Boundary $\mathcal{N} = 2$ Theory, Floer Homologies, Affine Algebras, and the Verlinde Formula

Meng-Chwan Tan

National University of Singapore

Gauge Theories and Differential Invariants, ICTP, 2019

Outline of Talk

- Introduction and Motivation
- Summary of Results
- Main Body of the Talk
- Conclusion

In this talk, we will discuss a topologically-twisted $\mathcal{N} = 2$ gauge theory on a **four-manifold with boundary**.

By exploiting the fact that **physical states** of a TQFT are **insensitive to topological deformations**, 2d and 3d mathematical invariants can be identified.

The motivations for doing so are to:

- Provide **physical proofs** of known mathematical conjectures and theorems.
- Obtain **physical derivations and generalizations** of *mathematically novel* identities between 2d and 3d invariants, and more.

Introduction and Motivation

This talk is based on

- M.-C. Tan et al., *Boundary $\mathcal{N} = 2$ Theory, Floer Homologies, Affine Algebras, and the Verlinde Formula*, arXiv preprint hep-th/1909.04058 (2019).

Built on earlier insights in

- M.-C. Tan, *Supersymmetric surface operators, four-manifold theory and invariants in various dimensions*, Adv.Theor.Math.Phys. 15, 71-129 (2011).

Related is an earlier work by Lozano-Marino which provides computational proofs of some of our results in

- C. Lozano and M. Marino, *Donaldson Invariants of Product Ruled Surfaces and Two-Dimensional Gauge Theories*, Comm.Math.Phys, vol. 220, no. 2, pp. 231-261, 2001.

Summary of Results

1. In Donaldson-Witten (DW) theory on a four-manifold M_4 with boundary Y_3, and in particular, if the moduli space of instantons is zero-dimensional, the partition function on M_4 is expressed as a sum of instanton Floer homology classes Ψ_{inst}

$$\boxed{Z_{M_4} = \langle 1 \rangle_{\Psi(\Phi_{Y_3})} = \sum_i \Psi_{\text{inst}}(\Phi^i_{Y_3})} \tag{1}$$

2. In DW theory on $M_4 = \Sigma \times C$, shrinking C leads to a sigma model on Σ, with target moduli space of flat connections on C, $\mathcal{M}_{\text{flat}}(C)$, that is characterized by

$$\boxed{F_{ab} = 0} \tag{2}$$

Summary of Results

3. Moreover, upon shrinking C, since the topological term of the form $F \wedge F$ leads to the (holomorphic) pullback, i.e.

$$\boxed{\frac{1}{8\pi^2}\int_{M_4} \mathrm{Tr}\,(F \wedge F) = \int_{\Sigma} X^* \omega_{\text{flat}}} \tag{3}$$

the 2d model on Σ must be an A-twisted sigma model with target $\mathcal{M}_{\text{flat}}(C)$, and action

$$\boxed{\begin{aligned} S'_{DW} = \frac{1}{e^2}\int_{\Sigma} d^2z \ \Bigg(G^{\text{flat}}_{I\bar{J}} \bigg(\frac{1}{2}\partial_z X^I \partial_{\bar{z}} X^{\bar{J}} + \frac{1}{2}\partial_{\bar{z}} X^I \partial_z X^{\bar{J}} \\ + \rho_z{}^{\bar{J}} \nabla_{\bar{z}} \chi^I + \rho_{\bar{z}}{}^I \nabla_z \chi^{\bar{J}} \bigg) \\ - R_{I\bar{J}K\bar{L}} \rho_{\bar{z}}{}^I \rho_z{}^{\bar{J}} \chi^K \chi^{\bar{L}} \Bigg) + i\theta \int_{\Sigma} X^* \omega_{\text{flat}} \end{aligned}} \tag{4}$$

Summary of Results

4. In Seiberg-Witten (SW) theory on a four-manifold M_4 with boundary Y_3, and in particular, if the moduli space of monopoles is zero-dimensional, the partition function on M_4 is expressed as a sum of monopole Floer homology classes Ψ_{mono}

$$\boxed{Z_{M_4} = \langle 1 \rangle_{\Psi(\Phi_{Y_3})} = \sum_i \Psi_{\mathrm{mono}}(\Phi^i_{Y_3})} \tag{5}$$

5. In SW theory on $M_4 = \Sigma \times C$, shrinking C leads to a sigma model on Σ with target moduli space of vortices on C, $\mathcal{M}^q_{\text{vort}}(C)$, that is characterized by

$$\boxed{\begin{aligned} &F_{w\overline{w}} = \frac{i}{4}\left(1 - |\varphi|^2\right) \\ &D_{\overline{w}}\varphi = 0 \end{aligned}} \tag{6}$$

6. Furthermore, upon shrinking C, the topological term of the form $\mathcal{F} \wedge \mathcal{F}$ (where $\mathcal{F}_{\mu\nu} = F_{\mu\nu} - i\,(\overline{\sigma}_{\mu\nu})^{\dot{\alpha}\dot{\beta}}\,\overline{M}_{(\dot{\alpha}}\ M_{\dot{\beta})}$) leads to the (holomorphic) pullback, i.e.

$$\boxed{\frac{1}{8\pi^2}\int_{M_4} \mathcal{F} \wedge \mathcal{F} = \int_{\Sigma} X^* \omega_{\text{vort}}} \tag{7}$$

Hence, the 2d sigma model on Σ must be an A-model with target $\mathcal{M}^q_{\text{vort}}(C)$, and action

$$\boxed{\begin{aligned} S'_{SW} = \frac{1}{e^2}\int_{\Sigma} d^2z \ \Bigg(G^{\text{vort}}_{I\bar{J}} \Bigg(\frac{1}{2}\partial_z X^I \partial_{\bar{z}} X^{\bar{J}} + \frac{1}{2}\partial_{\bar{z}} X^I \partial_z X^{\bar{J}} \\ + \rho_z{}^{\bar{J}} \nabla_{\bar{z}} \chi^I + \rho_{\bar{z}}{}^I \nabla_z \chi^{\bar{J}} \Bigg) \\ - R_{I\bar{J}K\bar{L}} \rho_{\bar{z}}{}^I \rho_z{}^{\bar{J}} \chi^K \chi^{\bar{L}} \Bigg) + i\theta \int_{\Sigma} X^* \omega_{\text{vort}} \end{aligned}} \tag{8}$$

7. Consider DW theory on $M_4 = \mathbb{R}^+ \times Y_3 \cong \mathbb{R}^+ \times I \times_f \Sigma$, of which the partition function sums classes of $\mathrm{HF}_*^{\mathrm{inst}}(Y_3)$.

On the other hand, shrinking Σ leads to the *open* A-model on $\mathbb{R}^+ \times I$ with target $\mathcal{M}_{\mathrm{flat}}(\Sigma)$, where the partition function of the A-model sums classes of the Lagrangian Floer homology $\mathrm{HF}_*^{\mathrm{Lagr}}(\mathcal{M}_{\mathrm{flat}}(\Sigma))$.

The physical equivalence between these partition functions (as well as 4d and 2d instantons) means that we have

$$\boxed{\mathrm{HF}_*^{\mathrm{inst}}(Y_3) \cong \mathrm{HF}_*^{\mathrm{Lagr}}(\mathcal{M}_{\mathrm{flat}}(\Sigma), L_0, L_1)} \tag{9}$$

which is the Atiyah-Floer conjecture.

8. Consider DW theory on $M_4 = \Sigma \times S^1 \times \mathbb{R}^+$, of which the partition function sums classes in $\mathrm{HF}^*_{\text{inst}}(\Sigma \times S^1)$.

On the other hand, shrinking Σ leads to the *closed* A-model on $S^1 \times \mathbb{R}^+$ with target $\mathcal{M}_{\text{flat}}(\Sigma)$, where the partition function of the A-model sums over classes of the symplectic Floer cohomology $\mathrm{HF}^*_{\text{symp}}(\mathcal{M}_{\text{flat}}(\Sigma))$.

Via the physical equivalence between the partition functions of DW theory and the A-model on Σ (as well as 4d and 2d instantons), and further making use of a result by Sadov in [1] which relates symplectic Floer and quantum cohomologies, we have

$$\boxed{QH^*(\mathcal{M}_{\text{flat}}(\Sigma)) \cong \mathrm{HF}^*_{\text{symp}}(\mathcal{M}_{\text{flat}}(\Sigma)) \cong \mathrm{HF}^*_{\text{inst}}\left(\Sigma \times S^1\right)} \tag{10}$$

which is Muñoz's theorem.

Summary of Results

9. The same analysis is carried out for SW theory, in which we consider monopole analogs of (9) and (10). In an analogous manner, we prove the monopole Atiyah-Floer conjecture, which takes the form

$$\boxed{\mathrm{HF}_*^{\mathrm{mono}}(q, Y_3) \cong \mathrm{HF}_*^{\mathrm{Heeg}}\left(\mathcal{M}_{\mathrm{vort}}^{q}(\Sigma), L_0, L_1\right)} \tag{11}$$

10. Our analysis of SW theory also allows us to deduce the *mathematically novel* monopole analog of Muñoz's theorem, which takes the form

$$\boxed{QH^*(\mathcal{M}_{\mathrm{vort}}^{q}(\Sigma)) \cong \mathrm{HF}^*_{\mathrm{symp}}\left(\mathcal{M}_{\mathrm{vort}}^{q}(\Sigma)\right) \cong \mathrm{HF}^*_{\mathrm{mono}}\left(q, \Sigma \times S^1\right)} \tag{12}$$

The results (9)–(12) can be generalized to higher rank gauge groups G, because the physical analysis is either independent of the choice of G, or simply involves a straightforward extension.

Summary of Results

11. Consider DW theory on $M_4 = \Sigma \times D \cong \Sigma \times S^1 \times \mathbb{R}^+$, with gauge group $G = SU(2)$. Instanton Floer homology can be defined on $Y_3 = \Sigma \times S^1$.

Moreover, shrinking D allows us to identify the target of the A-model on Σ with the based loop group ΩG.

Such an A-model is known to possess affine symmetry [2]. The corresponding A-model states form modules of an affine Lie algebra $\mathfrak{g}_{\text{aff}}$, which span the space of $\mathfrak{g}_{\text{aff}}$-modules on Σ that we denote by $\mathfrak{G}_{\text{mod}}(\Sigma)$.

We can identify the corresponding partition functions (as well as 4d and 2d instantons), and hence establish the *mathematically novel* isomorphism

$$\boxed{\mathrm{HF}^{\text{inst}}_*\left(\Sigma \times S^1\right) \cong \mathfrak{G}_{\text{mod}}(\Sigma)} \tag{13}$$

Summary of Results

12. Next, consider DW theory on $M_4 = \Sigma \times_f D \cong M_{g,p} \times \mathbb{R}^+$, where $M_{g,p}$ is a Seifert manifold, where 'g, p' refers to a Σ of genus g with an S^1-bundle that has Chern number p. This means $\Sigma \times S^1 = M_{g,0}$ is a trivially-fibered Seifert manifold.

By inserting p copies of the fibering operator $\mathcal{P}$ [3] in the partition function over $M_{g,0}$, which has the effect of shifting the Chern number by p, the partition function over $M_{g,p}$ is obtained. We may then generalize (13) to the case where $\Sigma \times S^1 \cong M_{g,0}$ is replaced by $M_{g,p}$.

Denoting $\mathfrak{G}_{\mathrm{mod},p}(\Sigma)$ as the space where each basis component is now acted upon by p copies of a suitable representation of $\mathcal{P}$, we similarly show that there is a *mathematically novel* isomorphism

$$\boxed{\mathrm{HF}_*^{\mathrm{inst}}(M_{g,p}) \cong \mathfrak{G}_{\mathrm{mod},p}(\Sigma)} \tag{14}$$

Summary of Results

13. Making use of (10) and (13), it is then straightforward to write down yet another *novel mathematical identity*

$$\boxed{QH^*(\mathcal{M}_{\text{flat}}(\Sigma)) \cong \mathfrak{G}_{\text{mod}}(\Sigma)} \quad (15)$$

14. As a preliminary step to deriving the Verlinde formula, we first study an A-model on $D \cong \mathbb{R}^+ \times S^1$ with target $\mathcal{M}_{\text{flat}}(\Sigma)$. We further shrink S^1 so that we get a QM model on $\mathcal{M}_{\text{flat}}(\Sigma)$ with action

$$\boxed{S_{QM} = \frac{1}{\hbar} \int d\tau \ \frac{1}{2} \dot{X}^I \dot{X}_I} \quad (16)$$

Thus, states of the A-model on D with target $\mathcal{M}_{\text{flat}}(\Sigma)$ are identified with QM states on $\mathcal{M}_{\text{flat}}(\Sigma)$. We can write the commutator relations for $X : \mathbb{R}^+ \to \mathcal{M}_{\text{flat}}(\Sigma)$ as

$$\boxed{[\widehat{X}^I, \widehat{P}^J] = \hbar \delta^{IJ}} \quad (17)$$

which amounts to quantizing $\mathcal{M}_{\text{flat}}(\Sigma)$.

15. Next, we make use of DW theory on $M_4 = \Sigma \times D$, to obtain, upon shrinking D, an A-model on Σ with target ΩG, and, upon shrinking Σ, an A-model on D with target $\mathcal{M}_{\text{flat}}(\Sigma)$. The latter A-model can be viewed as a QM model on $\mathcal{M}_{\text{flat}}(\Sigma)$, as we just saw.

In doing so, we can derive Falting's definition of the Verlinde formula [4, 5]

$$\boxed{V_\ell(\Sigma) \cong H^0(\mathcal{M}_{\text{flat}}(\Sigma), \mathcal{L}^\ell)} \tag{18}$$

where $V_\ell(\Sigma)$ is the space of zero-point conformal blocks of $\mathfrak{g}_{\text{aff}}$ at level ℓ on Σ.

The LHS of (18) is obtained from the A-model on Σ with target ΩG, and the RHS is obtained from the QM model on $\mathcal{M}_{\text{flat}}(\Sigma)$, where H^0 is the space of holomorphic sections of the determinant line bundle $\mathcal{L}$.

16. We also derive Pauly's definition of the Verlinde formula [6], which considers extra operator insertions on Σ. This derivation will proceed similarly to the case with *no* operator insertions, because we can exploit the position-independence of operator insertions in an A-model on Σ.

We are then able to derive the isomorphism

$$\boxed{V_\ell(\Sigma, \vec{p}) \cong H^0(\mathcal{M}_{\text{para}}(\Sigma, \vec{p}), \mathcal{L}^\ell)} \tag{19}$$

where $\vec{p} = (p_1, \cdots, p_n)$ are the operator insertion points on Σ, and $V_\ell(\Sigma, \vec{p})$ is the space of n-point conformal blocks of $\mathfrak{g}_{\text{aff}}$ at level ℓ on Σ.

LET'S EXPLAIN HOW WE GOT THESE RESULTS

DW Theory

In terms of a **scalar supercharge** $\mathcal{Q}$, the action of DW theory can be written in $\mathcal{Q}$-exact form as

$$S_{DW} = \frac{1}{e^2}\text{Tr}\{\mathcal{Q}, V_{DW}\} + \frac{i\theta}{8\pi^2}\int_{M_4}\text{Tr}(F\wedge F)\,. \tag{20}$$

Scalar supercharge $\mathcal{Q}$ generates the transformations

$$\begin{aligned}\delta A_\mu &= \zeta\chi_\mu\\ \delta\phi &= 0\\ \delta\phi^\dagger &= 2\sqrt{2}i\zeta\eta\\ \delta\eta &= i\zeta[\phi,\phi^\dagger]\\ \delta\chi_\mu &= 2\sqrt{2}\zeta D_\mu\phi\\ \delta\lambda_{\dot\alpha\dot\beta} &= i\zeta F^+_{\dot\alpha\dot\beta},\end{aligned} \tag{21}$$

where ζ is a Grassmannian parameter.

Observables of DW Theory

Observables of DW theory are n-point **correlation functions**

$$\langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle = \int \mathcal{D}\Phi \; \mathcal{O}_1 \dots \mathcal{O}_n e^{-S_{DW}}, \tag{22}$$

where $\mathcal{D}\Phi$ denotes the total path-integral measure over all fields. The operators $\mathcal{O}_r$, $r = 1, \dots, n$, are $\mathcal{Q}$-invariant, which means (22) is **independent of the gauge coupling** e.

Taking a Fourier expansion of Φ about its classical values Φ_0 – i.e. $\Phi = \Phi_0 + \sum_{s>0} \Phi_s$ – (22) can be written as

$$\begin{aligned} \langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle &= \left(\int d\Phi_0 \; \mathcal{O}_1 \dots \mathcal{O}_n \right) \left(\int \prod_{s>0} d\Phi_s \; e^{-S_{KE}} \right) \\ &= \left(\int d\Phi_0 \; \mathcal{O}_1 \dots \mathcal{O}_n \right), \end{aligned} \tag{23}$$

where S_{KE} contains only kinetic terms from the DW action.

Observables of DW Theory

To evaluate the Φ_0 part of (23), one studies the BPS equations, which turn out to be the **instanton equations**

$$F_{\mu\nu} + \frac{1}{2}\epsilon_{\mu\nu\rho\lambda}F^{\rho\lambda} = 0, \tag{24}$$

of which the solutions span the moduli space of instantons, $\mathcal{M}^k_{\text{inst}}(M_4)$. Due to $U(1)_R$ R-symmetry, the operator insertions $\mathcal{O}_1 \ldots \mathcal{O}_n$ are constrained to possess an R-charge equal to the virtual dimension of $\mathcal{M}^k_{\text{inst}}(M_4)$.

Let us study the case when the underlying four-manifold has a **boundary**, Y_3, so that observables take the form

$$\langle \mathcal{O}_1 \ldots \mathcal{O}_n \rangle_{\Psi(\Phi_{Y_3})} = \int \mathcal{D}\Phi \; \mathrm{e}^{-S} \; \mathcal{O}_1 \ldots \mathcal{O}_n \cdot \Psi(\Phi_{Y_3}), \tag{25}$$

where $\Psi(\Phi_{Y_3})$ imposes boundary conditions on the fields, and Φ_{Y_3} are fields restricted to Y_3. Importantly, Ψ can be identified with a class in the **instanton Floer homology** $\mathrm{HF}^{\text{inst}}_*(Y_3)$.

Observables of DW Theory

In this study, the **virtual dimension** will be taken to be **zero**, which means that the operators $\mathcal{O}_r$, $r = 1, \ldots, n$ *must* be replaced by the identity operator 1.

Then, observables are **partition functions** that sum classes of $\mathrm{HF}^{\mathrm{inst}}_{*}(Y_3)$, and take the form

$$\boxed{Z_{M_4} = \langle 1 \rangle_{\Psi(\Phi_{Y_3})} = \sum_i \Psi_{\mathrm{inst}}(\Phi^i_{Y_3})} \tag{26}$$

where i denotes the i^{th} gauge connection on Y_3 that descends from an instanton solution on M_4.

DW Theory and a 2d A-model

Let us take $M_4 = C \times \Sigma$, where C and Σ are compact Riemann surfaces with genera $g > 1$ and $h > 1$, respectively. Since we have a product manifold, the metric may be written in a **block diagonal form**

$$ds^2 = (G_\Sigma)_{AB}\, dx^A dx^B + (G_C)_{ab}\, dx^a dx^b. \tag{27}$$

Since we have a TQFT, we can shrink either Riemann surface, say C, without affecting its observables. This **topological deformation** is described by multiplying in a scaling factor ε, such that

$$ds^2 \to ds'^2 = (G_\Sigma)_{AB}\, dx^A dx^B + \varepsilon\, (G_C)_{ab}\, dx^a dx^b, \tag{28}$$

and then set $\varepsilon \to 0$.

Equivalently, we may view this as the limit in which we take Σ to be much bigger than C. This is why we have denoted indices on the small Riemann surface C as $a, b, \cdots$, and those on its large counterpart Σ as $A, B, \cdots$.

To ensure that the action remains finite, this deformation amounts to imposing the flatness condition

$$\boxed{F_{ab} = 0} \tag{29}$$

of which solutions span the **moduli space of flat connections** on C, $\mathcal{M}_{\text{flat}}(C)$.

We may define its **symplectic form** and **metric**, in terms of basis cotangent vectors α as [7]

$$\omega_{IJ}^{\text{flat}} = \int_C d^2w \ \text{Tr}\left(\alpha_{Iw}\alpha_{J\overline{w}} - \alpha_{I\overline{w}}\alpha_{Jw}\right), \tag{30}$$

$$G_{IJ}^{\text{flat}} = \int_C d^2w \ \text{Tr}\left(\alpha_{Iw}\alpha_{J\overline{w}} + \alpha_{I\overline{w}}\alpha_{Jw}\right), \tag{31}$$

where we have switched to complex coordinates on M_4, defined by

$$\begin{aligned} z &= x^1 + ix^2, \quad w = x^3 + ix^4, \\ \overline{z} &= x^1 - ix^2, \quad \overline{w} = x^3 - ix^4, \end{aligned} \tag{32}$$

DW Theory and a 2d A-model

Upon shrinking C, the gauge kinetic term of the DW action becomes

$$\frac{1}{4e^2}\int_{M_4} dx^4 \sqrt{G_{M_4}}\,\mathrm{Tr}\left(F^{\mu\nu}F_{\mu\nu}\right) \rightarrow \frac{1}{e^2}\int_{\Sigma} d^2z\ G^{\text{flat}}_{IJ}\partial_z X^I \partial_{\bar{z}} X^J, \tag{33}$$

where $X : \Sigma \rightarrow \mathcal{M}_{\text{flat}}(C)$. This is the free action for the **2d sigma model** on Σ with target $\mathcal{M}_{\text{flat}}(C)$.

On the other hand, the topological term of the DW action, $k = \frac{1}{8\pi^2}\int_{M_4} \mathrm{Tr}\left(F \wedge F\right)$, becomes

$$\boxed{\frac{1}{8\pi^2}\int_{M_4} \mathrm{Tr}\left(F \wedge F\right) = \int_{\Sigma} X^* \omega_{\text{flat}}} \tag{34}$$

which is the (holomorphic) **pullback** of ω_{flat}.

Moreover, shrinking on a compact Riemann surface generically breaks half of supersymmetry, which means that the 2d sigma model must have **4 supercharges**.

DW Theory and a 2d A-model

The only 2d model consistent with all these features is the **2d A-twisted sigma model**, which has the action

$$\boxed{\begin{aligned} S'_{DW} = \frac{1}{e^2}\int_\Sigma d^2z \ \Bigg(G^{\text{flat}}_{I\bar{J}} \bigg(&\frac{1}{2}\partial_z X^I \partial_{\bar{z}} X^{\bar{J}} + \frac{1}{2}\partial_{\bar{z}} X^I \partial_z X^{\bar{J}} \\ &+ {\rho_z}^{\bar{J}} \nabla_{\bar{z}} \chi^I + {\rho_{\bar{z}}}^I \nabla_z \chi^{\bar{J}} \bigg) \\ &- R_{I\bar{J}K\bar{L}} {\rho_{\bar{z}}}^I {\rho_z}^{\bar{J}} \chi^K \chi^{\bar{L}} \Bigg) + i\theta \int_\Sigma X^* \omega_{\text{flat}} \end{aligned}} \quad (35)$$

where $R_{I\bar{J}K\bar{L}}$ is the Riemann curvature tensor on $\mathcal{M}_{\text{flat}}(C)$, and $\nabla_{\bar{z}}\chi^I = \partial_{\bar{z}}\chi^I + \chi^J \Gamma^I_{JK} \partial_{\bar{z}} X^K$. Here, Γ^I_{JK} are the Christoffel symbols on $\mathcal{M}_{\text{flat}}(C)$.

Because of topological invariance, states of the A-model must be identified with those of DW theory.

SW Theory

The action of SW theory can also be written in $\mathcal{Q}$-exact form as

$$S_{SW} = \frac{1}{e^2}\{\mathcal{Q}, V_{SW}\} + \frac{i\theta}{8\pi^2}\int_{M_4}(F \wedge F)\,. \tag{36}$$

Scalar supercharge $\mathcal{Q}$ generates the transformations

$$\begin{aligned}
\delta A_\mu &= \zeta\chi_\mu, & \delta M_{\dot\alpha} &= -\sqrt{2}\zeta\mu_{\dot\alpha},\\
\delta\phi &= 0, & \delta\overline{M}_{\dot\alpha} &= \sqrt{2}\zeta\overline{\mu}_{\dot\alpha},\\
\delta\phi^\dagger &= 2\sqrt{2}i\zeta\eta, & \delta\mu_{\dot\alpha} &= 2i\zeta\phi M_{\dot\alpha},\\
\delta\eta &= i\zeta[\phi,\phi^\dagger], & \delta\overline{\mu}_{\dot\alpha} &= 2i\zeta\overline{M}_{\dot\alpha}\overline{\phi},\\
\delta\chi_\mu &= 2\sqrt{2}\zeta D_\mu\phi, & \delta\nu^\alpha &= -i\sqrt{2}\zeta D^{\dot\alpha\alpha}M_{\dot\alpha},\\
\delta\lambda_{\dot\alpha\dot\beta} &= i\zeta\left(F^+_{\dot\alpha\dot\beta} + 2i\overline{M}_{(\dot\alpha}M_{\dot\beta)}\right), & \delta\overline{\nu}_\alpha &= -i\sqrt{2}\zeta D_{\alpha\dot\alpha}\overline{M}^{\dot\alpha}.
\end{aligned} \tag{37}$$

where ζ is a Grassmannian parameter.

Observables of SW Theory

The BPS equations are **SW equations**

$$F^{+}_{\dot{\alpha}\dot{\beta}} = -2i\overline{M}_{(\dot{\alpha}}M_{\dot{\beta})} \tag{38a}$$

$$D^{\dot{\alpha}\alpha}M_{\dot{\alpha}} = 0, \tag{38b}$$

of which the solutions span the **moduli space of monopoles**, $\mathcal{M}^{q}_{\text{mono}}(M_4)$, where q is the monopole charge.

When M_4 has a **boundary**, Y_3, and the virtual dimension of $\mathcal{M}^{q}_{\text{mono}}(M_4)$ is zero, relevant observables are **partition functions** that sum classes of the **monopole Floer homology** $\text{HF}^{\text{mono}}_{*}(Y_3)$, that take the form

$$\boxed{Z_{M_4} = \langle 1 \rangle_{\Psi(\Phi_{Y_3})} = \sum_i \Psi_{\text{mono}}(\Phi^i_{Y_3})} \tag{39}$$

where i denotes the i^{th} gauge connection and monopole field on Y_3 that descends from a monopole solution on M_4.

SW Theory and a 2d A-model

Let us take $M_4 = C \times \Sigma$, where C and Σ are compact Riemann surfaces with genera $g > 1$ and $h > 1$, respectively. We can define a **modified gauge curvature** of the form

$$\mathcal{F}_{\mu\nu} = F_{\mu\nu} - i\,(\overline{\sigma}_{\mu\nu})^{\dot{\alpha}\dot{\beta}}\,\overline{M}_{(\dot{\alpha}}\,M_{\dot{\beta})}, \tag{40}$$

so that a kinetic term of the form $\mathcal{F} \wedge *\mathcal{F}$ can be written in the SW action.

Let us shrink C. To ensure that the action remains finite, this topological deformation amounts to imposing the constraints

$$\boxed{\begin{aligned} F_{w\overline{w}} &= \frac{i}{4}\left(1 - |\varphi|^2\right) \\ D_{\overline{w}}\varphi &= 0 \end{aligned}} \tag{41}$$

of which solutions span the **moduli space of charge q vortices connections** on C, $\mathcal{M}^q_{\text{vort}}(C)$.

We may define its **symplectic form** and **metric**, in terms of basis cotangent vectors α and $\tilde{\beta}$ as

$$G^{\text{vort}}_{IJ} = \int_C d^2w \; \left(\alpha_{Iw}\alpha_{J\overline{w}} + \widetilde{\beta}_{Iw}\widetilde{\beta}_{J\overline{w}} + \alpha_{I\overline{w}}\alpha_{Jw} + \widetilde{\beta}_{I\overline{w}}\widetilde{\beta}_{Jw}\right) \tag{42}$$

$$\omega^{\text{vort}}_{IJ} = \int_C d^2w \; \left(\alpha_{Iw}\alpha_{J\overline{w}} + \widetilde{\beta}_{Iw}\widetilde{\beta}_{J\overline{w}} - \alpha_{I\overline{w}}\alpha_{Jw} - \widetilde{\beta}_{I\overline{w}}\widetilde{\beta}_{Jw}\right). \tag{43}$$

Upon shrinking C, the **modified gauge kinetic term** of the SW action becomes

$$\frac{1}{4e^2}\int_{M_4} dx^4\sqrt{G_{M_4}}\;\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu} \to \frac{1}{e^2}\int_{\Sigma} d^2z\; G^{\text{vort}}_{IJ}\partial_z X^I \partial_{\bar{z}} X^J, \tag{44}$$

where $X : \Sigma \to \mathcal{M}^q_{\text{vort}}(C)$. This is the free action for the **2d sigma model** on Σ with target $\mathcal{M}^q_{\text{vort}}(\Sigma)$.

On the other hand, we have the topological term in the SW action

$$S_{top} = \frac{1}{8\pi^2} \int_{M_4} (\mathcal{F} \wedge \mathcal{F}) \,. \qquad (45)$$

Upon shrinking C of the SW action, S_{top}, becomes, in the path integral,

$$\boxed{\frac{1}{8\pi^2} \int_{M_4} \mathcal{F} \wedge \mathcal{F} = \int_{\Sigma} X^* \omega_{\text{vort}}} \qquad (46)$$

which is the (holomorphic) **pullback** of ω_{vort}.

SW Theory and a 2d A-model

The only 2d model consistent with all these features is the **2d A-twisted sigma model**, which has the action

$$\boxed{\begin{aligned} S'_{SW} = \frac{1}{e^2}\int_\Sigma d^2z \ \Bigg(G^{\text{vort}}_{I\bar{J}} \bigg(\frac{1}{2}\partial_z X^I \partial_{\bar{z}} X^{\bar{J}} + \frac{1}{2}\partial_{\bar{z}} X^I \partial_z X^{\bar{J}} \\ + {\rho_z}^{\bar{J}} \nabla_{\bar{z}} \chi^I + {\rho_{\bar{z}}}^I \nabla_z \chi^{\bar{J}} \bigg) \\ - R_{I\bar{J}K\bar{L}} {\rho_{\bar{z}}}^I {\rho_z}^{\bar{J}} \chi^K \chi^{\bar{L}} \Bigg) + i\theta \int_\Sigma X^* \omega_{\text{vort}} \end{aligned}} \quad (47)$$

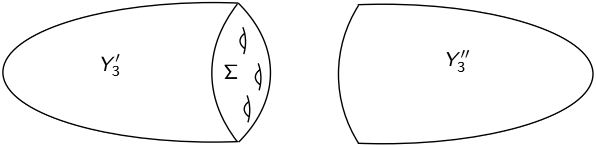
where $R_{I\bar{J}K\bar{L}}$ is now the Riemann curvature tensor on $\mathcal{M}^q_{\text{vort}}(C)$, and $\nabla_{\bar{z}}\chi^I = \partial_{\bar{z}}\chi^I + \chi^J \Gamma^I_{JK} \partial_{\bar{z}} X^K$. Here, Γ^I_{JK} are the Christoffel symbols on $\mathcal{M}^q_{\text{vort}}(C)$.

Because of topological invariance, observables of the A-model must be identified with those of SW theory.

The Atiyah-Floer Conjecture

Consider **DW theory** on $M_4 = \mathbb{R}^+ \times Y_3$, for which the partition function can be identified with $\mathrm{HF}_*^{\mathrm{inst}}(Y_3)$.

A **Heegaard split**, $Y_3 = Y_3' \cup_\Sigma Y_3''$, can be carried out along Σ.



Note that $Y_3^{\prime,\prime\prime} = I^{\prime,\prime\prime} \times_f \Sigma$, which means the metric of $M_4^{\prime,\prime\prime} = \mathbb{R}^+ \times Y_3^{\prime,\prime\prime}$ is a **warped metric** – i.e. it takes the form

$$ds_{M_4^{\prime,\prime\prime}}{}^2 = \left(dx^1\right)^2 + \left(dx^2\right)^2 + f(x^2)\,(G_\Sigma)_{ab}\,dx^a dx^b, \qquad (48)$$

where x^1 is identified with the time-direction $\mathbb{R}^+$.

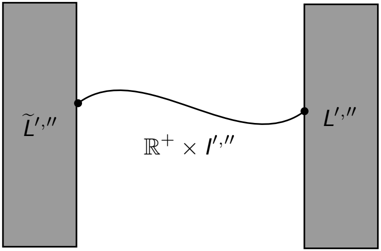
The Atiyah-Floer Conjecture

DW theory is defined on $M_4^{\prime,\prime\prime}$, whence one can carry out a **Weyl rescaling** on (48), so that the metric becomes

$$ds_{M_4^{\prime,\prime\prime}}{}^2 = \frac{1}{f(x^2)}\left[\left(dx^1\right)^2 + \left(dx^2\right)^2\right] + (G_\Sigma)_{ab}\, dx^a dx^b. \qquad (49)$$

Topologically, this describes a **product manifold** $M_4^{\prime,\prime\prime} = \mathbb{R}^+ \times I^{\prime,\prime\prime} \times \Sigma$

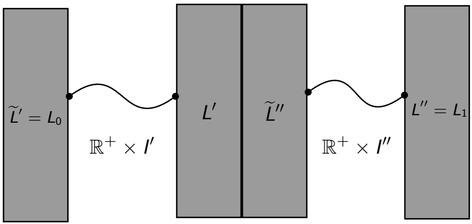
Upon shrinking Σ, we will obtain, from the DW theory on $M_4^{\prime,\prime\prime}$, an **open A-model** on $\mathbb{R}^+ \times I^{\prime,\prime\prime}$ with target $\mathcal{M}_{\text{flat}}(\Sigma)$, where the open string starts and ends on **two Lagrangian branes** $\widetilde{L}^{\prime,\prime\prime}$ and $L^{\prime,\prime\prime}$.



The Atiyah-Floer Conjecture

To relate to DW theory on M_4, we just need to **'glue'** them so that we get a ***single* sigma model** with a pair of ***different* Lagrangian branes** $\widetilde{L}'$ and L''.

We merge the adjacent Lagrangian branes L' and $\widetilde{L}''$, so that the two strings merge into a single open string, which now extends between $\widetilde{L}' = L_0$ and $L'' = L_1$ instead.



The Atiyah-Floer Conjecture

States of the A-model can then be identified with classes of the **Lagrangian Floer homology** $\mathrm{HF}_*^{\mathrm{Lagr}}\left(\mathcal{M}_{\mathrm{flat}}\left(\Sigma\right), L_0, L_1\right)$.

The physical equivalence between the **partition function of DW theory** on M_4 which sums classes in $\mathrm{HF}_*^{\mathrm{inst}}\left(Y_3\right)$, and the **partition function of the A-model** on $\mathbb{R}^+ \times I$ which sums classes in $\mathrm{HF}_*^{\mathrm{Lagr}}\left(\mathcal{M}_{\mathrm{flat}}\left(\Sigma\right), L_0, L_1\right)$, means that we have

$$\boxed{\mathrm{HF}_*^{\mathrm{inst}}\left(Y_3\right) \cong \mathrm{HF}_*^{\mathrm{Lagr}}\left(\mathcal{M}_{\mathrm{flat}}\left(\Sigma\right), L_0, L_1\right)} \tag{50}$$

which is the **Atiyah-Floer conjecture** [8]. Moreover, the grading of the LHS indeed corresponds to that of the RHS since 4d and 2d instanton numbers are equivalent.

The Monopole Analog of the Atiyah-Floer Conjecture

Consider **SW theory** on $M_4 = \mathbb{R}^+ \times Y_3$, for which the partition function can be identified with $\mathrm{HF}^{\mathrm{mono}}_*(q, Y_3)$.

We **Heegaard split** $Y_3 = Y_3' \cup_\Sigma Y_3''$, and take the **Weyl rescaled warped metric**, so Σ can be trivially shrunken away. Hence, an **open A-model** on $\mathbb{R}^+ \times I^{\prime,\prime\prime}$ with target $\mathcal{M}^q_{\mathrm{vort}}(\Sigma)$ is obtained.

States of the A-model are then identified with the classes of the **Heegaard Floer homology** $\mathrm{HF}^{\mathrm{Heeg}}_*\left(\mathcal{M}^q_{\mathrm{vort}}(\Sigma), L_0, L_1\right)$.

Since the **partition function of SW theory** on $M_4 = \mathbb{R}^+ \times Y_3$, which sums classes in $\mathrm{HF}^{\mathrm{mono}}_*(q, Y_3)$, can be identified with the **partition function of the A-model** on $\mathbb{R}^+ \times I$, which sums classes in $\mathrm{HF}^{\mathrm{Heeg}}_*\left(\mathcal{M}^q_{\mathrm{vort}}(\Sigma), L_0, L_1\right)$, we have

$$\boxed{\mathrm{HF}^{\mathrm{mono}}_*(q, Y_3) \cong \mathrm{HF}^{\mathrm{Heeg}}_*\left(\mathcal{M}^q_{\mathrm{vort}}(\Sigma), L_0, L_1\right)} \qquad (51)$$

which is the **monopole Atiyah-Floer conjecture** [9]. Moreover, the grading of the LHS and RHS correspond, as 2d and 4d instanton numbers are equivalent.

Muñoz's Theorem: Relating Instanton Floer to Quantum Cohomology

Consider **DW theory** on $M_4 = \Sigma \times S^1 \times \mathbb{R}^+$. Shrinking Σ away, a **closed A-model** on $S^1 \times \mathbb{R}^+$ with target $\mathcal{M}_{\text{flat}}(\Sigma)$ is obtained.

It is known from [1] that for *any closed* topological A-model with target T, there is an isomorphism between the **quantum cohomology** $QH^*(T)$, and **symplectic Floer cohomology** $\mathrm{HF}^*_{\text{symp}}(T)$. This tells us that our A-model possesses the isomorphism $QH^*(\mathcal{M}_{\text{flat}}(\Sigma)) \cong \mathrm{HF}^*_{\text{symp}}(\mathcal{M}_{\text{flat}}(\Sigma))$.

Since $\partial M_4 = \Sigma \times S^1$, the partition function of DW theory on M_4 will sum classes in the **instanton Floer cohomology** $\mathrm{HF}^*_{\text{inst}}(\Sigma \times S^1)$. Since the partition function of DW theory is equivalent to the partition function of the A-model on $S^1 \times \mathbb{R}^+$ which sums classes in $\mathrm{HF}^*_{\text{symp}}(\mathcal{M}_{\text{flat}}(\Sigma))$, we can write

$$\boxed{QH^*(\mathcal{M}_{\text{flat}}(\Sigma)) \cong \mathrm{HF}^*_{\text{symp}}(\mathcal{M}_{\text{flat}}(\Sigma)) \cong \mathrm{HF}^*_{\text{inst}}\left(\Sigma \times S^1\right)} \tag{52}$$

which is **Muñoz's theorem** [10]. Moreover, the grading of the LHS and RHS correspond, since 2d and 4d instanton numbers are equivalent.

Monopole Analog of Muñoz's Theorem

We will now consider **SW theory** on $M_4 = \Sigma \times S^1 \times \mathbb{R}^+$. Shrinking Σ, we obtain a **closed A-model** on $S^1 \times \mathbb{R}^+$ with target $\mathcal{M}^q_{\text{vort}}(\Sigma)$.

Since the partition function of SW theory on M_4 which sums classes in $\text{HF}^{\text{mono}}_*(q, \Sigma \times S^1)$, equals the partition function of the A-model on $S^1 \times \mathbb{R}^+$ which sums classes in $\text{HF}^*_{\text{symp}}(\mathcal{M}^q_{\text{vort}}(\Sigma))$, we can identify $\text{HF}^{\text{mono}}_*(q, \Sigma \times S^1)$ with $\text{HF}^*_{\text{symp}}(\mathcal{M}^q_{\text{vort}}(\Sigma))$.

Furthermore, note that from [1], $QH^*(\mathcal{M}^q_{\text{vort}}(\Sigma)) \cong \text{HF}^*_{\text{symp}}(\mathcal{M}^q_{\text{vort}}(\Sigma))$. Altogether, these relations can be written as

$$\boxed{QH^*(\mathcal{M}^q_{\text{vort}}(\Sigma)) \cong \text{HF}^*_{\text{symp}}(\mathcal{M}^q_{\text{vort}}(\Sigma)) \cong \text{HF}^*_{\text{mono}}\left(q, \Sigma \times S^1\right)} \qquad (53)$$

This furnishes a *mathematically novel*, **monopole version of Muñoz's theorem**, which is consistent with the suggestion in [11]. Moreover, the grading of the LHS and RHS correspond, since 2d and 4d instanton numbers are equivalent.

Higher Rank Generalizations: DW theory

Let us consider DW theory with **higher rank gauge group** G. All arguments about the relevant Floer homologies hold, since we may simply replace $SU(2)$ with G, and the rest of the analysis remains the same.

This suggests that the **Atiyah-Floer conjecture** can be generalized to G, whereby we obtain an isomorphism between instanton Floer homology and Lagrangian Floer homology for G.

Likewise, we should also be able to generalize **Muñoz's theorem** to G, by starting with higher rank DW theory, whilst noting that Sadov's results in [1] are valid for any G.

Higher Rank Generalizations: SW theory

We can also consider a **nonabelian SW theory**, for which the BPS solutions lead to nonabelian monopoles [12].

Taking $M_4 = \Sigma \times C$, and if a **modified nonabelian field strength** $\mathcal{F}$ can be written, shrinking C will require $\mathcal{F}_{ab} = 0$ to keep the action finite.

This condition should correspond to the **nonabelian vortex equations**, for which solutions will span the **moduli space of nonabelian vortices**. Then, the kinetic term $\mathcal{F} \wedge *\mathcal{F}$ will descend to the action of a sigma model on Σ with target moduli space of nonabelian vortices.

The equivalence of states of the A-model and nonabelian SW theory, then further allows us to obtain the relevant mathematical identities.

There should also be a **nonabelian monopole Floer homology** on Y_3, which will allow us to derive nonabelian versions of the **monopole analogs** of **Atiyah-Floer conjecture** and **Muñoz's theorem**.

Relating Instanton Floer Homology of $\Sigma \times S^1$ to Affine Algebras

Consider **DW theory** on $M_4 = \Sigma \times D \cong \Sigma \times S^1 \times \mathbb{R}^+$, and take $G = SU(2)$. The partition function sums classes in $\mathrm{HF}_*^{\mathrm{inst}}(\Sigma \times S^1)$.

Shrinking D, we obtain an **A-model** on Σ with target $\mathcal{M}_{\mathrm{flat}}(D) \cong \Omega G$, the **based loop group** of G [7, 13].

Such an A-model possesses an **affine Lie algebra** $\mathfrak{g}_{\mathrm{aff}}$ [2]. In addition, A-model states form **modules** of $\mathfrak{g}_{\mathrm{aff}}$.

Thus, since the partition functions of the A-model and DW theory can be equated, we obtain the *mathematically novel* isomorphism

$$\boxed{\mathrm{HF}_*^{\mathrm{inst}}\left(\Sigma \times S^1\right) \cong \mathfrak{G}_{\mathrm{mod}}(\Sigma)} \tag{54}$$

where $\mathfrak{G}_{\mathrm{mod}}(\Sigma)$ is the **space of $\mathfrak{g}_{\mathrm{aff}}$-modules** on Σ. Moreover, the grading of the LHS indeed corresponds to that of the RHS since 4d and 2d instanton numbers are equivalent.

Relating Instanton Floer Homology of Seifert Manifolds to Affine Algebras

Note that $Y_3 = \Sigma \times S^1$ is the **trivially-fibered Seifert manifold** $M_{g,0}$, where '$g, 0$' refers to a Σ of genus g and an S^1-bundle with **Chern number** equal to 0.

Let us generalize our previous discussion to $M_4 = \Sigma \times_f D \cong \Sigma \times_f S^1 \times \mathbb{R}_+$, which has a **nontrivial Seifert manifold** $\Sigma \times_f S^1 = M_{g,p}$.

By inserting p copies of the **fibering operator** [3], $\mathcal{P}$, into the DW partition function on $M_{g,0} \times \mathbb{R}_+$, where $\mathcal{P}$ shifts the Chern number $p_0 \to p_0 + 1$, the DW partition function on $M_{g,p} \times \mathbb{R}_+$ may be written as

$$\langle 1 \rangle_{M_{g,p}} = \langle \mathcal{P} \cdots \mathcal{P} \rangle_{M_{g,0}}. \tag{55}$$

Hence, a sum over classes in $\mathrm{HF}_*^{\mathrm{inst}}(M_{g,p})$ must be given by a sum over classes in $\mathrm{HF}_*^{\mathrm{inst}}(M_{g,0})$ which *each* has been acted upon by $\mathcal{P} \ldots \mathcal{P}$.

Relating Instanton Floer Homology of Seifert Manifolds to Affine Algebras

In turn, from (54), which we repeat here

$$\mathrm{HF}_*^{\mathrm{inst}}\left(\Sigma \times S^1\right) \cong \mathfrak{G}_{\mathrm{mod}}(\Sigma),$$

we have the *mathematically novel* isomorphism

$$\boxed{\mathrm{HF}_*^{\mathrm{inst}}\left(M_{g,p}\right) \cong \mathfrak{G}_{\mathrm{mod},p}(\Sigma)} \tag{56}$$

where p on the right hand side denotes that each basis component of the original space $\mathfrak{G}_{\mathrm{mod}}(\Sigma)$ of $\mathfrak{g}_{\mathrm{aff}}$-modules on Σ has been acted upon p times by a suitable representation of $\mathcal{P}$.

Relating Quantum Cohomology to Affine Algebras

Recall from (52) and (54), that we have the isomorphisms

$$QH^*(\mathcal{M}_{\text{flat}}(\Sigma)) \cong \text{HF}^*_{\text{symp}}(\mathcal{M}_{\text{flat}}(\Sigma)) \cong \text{HF}^*_{\text{inst}}\left(\Sigma \times S^1\right).$$

and

$$\text{HF}^{\text{inst}}_*\left(\Sigma \times S^1\right) \cong \mathfrak{G}_{\text{mod}}(\Sigma).$$

It is then straightforward to write down yet another *mathematically novel* identity

$$\boxed{QH^*(\mathcal{M}_{\text{flat}}(\Sigma)) \cong \mathfrak{G}_{\text{mod}}(\Sigma)} \tag{57}$$

where the grading on the LHS by degree of maps corresponds to the grading on the RHS by energy level.

In hindsight, this result is not surprising.

For every **4d instanton** of charge k on $M_4 = \Sigma \times D \cong \Sigma \times S^1 \times \mathbb{R}^+$, there is a corresponding **2d holomorphic map** $X : S^1 \times \mathbb{R}^+ \to \mathcal{M}_{\text{flat}}(\Sigma)$ of degree k [13, 14].

Relating Symplectic Floer Homology of $\Sigma \times S^1$ to Affine Algebras

This means that $\mathcal{M}^k_{\text{inst}}(M_4) \cong \mathcal{M}^k_{\text{maps}}(S^1 \times \mathbb{R}^+ \to \mathcal{M}_{\text{flat}}(\Sigma))$.

The **homology cycles of $\mathcal{M}^k_{\textbf{inst}}(M_4)$** ought to furnish a module for $\mathfrak{g}_{\text{aff}}$ [arXiv:1701.03298].

This is thus true of **homology cycles of $\mathcal{M}^k_{\textbf{maps}}(S^1 \times \mathbb{R}^+ \to \mathcal{M}_{\textbf{flat}}(\Sigma))$**.

By Poincaré duality, **homology cycles** of $\mathcal{M}^k_{\text{maps}}(S^1 \times \mathbb{R}^+ \to \mathcal{M}_{\text{flat}}(\Sigma))$, must correspond to differential forms on $\mathcal{M}^k_{\text{maps}}(S^1 \times \mathbb{R}^+ \to \mathcal{M}_{\text{flat}}(\Sigma))$, which in turn generate the **quantum cohomology** $QH^*(\mathcal{M}_{\text{flat}}(\Sigma))$.

Hence, $QH^*(\mathcal{M}_{\text{flat}}(\Sigma))$ ought to furnish a module for $\mathfrak{g}_{\text{aff}}$.

This gives an independent verification of the result in (57), which was deduced using purely physical arguments.

As a preliminary step to deriving the Verlinde formula, consider the A-model on $D \cong \mathbb{R}^+ \times S^1$ with target $\mathcal{M}_{\text{flat}}(\Sigma)$. We further shrink S^1 so that SQM is obtained.

Taking $N_I = 0$, the bosonic part of the SQM model can be viewed as a **QM model** on $\mathcal{M}_{\text{flat}}(\Sigma)$, which has the action

$$\boxed{S_{QM} = \frac{1}{\hbar} \int d\tau \ \frac{1}{2} \dot{X}^I \dot{X}_I} \tag{58}$$

where the **Planck's constant** of the QM model is identified as $\frac{1}{\hbar} = \frac{2}{e^2}$. We may define the conjugate momenta by $P^I = \partial L_{QM} / \partial \dot{X}_I = \dot{X}^I$.

Replacing (X, P) with operators $(\widehat{X}, \widehat{P})$, we obtain the commutator relations

$$\boxed{[\widehat{X}^I, \widehat{P}^J] = \hbar \delta^{IJ}} \tag{59}$$

The operators $\widehat{X}^I$ describe *quantized* coordinates, which amounts to **quantizing** $\mathcal{M}_{\text{flat}}(\Sigma)$.

Deriving the Verlinde Formula

The **Verlinde formula** computes the **dimension of the space of conformal blocks**, which can be defined in any 2d conformal field theory (CFT) with affine symmetry.

We shall use the definition by Faltings [4, 5], in which the dimension of the space of conformal blocks on Σ is the same as the **number of holomorphic sections** of (an integer power of) the **determinant line bundle** over $\mathcal{M}_{\text{flat}}(\Sigma)$.

To obtain a physical proof, consider DW theory on $M_4 = \Sigma \times D$, so that we can shrink Σ or D to obtain an A-model on $D \cong S^1 \times \mathbb{R}^+$ or Σ with target $\mathcal{M}_{\text{flat}}(\Sigma)$ or ΩG, which are equivalent.

Physical Proof of the Verlinde Formula – LHS

Let us first shrink D, so that an A-model on Σ with target ΩG can be obtained, which has $\mathfrak{g}_{\text{aff}}$ at level ℓ.

Its states, κ, will be **modules of $\mathfrak{g}_{\text{aff}}$ on** Σ, while the partition function is

$$\langle 1 \rangle = \sum_{v} \kappa_v \overline{\kappa}_v, \tag{60}$$

where v labels the energy eigenstates.

Since we are dealing with modules of $\mathfrak{g}_{\text{aff}}$ on Σ, we can also write (60) in terms of conformal blocks $\mathscr{F}$ [15] as

$$\sum_{v} \kappa_v \overline{\kappa}_v = \sum_{v} \mathscr{F}_v \overline{\mathscr{F}}_v. \tag{61}$$

Thus, κ_v can be identified with the holomorphic conformal block $\mathscr{F}_v$, that spans the **space of zero-point conformal blocks** on Σ, which we denote by $V_\ell(\Sigma)$.

Physical Proof of the Verlinde Formula – RHS

Shrinking Σ, we obtain an A-model on $D \cong \mathbb{R}^+ \times S^1$ with target $\mathcal{M}_{\text{flat}}(\Sigma)$.

Further dimensionally reducing on S^1, as seen earlier, a QM model on $\mathcal{M}_{\text{flat}}(\Sigma)$ is obtained.

The **QM space of states** can be identified as the **space of holomorphic sections** of $\mathcal{L}$, raised to a power $k \in \mathbb{Z}^+$ [16, 17], where $\mathcal{L}$ is the **determinant line bundle** over $\mathcal{M}_{\text{flat}}(\Sigma)$. We shall denote this space by $H^0(\mathcal{M}_{\text{flat}}(\Sigma), \mathcal{L}^k)$.

Define $k = \frac{1}{\hbar} = \frac{2}{e^2}$, so that the QM action can be rewritten as

$$S_{QM} = k \int d\tau\, \dot{X}^I \dot{X}_I, \tag{62}$$

so that k can now be interpreted as the **coupling of the QM model**.

The Verlinde Formula

One can argue that tuning the value of $\frac{1}{e^2}$ also tunes ℓ of $\mathfrak{g}_{\text{aff}}$ from the the A-model on Σ. Since $\hbar$ also descended from the 4d coupling, ℓ and k can indeed be shown to be related.

Since the two spaces of states are equivalent, we obtain the relation

$$\boxed{V_\ell(\Sigma) \cong H^0(\mathcal{M}_{\text{flat}}(\Sigma), \mathcal{L}^\ell)} \tag{63}$$

which is **Faltings's result** [4, 5], from which the **Verlinde formula** is obtained by $\dim V_\ell = \dim H^0(\mathcal{M}_{\text{flat}}(\Sigma), \mathcal{L}^\ell)$.

With Extra Operator Insertions

There is also a more general result found by Pauly [6], in which n operators insertions at $\vec{p} = (p_1, \cdots, p_n)$, where $p_1, \cdots, p_n \in \Sigma$.

There is an isomorphism between the space $V_\ell(\Sigma, \vec{p})$ of n-**point conformal blocks** on Σ, and holomorphic sections of $\mathcal{L}$ over the **moduli space of parabolic vector bundles** on Σ, which we denote by $\mathcal{M}_{\text{para}}(\Sigma, \vec{p})$.

To show this relation physically, let us now insert n **scalar operators** in DW theory on $M_4 = \Sigma \times D$, taking $N_l = 0$ and insisting that $\mathcal{O}_1, \cdots, \mathcal{O}_n$ are inserted at $\vec{p} \in \Sigma$.

Further note that a CFT on Σ with operator insertions at $\vec{p}$, is the same as a CFT *without insertions*, albeit on $\Sigma - \vec{p}$.

We may then carry out the same shrinking procedure on $D \cong \mathbb{R}^+ \times S^1$ or $\Sigma - \vec{p}$ to obtain a 2d sigma model on Σ or $\mathbb{R}^+ \times S^1$, respectively.

With Extra Operator Insertions

Shrinking D, an A-model on Σ with target ΩG is obtained, which gives rise to **correlation functions** $\langle \prod_{r=1}^{n} \mathcal{O}_r(z_r, \bar{z}_r) \rangle$, where $z, \bar{z} \in \Sigma$.

The **position-independence** of operator insertions $\vec{p}$ in a TQFT, means a pair of operators can be merged into a single operator via **fusion rules**. This can be repeated until all n operators coalesce into a single operator $\mathcal{O}'$.

Furthermore, the position-independence of operator insertions also implies that the scalar operator $\mathcal{O}'$ is a **constant**, which means $\langle \prod_{r=1}^{n} \mathcal{O}_r(z_r, \bar{z}_r) \rangle = \langle \mathcal{O}' \rangle = \langle 1 \rangle = \sum_v \kappa_{v;n} \overline{\kappa}_{v;n}$, where $\kappa_{v;n}$ are **A-model eigenstates**, that can be identified with **conformal blocks** [15], wherein

$$\sum_v \kappa_{v;n} \overline{\kappa}_{v;n} = \sum_v \mathscr{F}_{v;n} \overline{\mathscr{F}}_{v;n}. \tag{64}$$

With Extra Operator Insertions

Shrinking $\Sigma - \vec{p}$, an A-model on $D \cong \mathbb{R}^+ \times S^1$ with target $\mathcal{M}_{\text{flat}}(\Sigma - \vec{p})$ is obtained, which can be quantized upon further shrinking S^1.

Note that $\mathcal{M}_{\text{flat}}(\Sigma - \vec{p})$, is the same as the **moduli space of parabolic bundles** on Σ, $\mathcal{M}_{\text{para}}(\Sigma, \vec{p})$ – i.e. $\mathcal{M}_{\text{flat}}(\Sigma - \vec{p}) \cong \mathcal{M}_{\text{para}}(\Sigma, \vec{p})$.

Like before, $\mathcal{L}$ is raised to a power k, whereby k is identified with the coupling constant of the QM model on $\mathcal{M}_{\text{para}}(\Sigma, \vec{p})$. The space of QM states can be identified with $H^0(\mathcal{M}_{\text{para}}(\Sigma, \vec{p}), \mathcal{L}^k)$.

It can also be argued that $k = \ell$, and since we can identify both spaces of states, we can write down the relation

$$\boxed{V_\ell(\Sigma, \vec{p}) \cong H^0(\mathcal{M}_{\text{para}}(\Sigma, \vec{p}), \mathcal{L}^\ell)} \tag{65}$$

which is just **Pauly's result**, from which the **Verlinde formula** is obtained by $\dim V_\ell(\Sigma, \vec{p}) = \dim H^0(\mathcal{M}_{\text{para}}(\Sigma, \vec{p}), \mathcal{L}^\ell)$.

Conclusion

- We exploited properties of 4d $\mathcal{N} = 2$ TQFT in the manner described above, so that physical proofs of known mathematical conjectures and theorems as well as derivations of mathematically novel identities between 3d and 2d invariants, and more, can be obtained.

- Notably, we furnished purely physical proofs of the Atiyah-Floer conjecture and its monopole analog, and Muñoz's theorem relating quantum and instanton Floer cohomology.

- We also physically derived the monopole analog of Muñoz's theorem, and described the higher rank generalizations of relevant mathematical identities.

Conclusion

- For a Seifert manifold, one can also relate its instanton Floer homology to modules of an affine algebra.

- In turn, we uncovered an action of the affine algebra on the quantum cohomology of the moduli space of flat connections on a Riemann surface.

- We also derived the Verlinde formula as defined by Faltings and Pauly, respectively.

THANKS FOR LISTENING!

[1] V. Sadov, "On equivalence of Floer's and quantum cohomology," *Communications in mathematical physics*, vol. 173, no. 1, pp. 77–99, 1995.

[2] M. Ashwinkumar, J. Cao, Y. Luo, M.-C. Tan, and Q. Zhao, "Little strings, quasi-topological sigma model on loop group, and toroidal Lie algebras," *Nuclear Physics B*, vol. 928, pp. 469–498, 2018.

[3] M. Blau and G. Thompson, "Chern-Simons theory on S^1-bundles: Abelianisation and q-deformed Yang-Mills theory," *Journal of High Energy Physics*, vol. 2006, no. 05, p. 003, 2006.

[4] G. Faltings, "A proof for the Verlinde formula," *Journal of Algebraic Geometry*, vol. 3, no. 2, p. 347, 1994.

[5] A. Beauville, "Conformal blocks, fusion rules and the Verlinde formula," *arXiv preprint alg-geom/9405001*, 1994.

[6] C. Pauly, "Espaces de modules de fibrés paraboliques et blocs conformes," *Duke Mathematical Journal*, vol. 84, no. 1, pp. 217–235, 1996.

[7] M. Bershadsky, A. Johansen, V. Sadov, and C. Vafa, "Topological reduction of 4D SYM to 2D σ-models," *Nuclear Physics B*, vol. 448, no. 1-2, pp. 166–186, 1995.

[8] M. Atiyah, "New invariants of 3-and 4-dimensional manifolds," *The mathematical heritage of Hermann Weyl (Durham, NC, 1987)*, vol. 48, pp. 285–299, 1988.

[9] C. Kutluhan, Y.-J. Lee, and C. Taubes, "HF= HM I: Heegaard Floer homology and Seiberg–Witten Floer homology," *arXiv preprint arXiv:1007.1979*, 2010.

[10] V. Muñoz, “Ring structure of the Floer cohomology of $\Sigma \times S^1$,” *Topology*, vol. 38, no. 3, pp. 517–528, 1999.

[11] V. Muñoz and B.-L. Wang, “Seiberg–Witten–Floer homology of a surface times a circle for non-torsion spinC structures,” *Mathematische Nachrichten*, vol. 278, no. 7-8, pp. 844–863, 2005.

[12] J. Labastida and M. Mariño, *Topological quantum field theory and four manifolds*, vol. 25. Springer, 2005.

[13] A. Popov, “Loop groups in Yang–Mills theory,” *Physics Letters B*, vol. 748, pp. 439–442, 2015.

[14] S. Dostoglou and D. Salamon, “Self-dual instantons and holomorphic curves,” *Annals of Mathematics*, vol. 139, no. 3, pp. 581–640, 1994.

[15] P. Francesco, P. Mathieu, and D. Sénéchal, *Conformal field theory*. Springer Science & Business Media, 2012.

[16] E. Witten, “Topological quantum field theory,” *Communications in Mathematical Physics*, vol. 117, no. 3, pp. 353–386, 1988.

[17] G. Segal, “The definition of conformal field theory,” in *Differential geometrical methods in theoretical physics*, pp. 165–171, Springer, 1988.