}{k_{3122}}\Big) + \frac{2\pi}{\varepsilon^2}\left(\frac{\partial V}{\partial x_2}\cos x_3 - \frac{\partial V}{\partial x_1}\sin x_3\right)
$$

## Cubic equation with respect to $a_{122}$

$$
\begin{aligned}
&\frac{\varepsilon^3 a_{122}^3}{16\pi^2 k_{2122}k_{3122}}\cos(x_3) + \frac{\varepsilon^3}{16}\frac{3k_{1122}^2 + 10k_{2122}k_{3122}}{\pi^2 k_{2122}^2 k_{3122}^2} a_{122}^2 a_1 \cos(x_3) \\
&+ \frac{\varepsilon^3 a_{122}}{4\pi}\left(\frac{2(k_{2122}+k_{3122})}{\varepsilon k_{2122}k_{3122}} a_1 \sin(x_3) + \frac{(k_{2122}+k_{3122})a_1 a_2}{k_{2122}k_{3122}} + \frac{a_2^2}{\pi k_{1122}^2}\right. \\
&\left.+\frac{(k_{2122}+k_{3122})a_1 a_{12}}{\pi k_{2122}k_{3122}k_{12}} - \frac{(k_{2122}+k_{3122})a_{12}^2}{\pi k_{2122}k_{3122}k_{12}}\right) \\
&+ \frac{\varepsilon^3}{4\pi k_{12}}\left(\frac{2\pi_{12}}{3} a_1 a_2^2 + a_1 a_2 a_{12} + \frac{a_2^2 a_{12}}{\pi k_{12}} + \frac{3 a_1 a_{12}^2}{4\pi k_{12}}\right. \\
&\left.- \frac{a_2 a_{12}^2}{2} - \frac{a_{12}^3}{2\pi k_{12}}\right)\cos(x_3) + \frac{\varepsilon^2}{4\pi k_{12}}\left(2\pi k_{12} a_1 a_2 + 2a_1 a_{12} - a_{12}^2\right) \\
&- \varepsilon a_1 \cos(x_3) = -\frac{\partial V(x)}{\partial x_1}.
\end{aligned}
$$

## Time plots of the $\pi_\varepsilon$-solution of system (13) with controls (14)




## Initial conditions and parameters

$$x_1(0) = x_2(0) = 1, \; x_3(0) = x_4(0) = \pi/4, \; \varepsilon = 1.$$

$$\dot{x} = f_0(x) + \sum_{j=1}^{m} u_j f_j(x) \equiv f(x,u), \quad x \in D \subset \mathbb{R}^n,\ u \in \mathbb{R}^m,\ m < n. \qquad (\Sigma)$$

**Lie bracket of $f_i(x)$ and $f_j(x)$:**

$$[f_i, f_j](x) = L_{f_i} f_j(x) - L_{f_j} f_i(x),\ L_{f_i} f_j = \frac{\partial f_j(x)}{\partial x} f_i(x).$$

**Step-3 bracket generating condition at $x = 0 \in D$:**

$$\mathrm{span}\Big\{ f_i(x),\ [f_{i_1}, f_{i_2}](x),\ \big[f_{j_1}, [f_{j_2}, f_{j_3}]\big](x), [f_l, f_0](x),\ \big[f_{l_1}, [f_{l_2}, f_0]\big](x) \Big\} = \mathbb{R}^n,$$

$$i \in S_1,\ (i_1, i_2) \in S_2,\ (j_1, j_2, j_3) \in S_3,\ (l_1, l_2) \in S_{20},$$

$$S_1, S_{10} \subseteq \{1, 2, \dots, m\},\ S_2, S_{20} \subseteq \{1, 2, \dots, m\}^2,\ S_3 \subseteq \{1, 2, \dots, m\}^3,$$

$$|S_1| + |S_2| + |S_3| + |S_{10}| + |S_{20}| = n.$$

STLC of $(\Sigma)$ at $x = 0$: Sussmann (1987), Agrachev and Sarychev (2005).

**Theorem 3.** *Let $0 \in D$, $f_i \in C^4(D)$, $i = 0, \dots, m$. Assume that:*

$\mathcal{F}(x) = \Big( f_i(x)_{i \in S_1},\ \big[f_{l_1}, [f_{l_2}, f_0]\big](x)_{l \in S_{20}} \Big)$ *is of full rank for all* $x \in D$,

$S_1 \subseteq \{1, ..., m\},\ S_{20} \subseteq \{1, ..., m\}^2,\ |S_1| + |S_{20}| = n,$

$f_0(0) = L_{f_0} f_0(0) = 0 = L_{f_0} L_{f_0} f_0(0) = \big[f_0, [f_0, f_k]\big](0) = 0,$

$\big[f_{l_1}, [f_{l_1}, f_0]\big](x) + \big[f_{l_2}, [f_{l_2}, f_0]\big](x) = O(\|x\|^\mu),\ \big[f_0, [f_{l_1}, f_{l_2}]\big](x) = O(\|x\|^\mu),\ \mu > 0,$

*for any* $(l_1, l_2) \in S_{20}$ *and any* $k : (l_1, k) \in S_{20}$ *or* $(k, l_2) \in S_{20}$.

*Then the time-varying feedback control*

$$u_k^\varepsilon(t, x) = \sum_{i \in S_1} \delta_{ki} a_i + \tfrac{4\pi}{\varepsilon} \sum_{(l_1, l_2) \in S_{20}} \kappa_{l_1 l_2} \big(\delta_{k l_1} + \delta_{k l_2} \mathrm{sign}(a_{l_1 l_2})\big) \sqrt{|a_{l_1 l_2}|} \cos\Big(\tfrac{2\pi \kappa_{l_1 l_2} t}{\varepsilon}\Big),$$

$$\begin{pmatrix} a_i \\ a_{l_1 l_2} \end{pmatrix}_{i \in S_1, (l_1, l_2) \in S_{20}} = -\mathcal{F}^{-1}(x)\big(Qx + f_0(x)\big), \quad Q = Q^\top > 0, \quad (C)$$

*ensures asymptotic stability of the trivial solution of* $(\Sigma)$, *if* $\varepsilon > 0$ *is small enough and positive integers* $\kappa_{l_1 l_2}$ *have no resonances of order up to 3.*

## Euler's equations with 2 control torques

$$\dot{x} = f_0(x) + \sum_{j=1}^{2} u_j f_j(x), \quad x \in \mathbb{R}^3,\ u \in \mathbb{R}^2, \qquad (\Sigma_3)$$

$$f_0(x) = (\alpha_1 x_2 x_3, \alpha_2 x_1 x_3, \alpha_3 x_1 x_2)^\top,\ f_1 = (1,0,0)^\top,\ f_2 = (0,1,0)^\top,\ \alpha_3 \neq 0.$$

Bracket generating condition:

$$\mathrm{span}\,\{f_1(x), f_2(x), [f_1,[f_2,f_0]](x)\} = \mathbb{R}^3 \quad \text{at each } x \in \mathbb{R}^3.$$

## Matrix notation

$$\mathcal{F}(x) = \Big(f_i(x)_{i\in S_1}, [f_{l_1},[f_{l_2},f_0]](x)_{(l_1,l_2)\in S_{20}}\Big),\ S_1 = \{1,2\},\ S_{20} = \{(1,2)\}.$$

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$$\operatorname{span}\{f_1(x), f_2(x), [f_1,[f_2,f_0]](x)\} = \mathbb{R}^3 \quad \text{at each } x \in \mathbb{R}^3.$$

## Time-varying controls (Exponential stabilization)

$$u_1 = a_1 + \frac{4\pi\sqrt{|a_{12}|}}{\varepsilon}\cos\left(\frac{2\pi t}{\varepsilon}\right),\ u_2 = a_2 + \frac{4\pi\sqrt{|a_{12}|}}{\varepsilon}\operatorname{sign}(a_{12})\cos\left(\frac{2\pi t}{\varepsilon}\right). \quad (C)$$

Control design with $(a_1, a_2, a_{12})^\top = a(x)$:

$$a(x) = -\mathcal{F}^{-1}(x)\Big(\gamma x + f_0(x)\Big),\ \gamma > 0.$$

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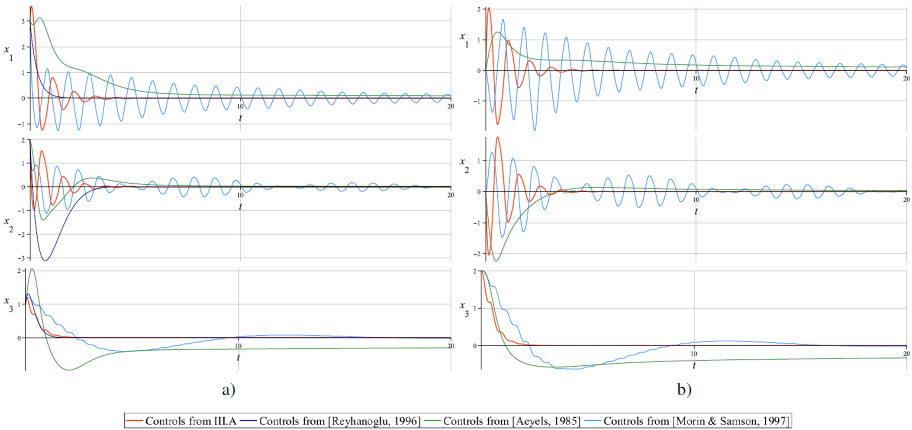
Control design with $(a_1, a_2, a_{12})^\top = a(x)$:

$$a_1 = \gamma x_1 + \alpha_1 x_2 x_3,\ a_2 = \gamma x_2 + \alpha_2 x_1 x_3,\ a_{12} = \frac{\gamma}{2\alpha_3} x_1 + \frac{1}{2} x_1 x_2.$$

## Simulation results




Solutions of $(\Sigma_3)$ with (C). Fig. a): $x(0) = (3,2,1)^\top$; fig. b): $x(0) = (0,0,2)^\top$.

Cf.: Reyhanoglu (1996), Aeyels (1985), Morin and Samson (1997).

**The Navier–Stokes Equations on $\mathbb{T}^2$ (case of incompressible fluid)**

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)\, v + \frac{1}{\rho}\nabla p - \nu \Delta v = \sum_{j=1}^{m} u_j(t) F_j(y), \quad y = (y_1, y_2) \in \mathbb{T}^2,$$

$$\nabla \cdot v = 0, \qquad \textit{(continuity equation)}$$

where $v = (v_1(t,y), v_2(t,y))$ – velocity, $p = p(t,y)$ – pressure.

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## Reduction Scheme

$$v(t,y) \leftrightarrow w(t,y) = \nabla^{\perp} \cdot v = \frac{\partial v_2}{\partial y_1} - \frac{\partial v_1}{\partial y_2}$$

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## Galerkin Approximations with $m$ inputs

$$\dot{q}_k = \sum_{l+n=k} (l_1 n_2 - l_2 n_1)|l|^{-2} q_l q_n - \nu |k|^2 q_k + \sum_{j=1}^{m} u_j \phi_{jk}, \quad k, l, n \in G. \qquad (\Gamma)$$

A. Agrachev and A. Sarychev (2005) "Navier–Stokes Equations: Controllability by means of Low Modes Forcing", *Journal of Mathematical Fluid Mechanics*, Vol. **7**: 108–152.

# Stabilization of the Galerkin System

## Galerkin approximation of the Euler equations ($\nu = 0$)

$$\dot{x} = f_0(x) + \sum_{j=1}^{m} u_j f_j(x), \quad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m. \tag{$\Gamma$}$$

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$$\dot{x} = f_0(x) + \sum_{j=1}^{m} u_j f_j(x), \quad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m. \tag{$\Gamma$}$$

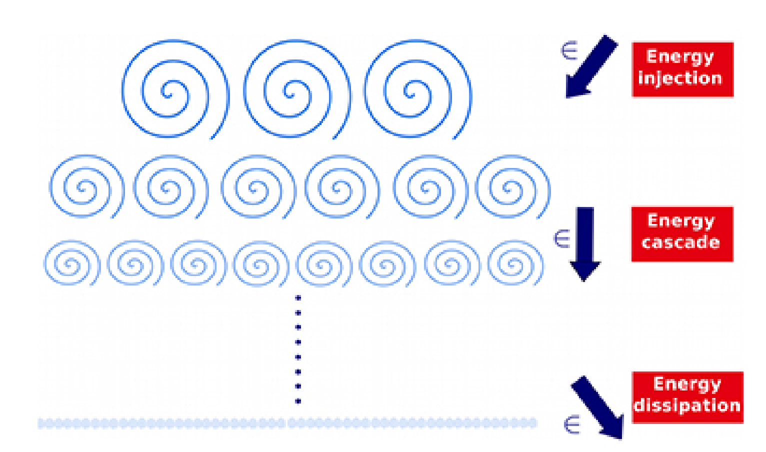
## Step-3 bracket generating condition

$$\mathrm{span}\{f_j(x),\ [f_\alpha,[f_\beta,f_0]](x) : j \in S_1,\ (\alpha\,(\beta\,0)) \in S_3\} = \mathbb{R}^n \text{ for all } x \in \mathbb{R}^n, \tag{$B_3$}$$

$$S_1 \subset \{1,...,m\},\ S_2 \subset \{(\alpha\,(\beta\,0)) : \alpha,\beta \in \{1,...,m\}\},\ |S_1| + |S_3| = n.$$

- W. E and J. Mattingly (2001)
- A. Agrachev and A. Sarychev (2005)

## Energy cascade




U. Frisch (2018): *Turbulence: The Legacy of A. N. Kolmogorov*, Cambridge University Press.

## Step-3 case: parametrization of controls

$$u(x,t) = \sum_{j\in S_1} u^j + \sum_{j\in S_3} u^j, \qquad (C)$$

$$u^j = a_j e_j \quad \text{for} \;\; j \in S_1,$$

$$u^j = \frac{4\pi\sqrt{|a_j|}}{\varepsilon} \cos\left(\frac{2\pi K_j t}{\varepsilon}\right)(e_\alpha + \mathrm{sign}\,(v_j) e_\beta) \;\; \text{for} \;\; j = (\alpha\,(\beta\, 0)) \in S_3,$$

where $e_j$ is the $j$-th unit vector in $\mathbb{R}^m$.

## Stabilizing control design with $v_j = v_j(x)$

$$\sum_{j\in S_1} a_j f_j(x) + \sum_{j\in S_3} a_j [f_\alpha, [f_\beta, f_0]](x) = -\left(\frac{\partial V(x)}{\partial x}\right)^* - f_0(x). \qquad (\Sigma_A)$$

## Step-3 case: parametrization of controls

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where $e_j$ is the $j$-th unit vector in $\mathbb{R}^m$.

## Theorem 4.

Let the control system $(\Gamma)$ satisfy $(B_3)$ and $f_i = \mathrm{const}$ for $i = 1, 2, ..., m$. Then, for any positive definite quadratic form $V(x)$ and any non-resonant set of integers $\{K_j \mid j \in S_3\}$, the controls $(C)$ with $a_j = a_j(x)$ stabilize the solution $x = 0$ of $(\Gamma)$ asymptotically, provided that $\varepsilon > 0$ is small enough.

## An Example: Galerkin approximation with $n = 8$, $m = 4$

$$\dot{x} = f_0(x) + \sum_{j=1}^{4} u_j f_j(x), \quad x = (x_1, x_2, ..., x_8)^* \in \mathbb{R}^8, \tag{\Gamma}$$

$$G = \{(k_1, k_2) \in \mathbb{Z}^2 |\ |k_1| \leq 1,\ |k_2| \leq 1\},$$

$$q_{1,1} = x_1 + ix_2,\ q_{1,-1} = x_3 + ix_4,\ q_{1,0} = x_5 + ix_6,\ q_{0,1} = x_7 + ix_8.$$

## Controlled modes

$$f_1 = (1, 0, 0, 0, 0, 0, 0, 1)^*,\ f_2 = (0, 1, 0, 0, 0, 0, 1, 0)^*,$$

$$f_3 = (0, 0, 1, 0, 0, 1, 0, 0)^*,\ f_4 = (0, 0, 0, 1, 1, 0, 0, 0)^*.$$

Bracket generating condition:

$$\mathrm{span}\{f_1, ..., f_4, [f_2, [f_1, f_0]], [f_3, [f_1, f_0]], [f_4, [f_1, f_0]], [f_4, [f_2, f_0]]\} = \mathbb{R}^8.$$

## Controls for the case $(B_3)$ with $n = 8$, $m = 4$

$$
\begin{aligned}
u_1 =& a_1 + \frac{4\pi}{\varepsilon}\left(\sqrt{|a_{210}|}\operatorname{sign}(a_{210})\cos(\omega_1 t) + \sqrt{|a_{310}|}\operatorname{sign}(a_{310})\cos(\omega_2 t)\right.\\
&\left. + \sqrt{|a_{410}|}\operatorname{sign}(a_{410})\cos(\omega_3 t)\right),\\
u_2 =& a_2 + \frac{4\pi}{\varepsilon}\left(\sqrt{|a_{210}|}\cos(\omega_1 t) + \varepsilon^2\sqrt{|a_{420}|}\operatorname{sign}(a_{420})\cos(\omega_4 t)\right),\\
u_3 =& a_3 + \frac{4\pi}{\varepsilon}\sqrt{|a_{310}|}\cos(\omega_2 t),\\
u_4 =& a_4 + \frac{4\pi}{\varepsilon}\left(\sqrt{|a_{410}|}\cos(\omega_3 t) + \sqrt{|a_{420}|}\cos(\omega_4 t)\right),
\end{aligned}
$$

where $\varepsilon > 0$,

$$
\omega_1 = \frac{2\pi K_{210}}{\varepsilon},\ \omega_2 = \frac{2\pi K_{310}}{\varepsilon},\ \omega_3 = \frac{2\pi K_{410}}{\varepsilon},\ \omega_4 = \frac{2\pi K_{420}}{\varepsilon},
$$

$v = (v_1, v_2, v_3, v_4, v_{210}, v_{310}, v_{410}, v_{420}) = -M^{-1}\left(Qx + f_0(x)\right)$, $Q$−positive definite.

## Higher order controllability conditions

$$\dot{x} = f_0(x) + \sum_{j=1}^{m} u_j f_j(x), \quad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m. \qquad (\Gamma)$$

Consider the following sets of "words of indices":

$$\mathcal{W}_1 = \{(\alpha) : \alpha \in \{1, 2, ..., m\}\},$$
$$\mathcal{W}_k = \{(\alpha\,(\beta\,0)) :\ \alpha \in \mathcal{W}_l,\ \beta \in \mathcal{W}_p,\ l, p - \text{odd},\ l + p = k - 1\},\ k = 3, 5, 7, \dots .$$

Define the map $B(f_i, f_j) := [f_i, [f_j, f_0]]$ and introduce iterated Lie brackets with the indices from $\mathcal{W}_k$:

$$f_j = B(f_\alpha, f_\beta) \quad \text{for } j = (\alpha\,(\beta\,0)) \in \mathcal{W}_k.$$

Bracket generating condition:

$$\text{span}\,\{f_j(x) : j \in S\} = \mathbb{R}^n \quad \text{for all } x \in \mathbb{R}^n, \qquad (B)$$

where $S = S_1 \cup S_3 \cup ... \cup S_N,\ S_k \subset \mathcal{W}_k,\ |S| = n.$

## Control design scheme under $(B)$

$$u = \sum_{j \in S_1 \cup S_3 \cup \ldots \cup S_N} u^j,$$

where $u^j = v_j$ for $j \in S_1$, and $u^j = u^j_{\varepsilon_j}(v_j, t)$ is defined recursively in terms of controls implementing along $f_\alpha$ and $f_\beta$ for $j = (\alpha\,(\beta\,0)) \in S_k$, $k \geq 3$.
Main idea: define $v_j = v_j(x)$ from the condition $x(\varepsilon) \approx x_0 - \kappa\varepsilon \nabla V(x_0)$ under a suitable choice of small parameters $\varepsilon > 0$ and $\varepsilon_j > 0$.

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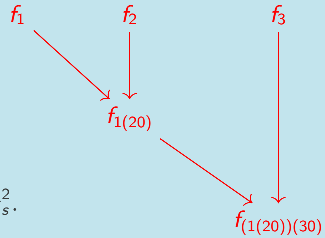
Main idea: define $v_j = v_j(x)$ from the condition $x(\varepsilon)\approx x_0 - \kappa\varepsilon\nabla V(x_0)$ under a suitable choice of small parameters $\varepsilon>0$ and $\varepsilon_j>0$.

## The motion along $f_j = [[f_1,[f_2,f_0]],[f_3,f_0]]$, $j=((1(20))(30))$

$$u^j_{\varepsilon_j}(v_j,t): \begin{aligned} u_1 &= \frac{2|v_j|^{1/4}}{\varepsilon_f}\frac{d}{dt}\left(\sin\left(\frac{t}{\varepsilon_f^2}\right)\cos\left(\frac{t}{\varepsilon_s^2}\right)\right),\\ u_2 &= \frac{\sqrt{2}|v_j|^{1/4}}{\varepsilon_f\varepsilon_s^3}\cos\left(\frac{t}{\varepsilon_f^2}\right),\\ u_3 &= \frac{\sqrt{2|v_j|}}{\varepsilon_s}\mathrm{sign}(v_j)\cos\left(\frac{t}{\varepsilon_s^2}\right),\ \varepsilon_s\asymp\varepsilon,\ \varepsilon_f\asymp\varepsilon_s^2.\end{aligned}$$



# Thank you for your attention!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