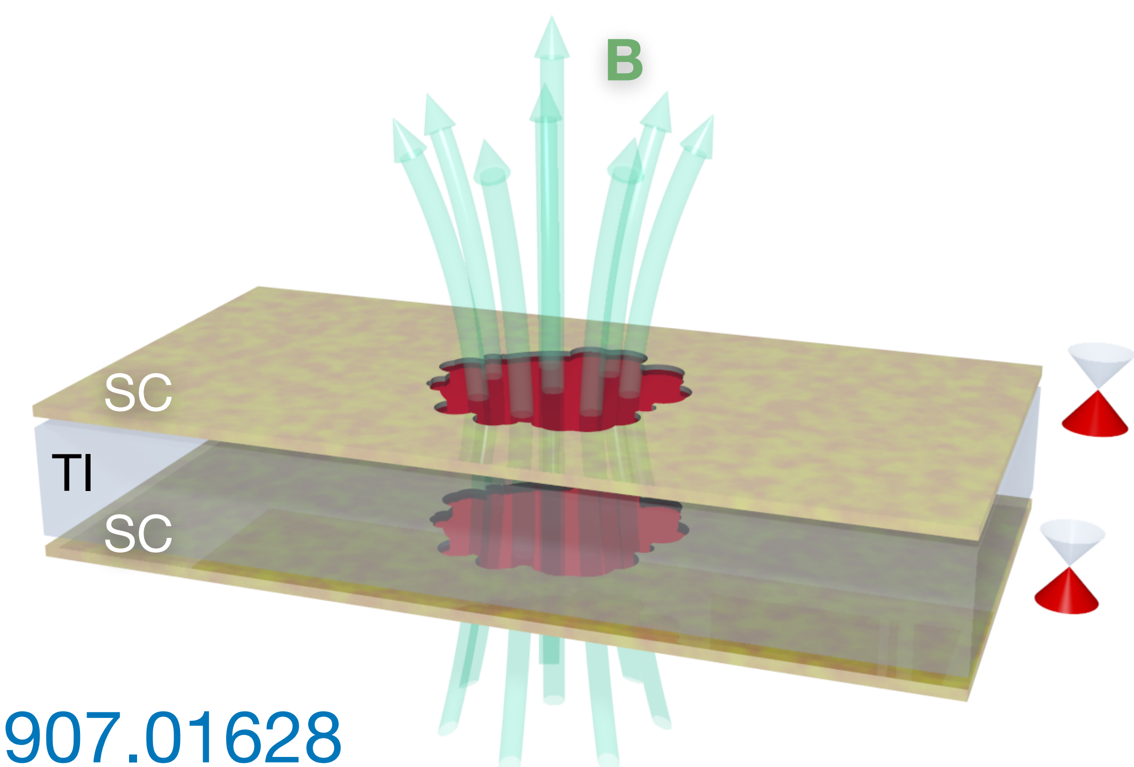
Diagnosing quantum chaos using entanglement

E´tienne Lantagne-Hurtubise, Stephan Plugge, Oguzhan Can, and Marcel Franz



NSERC CRSNG

arXiv:1907.01628

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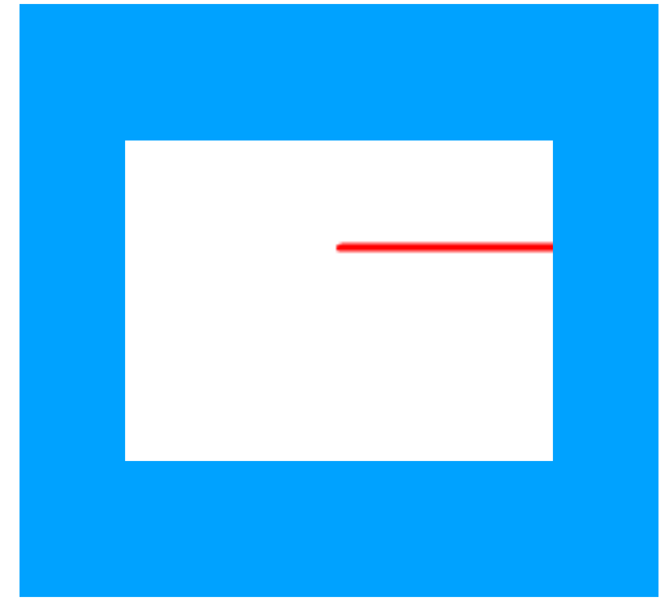
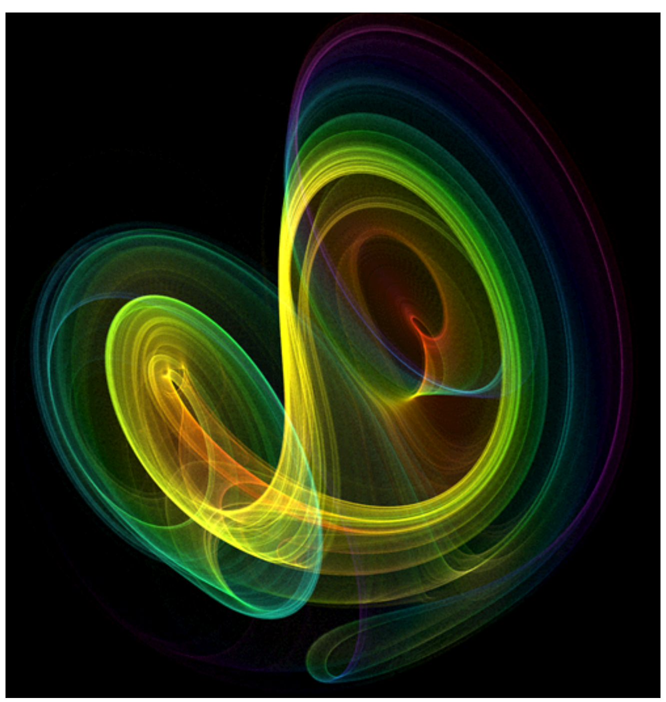
Classical chaos

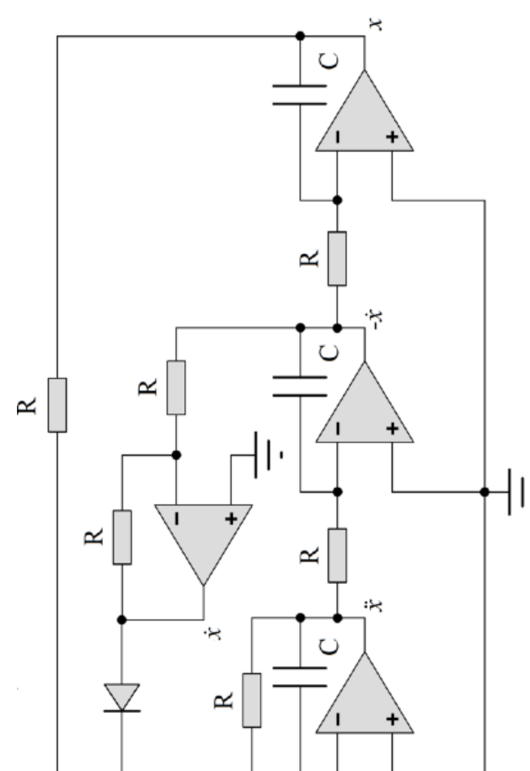
- Exponential sensitivity of classical trajectories to small changes in initial conditions.
- Butterfly effect, breakdown of predictability in deterministic systems
- Many examples of classically chaotic systems exist in mathematics, physics and natural phenomena



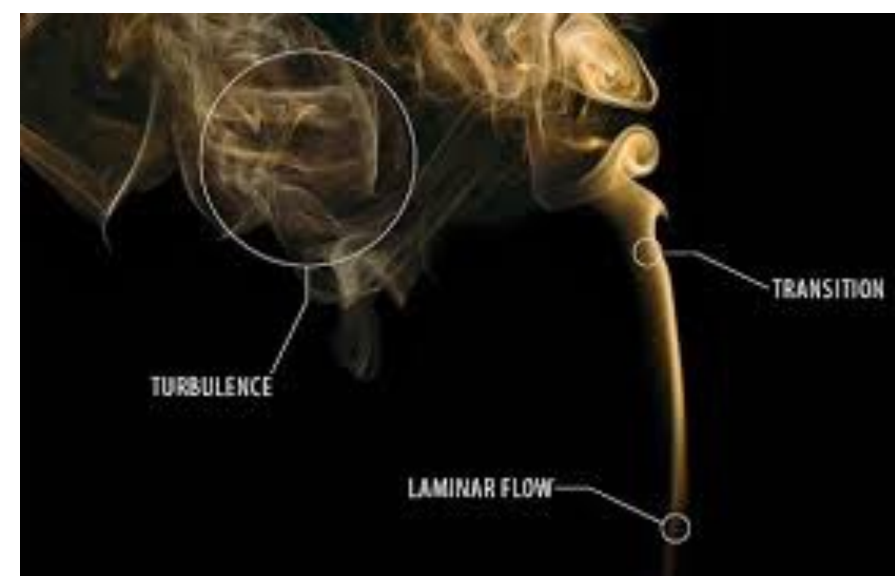
Double-rod pendulum simulation



Lorenz attractor



jerk circuit



turbulent flow

Quantum chaos

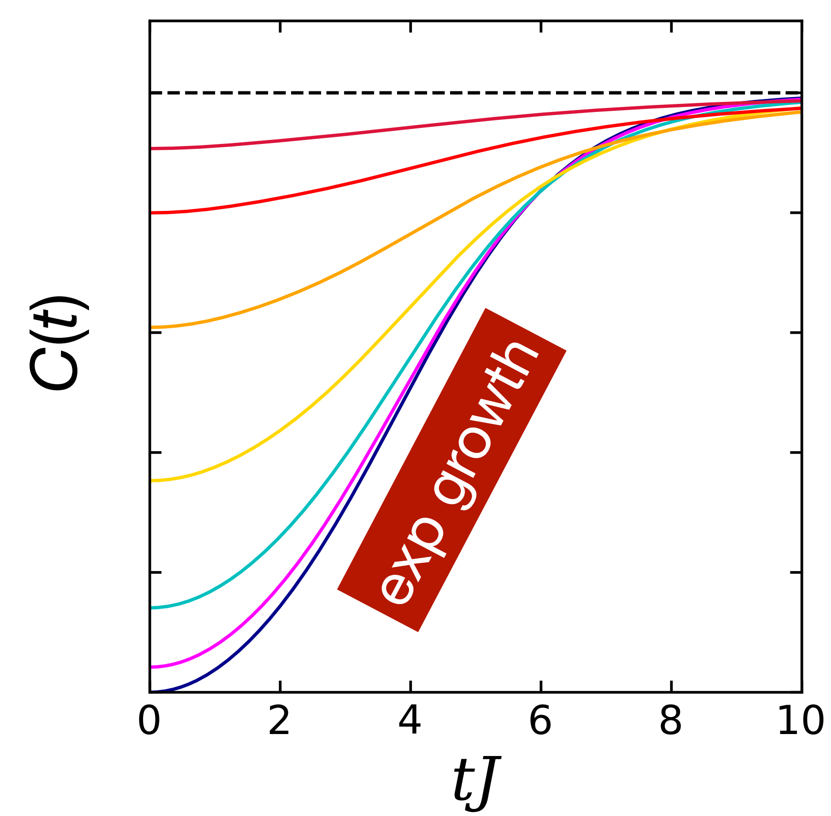
- There are no classical trajectories in quantum systems.
- **Quantum chaos** is therefore quantified using the idea of **operator growth.**

Consider an expectation value

$$C(t) = -\langle [W(t), V(0)]^2 \rangle$$

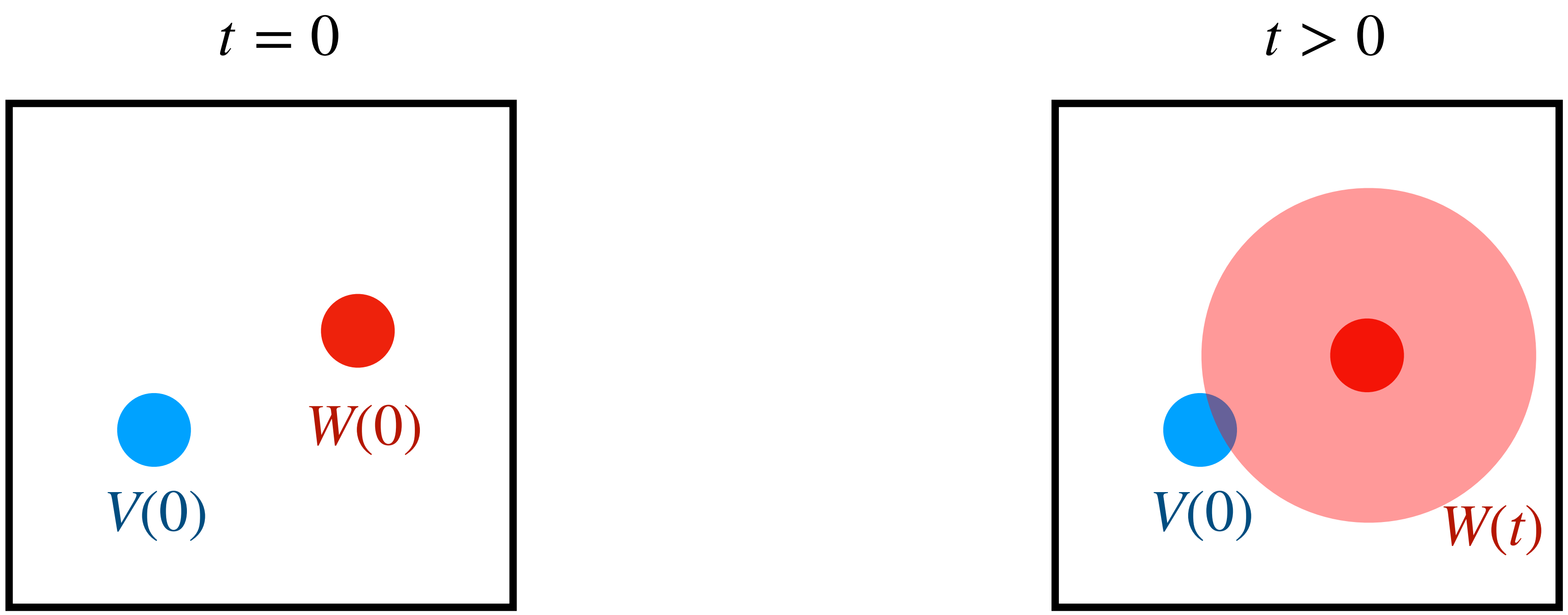
with two initially commuting hermitian operators $[W, V] = 0$.

Time evolution is according to
$W(t) = e^{iHt} W e^{-iHt}$
and the expectation value is with respect to a thermal ensemble at temperature T.



Operator growth intuitive picture:

- As the initially “simple” operator W becomes more complex due to the time evolution it eventually fails to commute with V.



$[V(0), W(0)] = 0$ $\qquad$ $[V(0), W(t)] \neq 0$

- For quantum-chaotic systems the growth of $C(t) = -\langle [W(t), V(0)]^2 \rangle$ is exponential at intermediate times,

$$C(t) \sim e^{\lambda_L t}$$

- Here λ_L is the quantum Lyapunov exponent

The chaos bound

- The Lyapunov exponent λ_L characterizing the exponential growth of $C(t) = -\langle [W(t), V(0)]^2 \rangle$ at intermediate times is subject to a fundamental upper bound

$$\lambda_L \leq 2\pi T$$

[Maldacena, Stanford, Shenker 2017]

- Quantum systems saturating the chaos bound $(\lambda_L = 2\pi T)$ are called "maximally chaotic"
- Such systems are usually holographically dual to a quantum gravity theory in a geometry with a black hole.
- *Black holes* are also maximally chaotic, i.e. they thermalize in shortest possible time consistent with causality and unitarity.

Why is understanding of quantum chaos important?

- Chaotic behaviour is essential to thermalization of closed quantum systems
- It underlies our understanding of important concepts including many-body localization and eigenstate thermalization hypothesis (ETH)
- Quantum chaos is important in attempts to reconcile quantum mechanics with general relativity: It points to a resolution of fundamental open questions such as the Hawking black hole information paradox

Out-of-time-order correlators

Expand the commutator squared:

naturally time ordered (NTOC)

$$C(t) = \langle W(t)VVW(t)\rangle + \langle VW(t)W(t)V\rangle$$
$$-\langle VW(t)VW(t)\rangle - \langle W(t)VW(t)V\rangle$$

"out-of-time-ordered" (OTOC)

- **NTOC** — correspond to *measurable* (at least in principle) quantities

Consider $\langle V(0)W(t)W(t)V(0)\rangle = \langle \Psi_0 V(0) \,|\, W(t)W(t) \,|\, V(0)\Psi_0\rangle$

(ii) evolve the perturbed state forward in time and
(iii) perform a measurement of the quantity represented by $W(t)^2$

(i) create a perturbation at time t=0

- **OTOCs**— correspond to quantities that require backward time evolution to measure

Consider $\langle W(t)V(0)W(t)V(0)\rangle = \langle \Psi_0 W(t)V(0) \,|\, W(t)V(0)\Psi_0\rangle$

This inner product can be interpreted as comparing two quantum states:

$|\Psi_1(t)\rangle = |W(t)V(0)\Psi_0\rangle$ and $|\Psi_2(t)\rangle = |V(0)^\dagger W(t)^\dagger \Psi_0\rangle$

Creating $|\Psi_2(t)\rangle$ clearly requires evolving the system **backward in time**!

- Backward time evolution can be achieved by reversing the sign of the Hamiltonian

$$W(t) = e^{iHt} W e^{-iHt} \rightarrow e^{-iHt} W e^{iHt}$$

- In most systems however this is difficult or impossible to achieve

nature physics

ARTICLES

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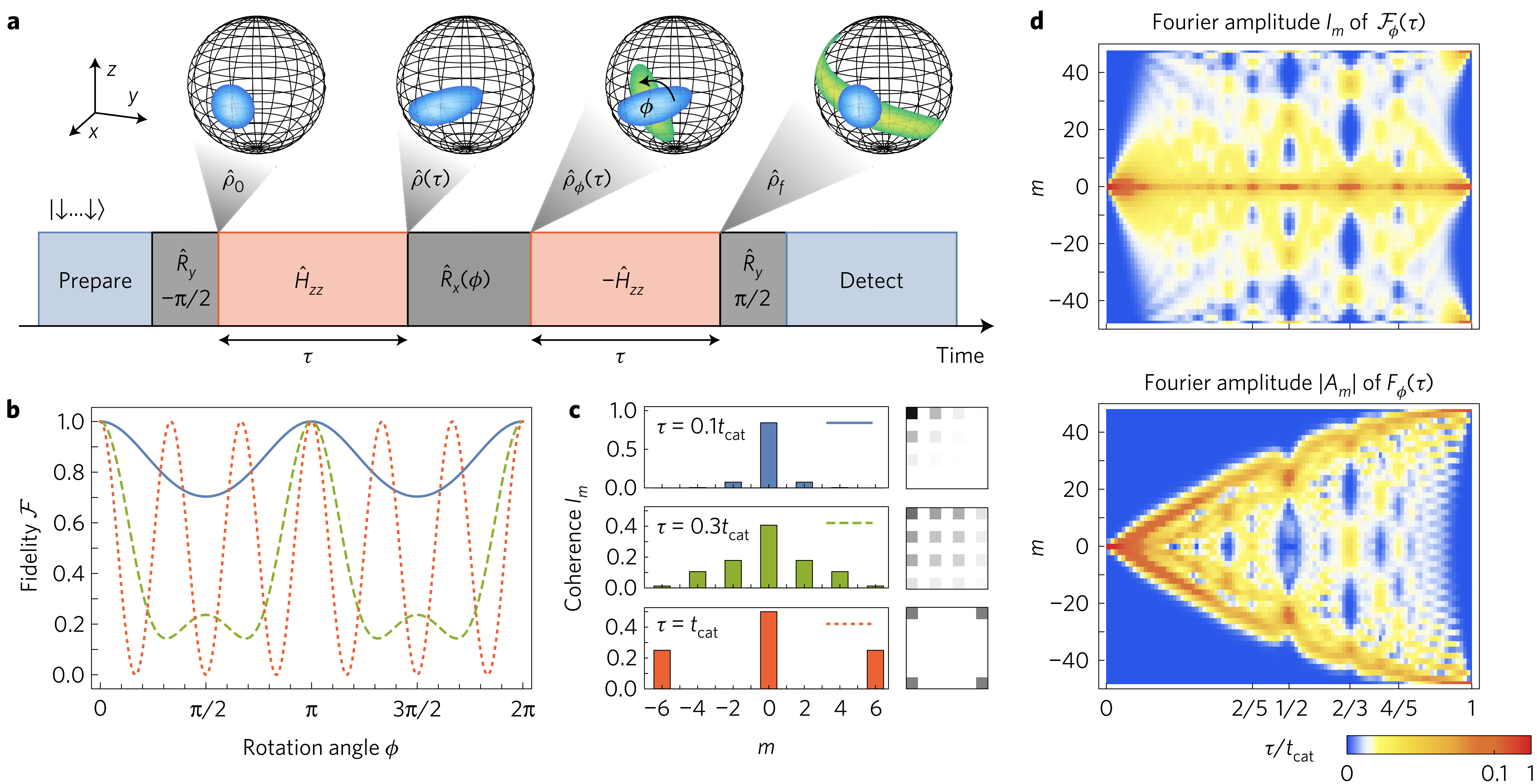
Measuring out-of-time-order correlations and multiple quantum spectra in a trapped-ion quantum magnet

Martin Gärttner[1†], Justin G. Bohnet[2†], Arghavan Safavi-Naini[1], Michael L. Wall[1], John J. Bollinger[2] and Ana Maria Rey[1*]



Selected for a Viewpoint in *Physics*
PHYSICAL REVIEW X **7,** 031011 (2017)

Measuring Out-of-Time-Order Correlators on a Nuclear Magnetic Resonance Quantum Simulator

Jun Li,[1] Ruihua Fan,[2,3] Hengyan Wang,[3] Bingtian Ye,[3] Bei Zeng,[4,5,2,*] Hui Zhai,[2,6,†] Xinhua Peng,[7,8,9,‡] and Jiangfeng Du[7,8]
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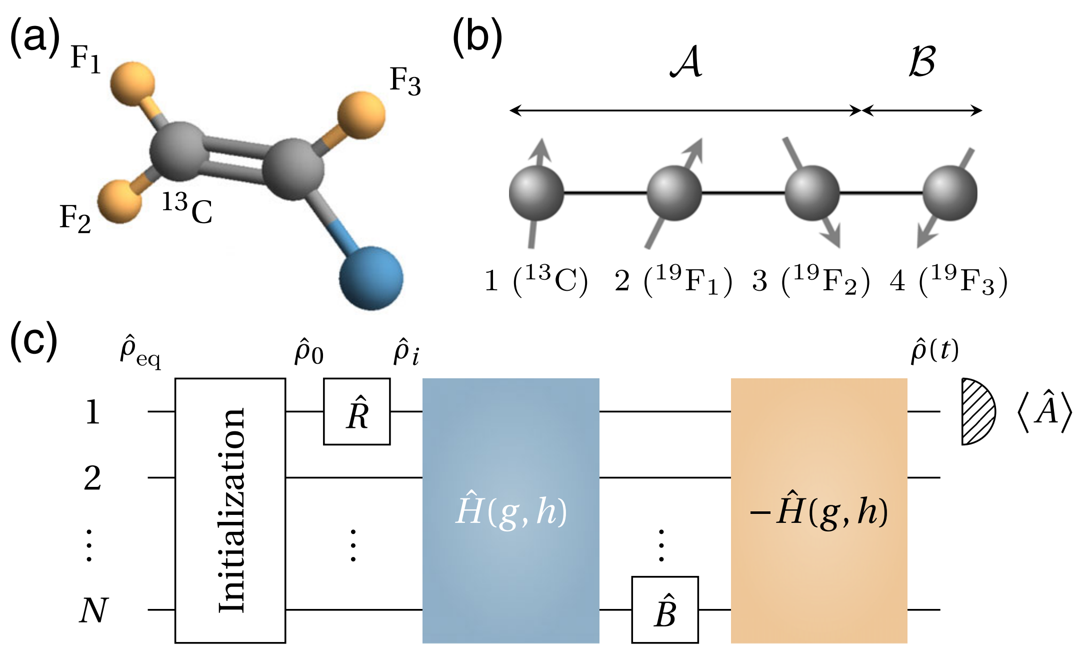
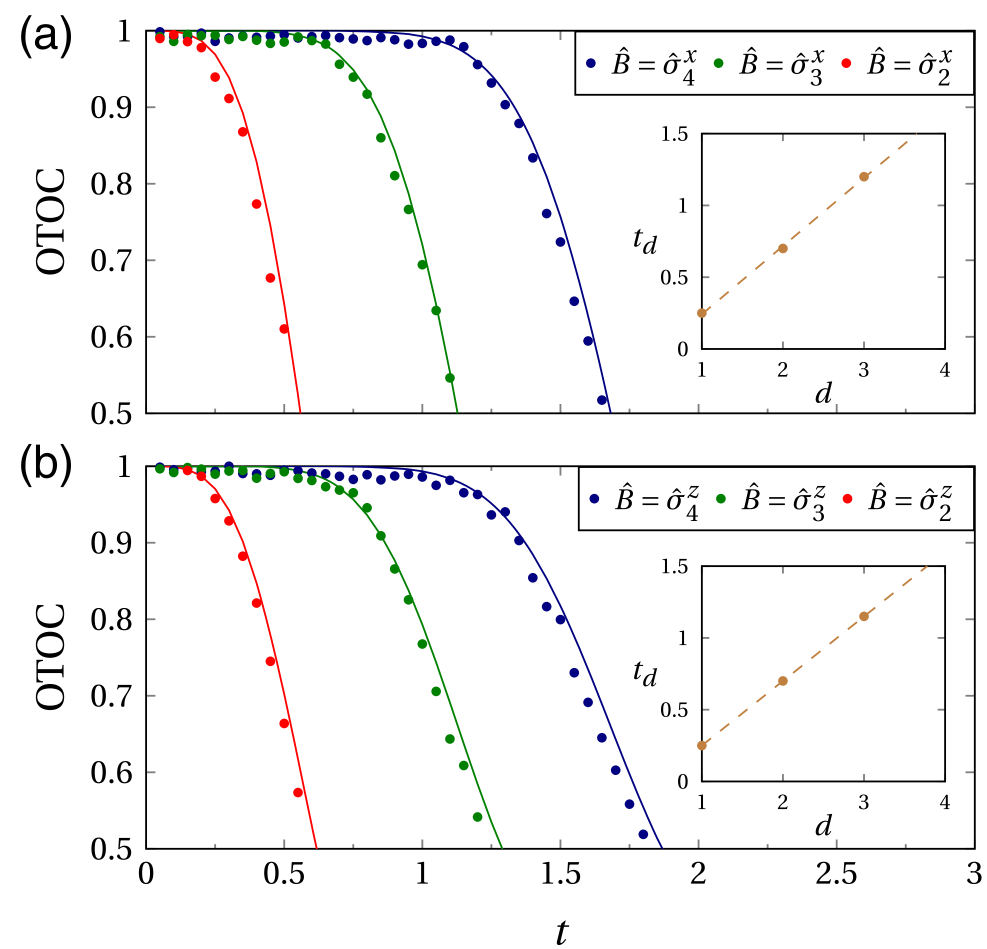


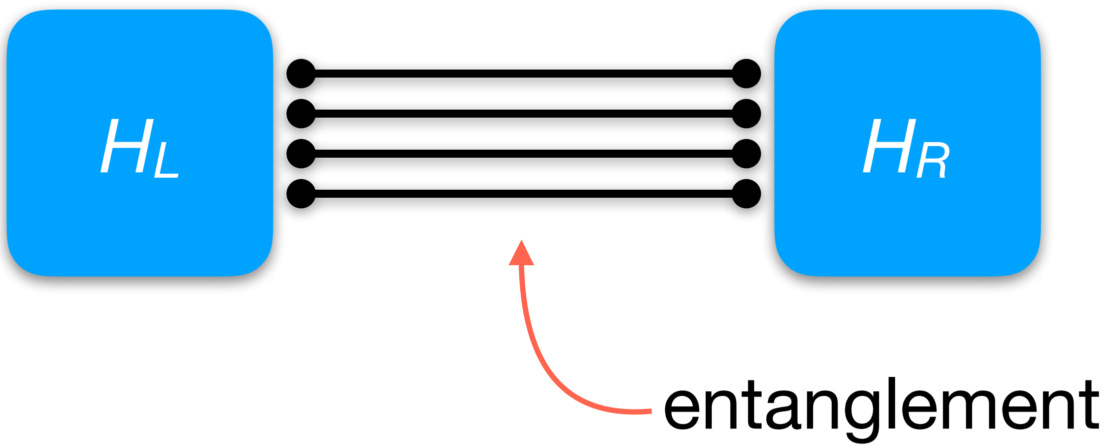


Conventional quantum chaos diagnosis requires backward time evolution and is therefore *hard*.

Is there an alternative that could be used in complex many-body systems?

Quantum chaos diagnosis using entangled states

Consider two identical copies of a quantum system



Specifically, we want the "thermofield double state" defined as

$$|\mathrm{TFD}_\beta\rangle = \frac{1}{\sqrt{Z_\beta}} \sum_n e^{-\beta E_n/2} |\bar{n}\rangle_L \otimes |n\rangle_R$$

where $|n\rangle_{L/R}$ is an eigenstate with energy E_n of $H_{L/R}$ and $|\bar{n}\rangle = \Theta|n\rangle$.

Properties of TFD state

aka "traversable wormhole"

$$|\mathrm{TFD}_\beta\rangle = \frac{1}{\sqrt{Z_\beta}} \sum_n e^{-\beta E_n/2} |\bar{n}\rangle_L \otimes |n\rangle_R$$

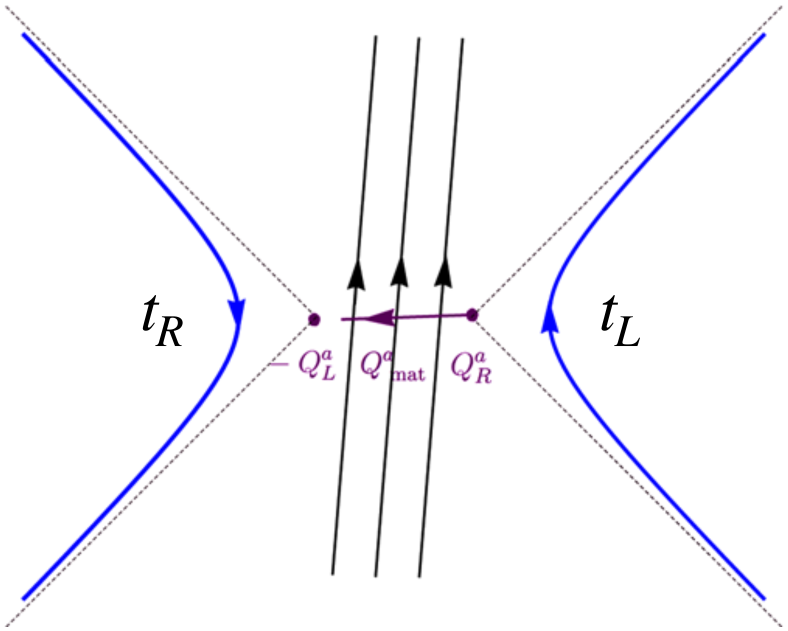
1. Expectation value of any one-sided operator is given by a thermal average:

$$\langle \mathcal{O}_L \rangle_{\mathrm{TFD}} = Z_\beta^{-1} \sum_n e^{-\beta E_n} \, {}_L\langle n | \mathcal{O}_L | n \rangle_L \, .$$

2. TFD is not an eigenstate of the total Hamiltonian $H = H_L + H_R$, but it is an eigenstate with eigenvalue zero of $H\_ = H_L - H_R$.

Item 2 above has important consequence for the time-translation invariance in the TFD state:

Time effectively flows in the **opposite direction** in two subsystems forming the TFD pair!



$$\mathcal{F}(t_1, t_2) = \langle \mathcal{O}_L(t_1) \mathcal{O}_R(t_2) \rangle_{\mathrm{TFD}} = \langle \mathcal{O}_L(t_1 + t) \mathcal{O}_R(t_2 - t) \rangle_{\mathrm{TFD}} = \mathcal{F}(t_1 + t_2)$$

Probing OTOCs by means of conventional measurement in the TFD state

Consider the following 4-point NTOC correlator evaluated in the TFD state:

$$\tilde{F}(t,t') = \langle \mathcal{T}[V_L(t)W_R(t)V_R(t')W_L(t')]\rangle_{\mathrm{TFD}}$$

By using the definition of $|\mathrm{TFD}_\beta\rangle$ it is straightforward to show (about one page of calculation, details in arXiv:1907.01628) that

$$\tilde{F}(t,-t) = \mathrm{tr}[W(2t)V(0)y^2W(2t)V(0)y^2]$$

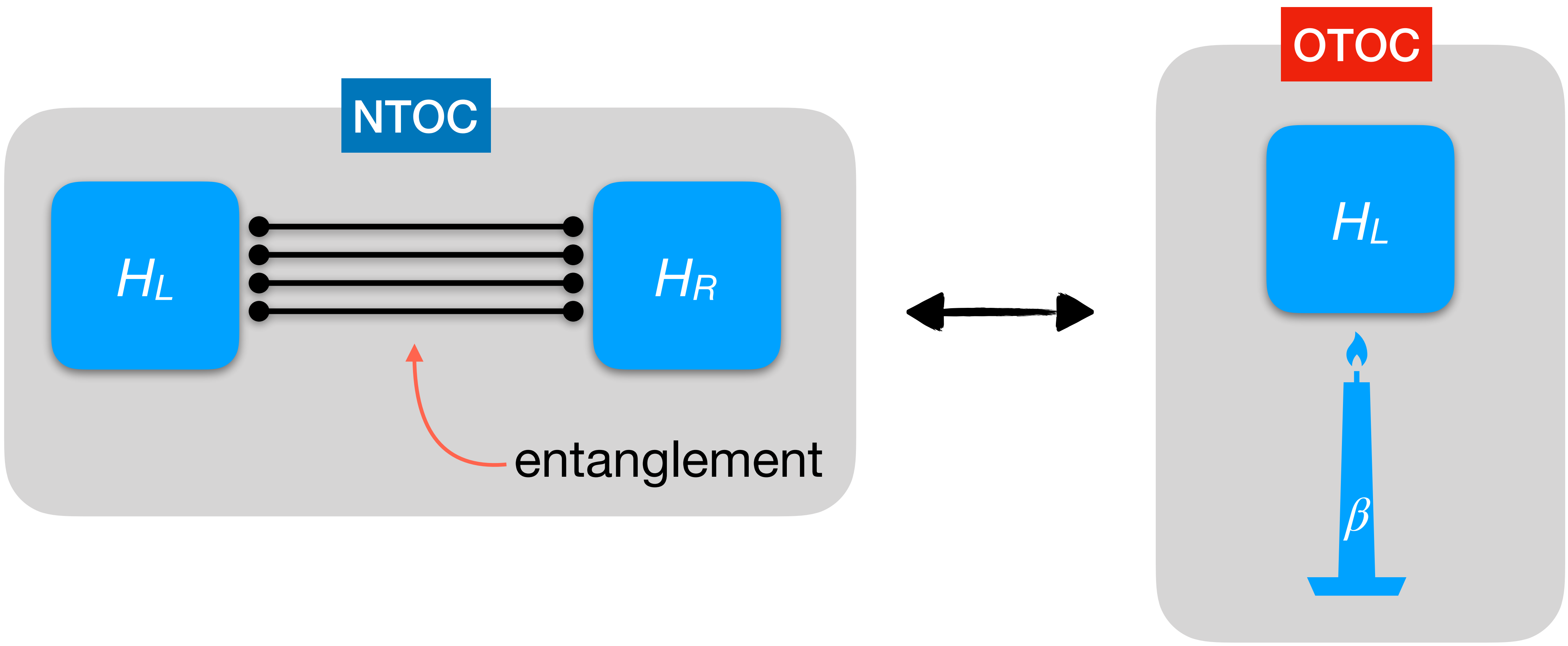
Here $y^4 = e^{-\beta H}/Z_\beta$ and the trace is with respect to the eigenstates $|n\rangle_L$ of single subsystem.

The expression for $\tilde{F}(t,-t)$ above with density matrix insertions y^2 is called "thermally regularized OTOC" and has been argued in the literature to most directly diagnose quantum chaos.

Summary of the main result:

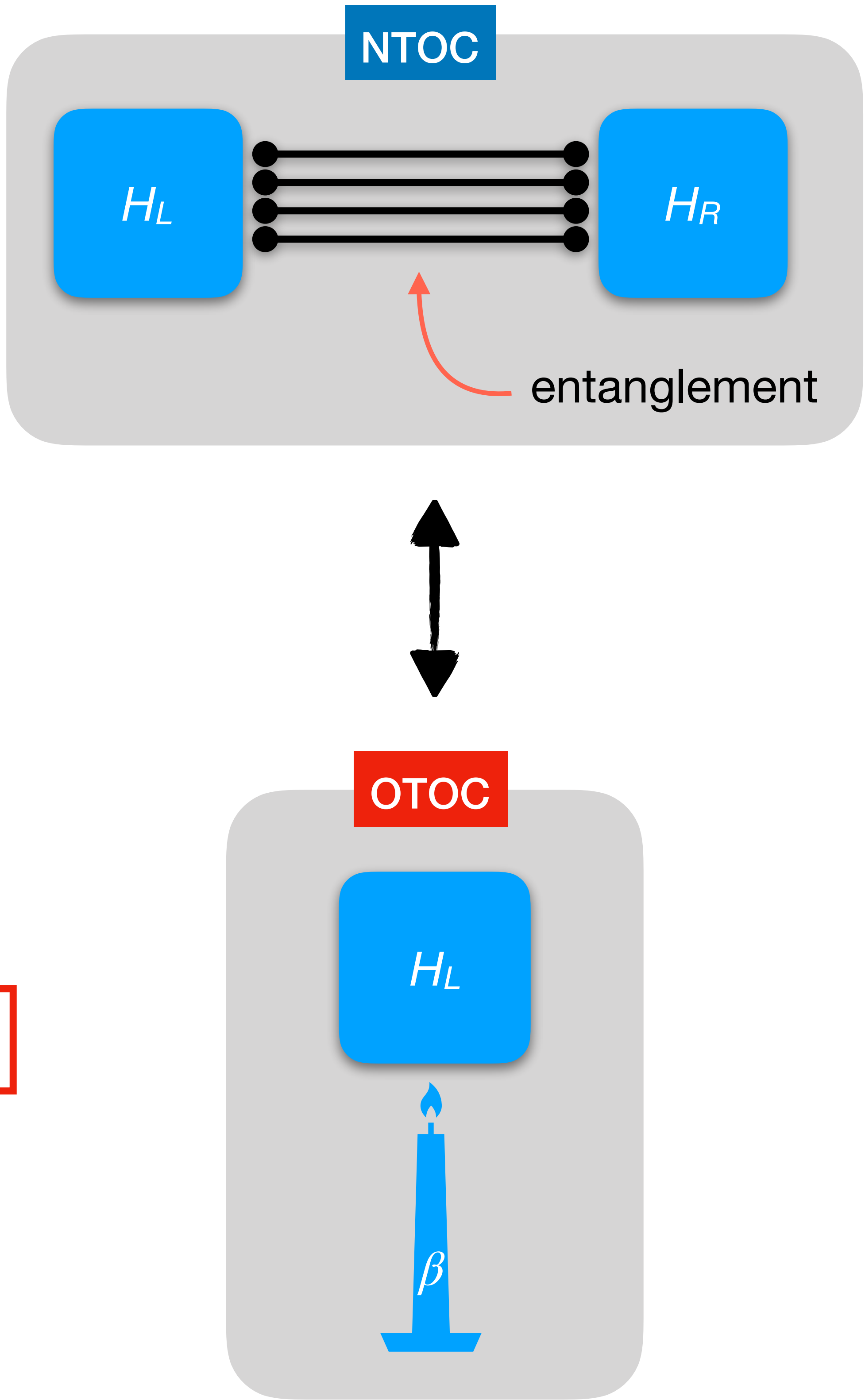
Certain naturally ordered 4-point correlators evaluated in the TFD entangled state map onto **regularized OTOCs**.

$$\langle \mathcal{T}[V_L(t)W_R(t)V_R(-t)W_L(-t)]\rangle_{\mathrm{TFD}} = \mathrm{tr}[W(2t)V(0)y^2W(2t)V(0)y^2]$$



New protocol for probing OTOCs

1. Construct a pair of identical systems
2. Prepare them in the TFD state
3. Perform an ordinary measurement



The challenge of backward time evolution has been replaced by the challenge to prepare a TFD entangled state.

Is it possible to efficiently prepare the TFD state?

TFD state preparation

Recent theoretical work showed how to construct a Hamiltonan H_S which admits $|\mathrm{TFD}_\beta\rangle$ as its ground state.

[W. Cottrell, B. Freivogel, D.M. Hofman, and S.F. Lokhande, J. High Energy Phys. 2019, 58 (2019)]

A simple Hamiltonian

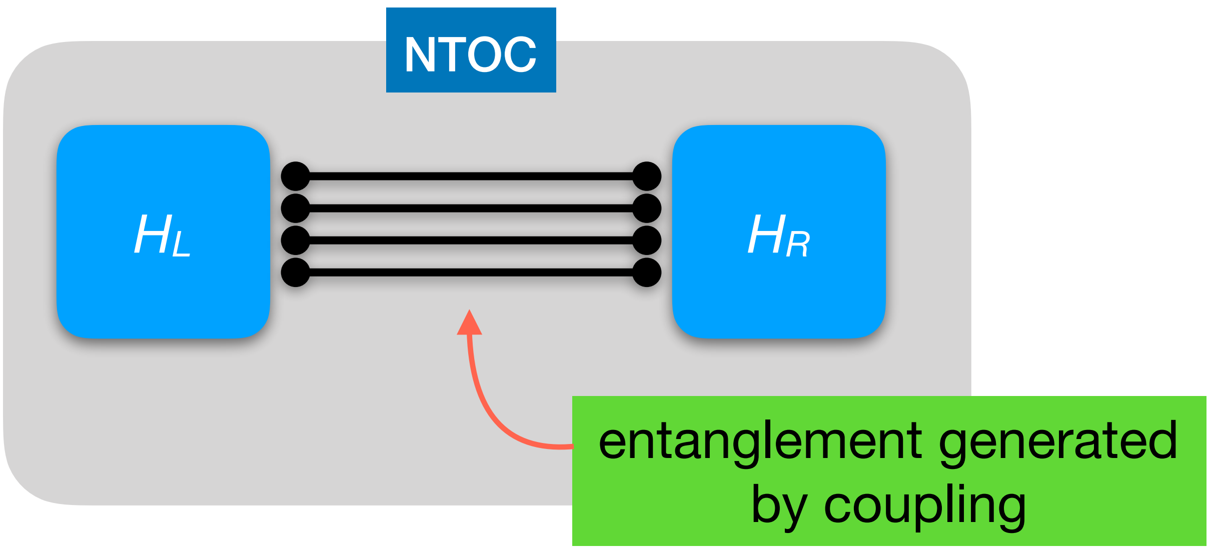
[J. Maldacena and X.-L. Qi, arXiv:1804.00491]

$$H_S = H_L + H_R + H_I \quad \text{with} \quad H_I = i\mu \sum_j \mathcal{O}^j_L \mathcal{O}^j_R$$

has a ground state that is **very close** to $|\mathrm{TFD}_\beta\rangle$, that is $\langle\Psi_0|\mathrm{TFD}_\beta\rangle \simeq 1$.

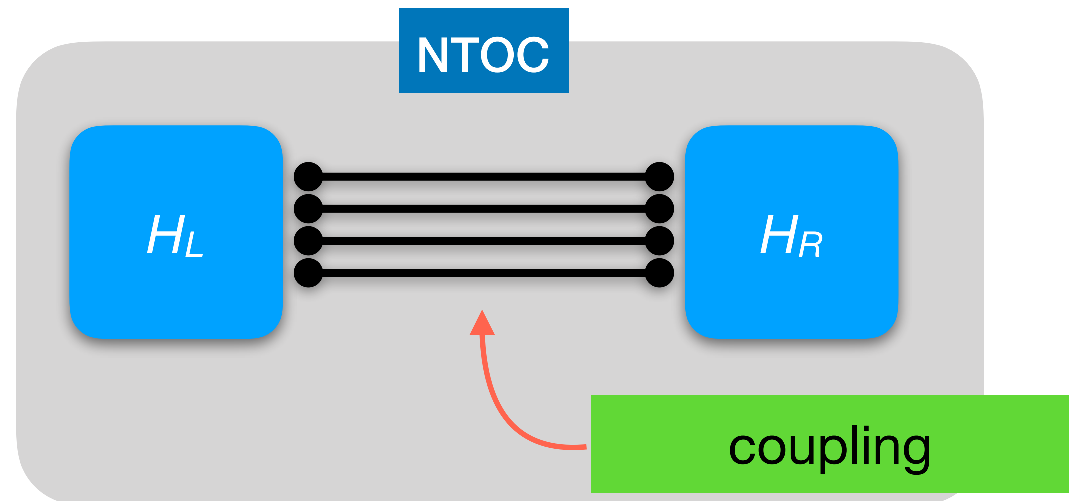
Strategy to prepare TFD state:

1. Engineer a system with Hamiltonian H_S
2. Cool the system to its ground state $|\Psi_0\rangle \simeq |\mathrm{TFD}_\beta\rangle$



Measurement strategy

1. Construct a pair of identical systems
2. Prepare them in the TFD state
3. Perform an ordinary measurement



It turns out to be sufficient to measure an ordinary two-sided correlator $iG^{\text{ret}}_{LR}(t,t') = \theta(t-t')\langle\{V_L(t), V_R(t')\}\rangle_0$ in the TFD ground state.

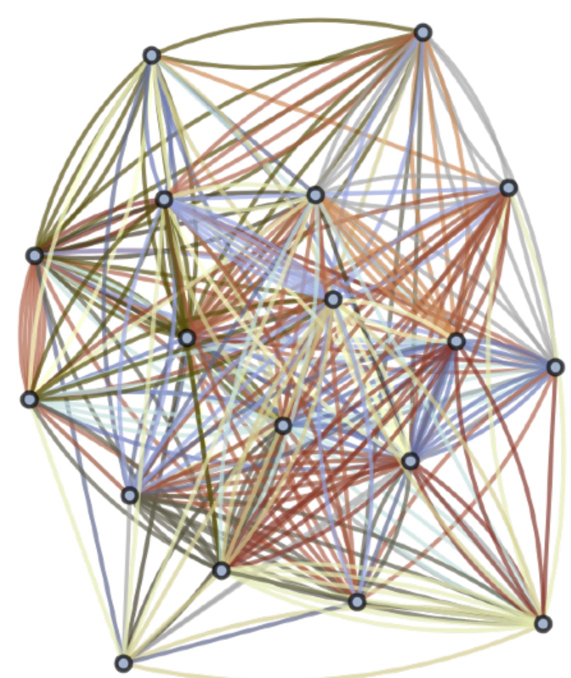
One can show that, at intermediate times $t \ll \mu^{-1}$ it holds

OTOC

$$iG^{\text{ret}}_{LR}(t,-t) \simeq 2\mu \sum_j \int_0^t ds \,\text{Re}\, \text{tr}\left[\mathcal{O}^j(t+s)V(0)y^2\mathcal{O}^j(t-s)V(0)y^2\right] + \text{NTOC}\,.$$

We thus expect $iG^{\text{ret}}_{LR}(t,-t) \simeq A + Be^{2\lambda_L t}$

Example: Black holes, wormholes and the Sachdev-Ye-Kitaev model



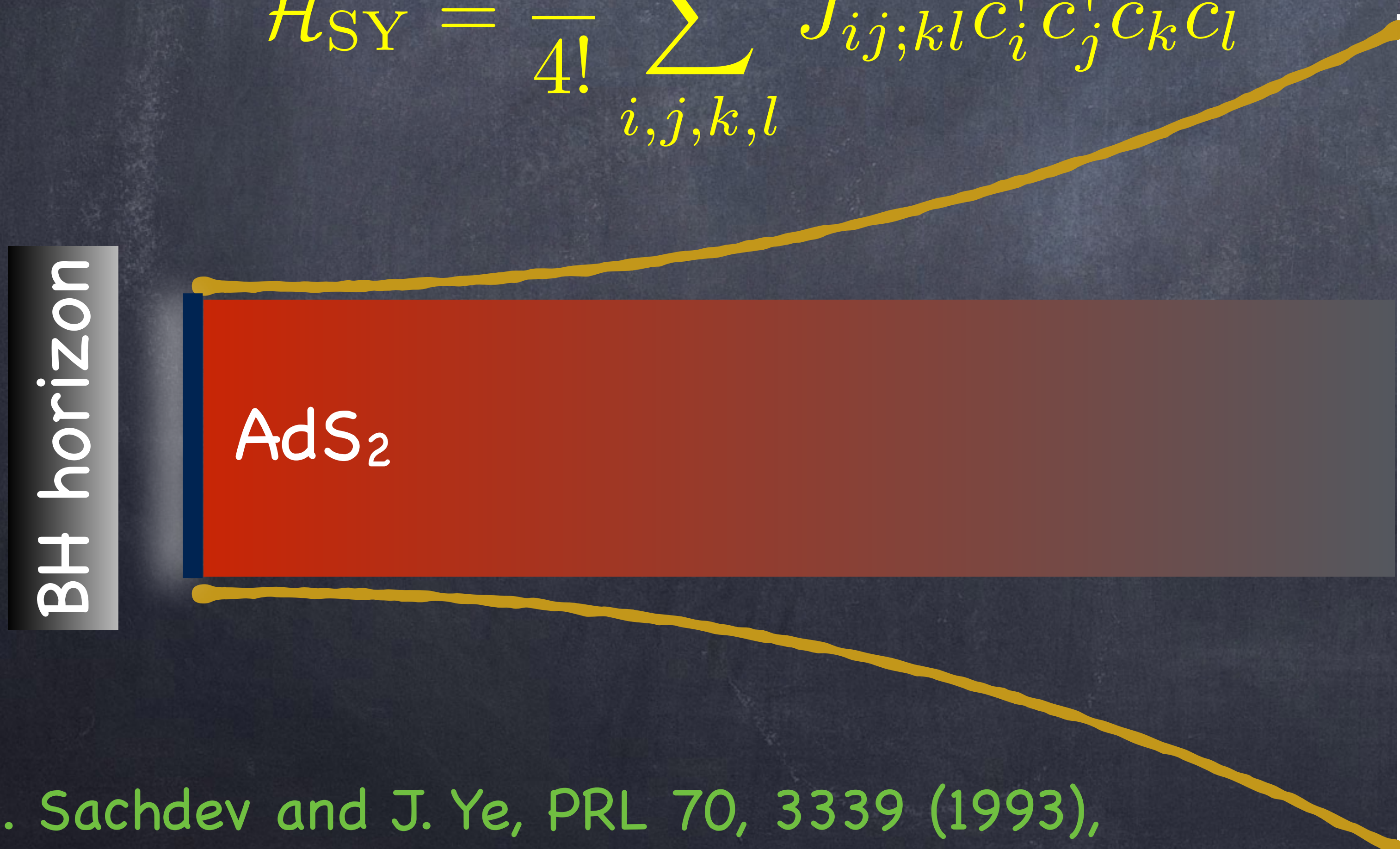
Sachdev-Ye-Kitaev (SYK) Model review:

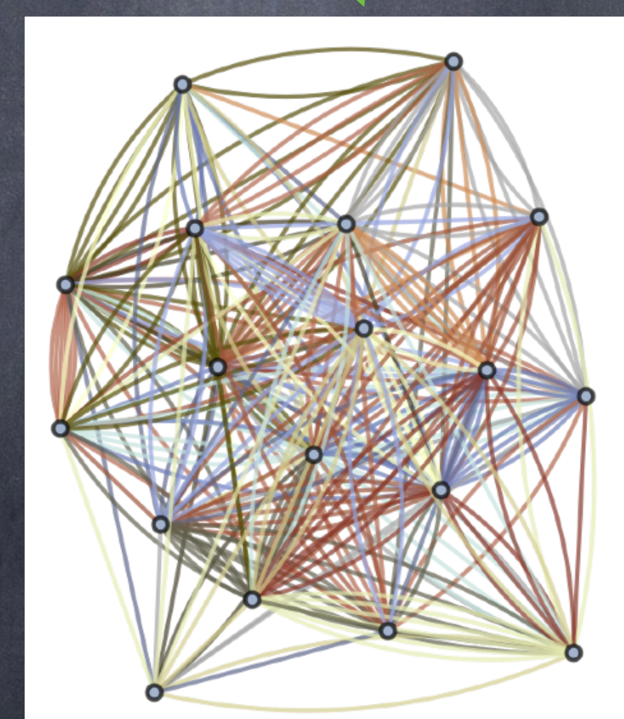
A toy model that is both a black hole and a "strange metal."

$$\mathcal{H}_{\mathrm{SYK}} = \frac{1}{4!}\sum_{i,j,k,l} J_{ijkl}\chi_i\chi_j\chi_k\chi_l$$

$$\mathcal{H}_{\mathrm{SY}} = \frac{1}{4!}\sum_{i,j,k,l} J_{ij;kl}c_i^\dagger c_j^\dagger c_k c_l$$

A system of N (Majorana) fermions with random all-to-all interactions



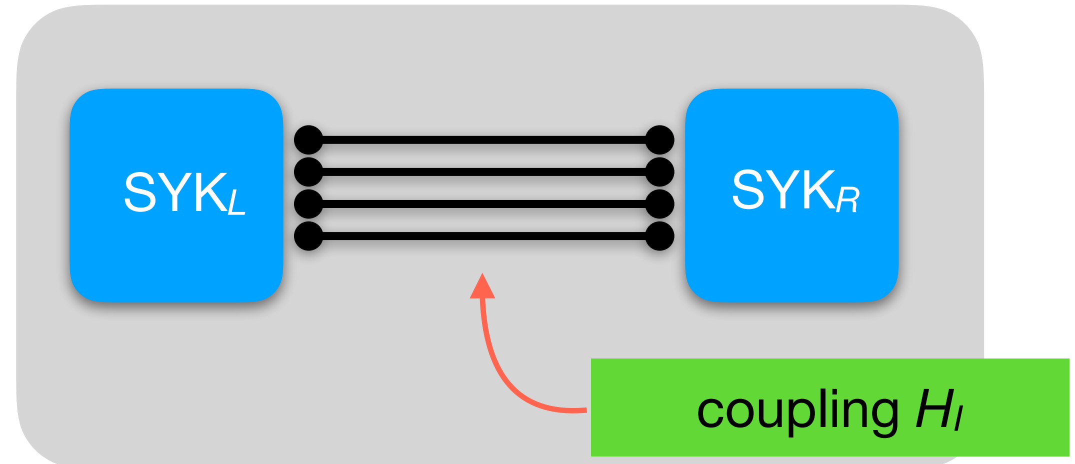


S. Sachdev and J. Ye, PRL 70, 3339 (1993),
O. Parcollet and A. Georges, PRB 59, 5341 (1999), A. Kitaev (unpublished, 2015).

Maldacena-Qi model and the TFD

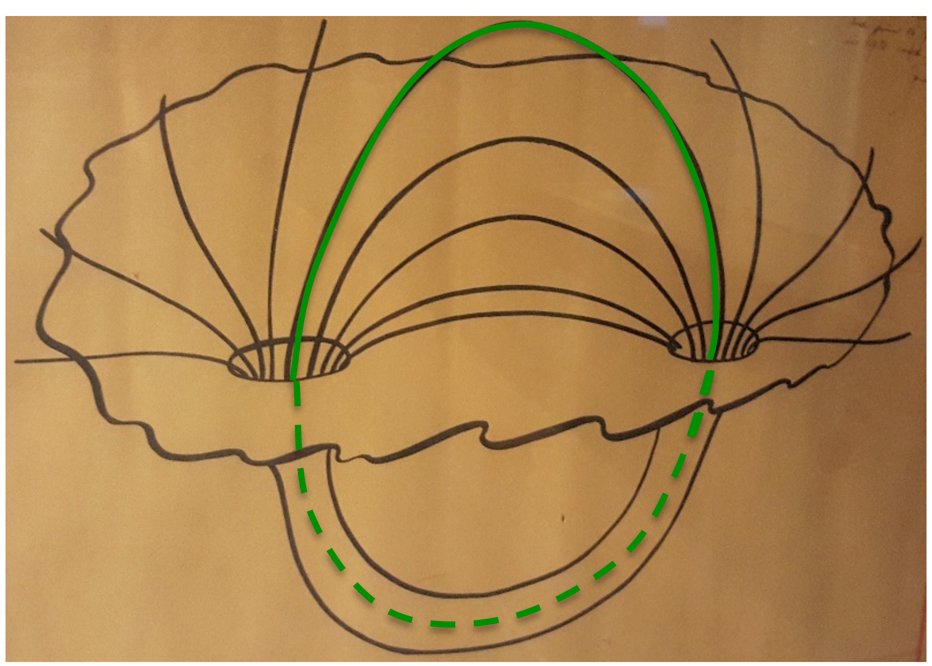
$$H = H_L^{\mathrm{SYK}} + H_R^{\mathrm{SYK}} + i\mu \sum_j \chi_L^j \chi_R^j$$

$$H_\alpha^{\mathrm{SYK}} = \sum_{i<j<k<l} J_{ijkl}\chi_\alpha^i \chi_\alpha^j \chi_\alpha^k \chi_\alpha^l$$

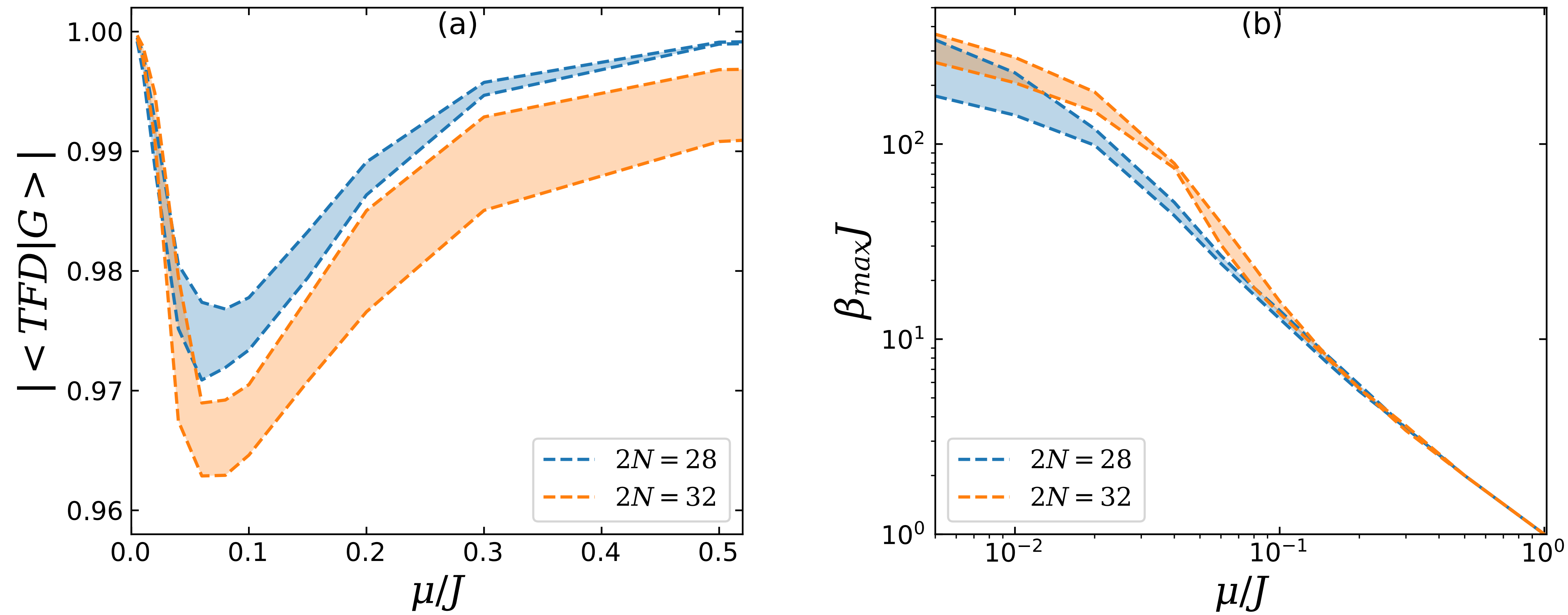


[J. Maldacena and X.-L. Qi, arXiv:1804.00491]

Two identical SYK models coupled via simple bilinear term realize holographically an “eternal traversable wormhole” and are therefore of great current interest in the quantum gravity community.



For our purposes it is important that the ground state of the Maldacena-Qi model is very close to the TFD:



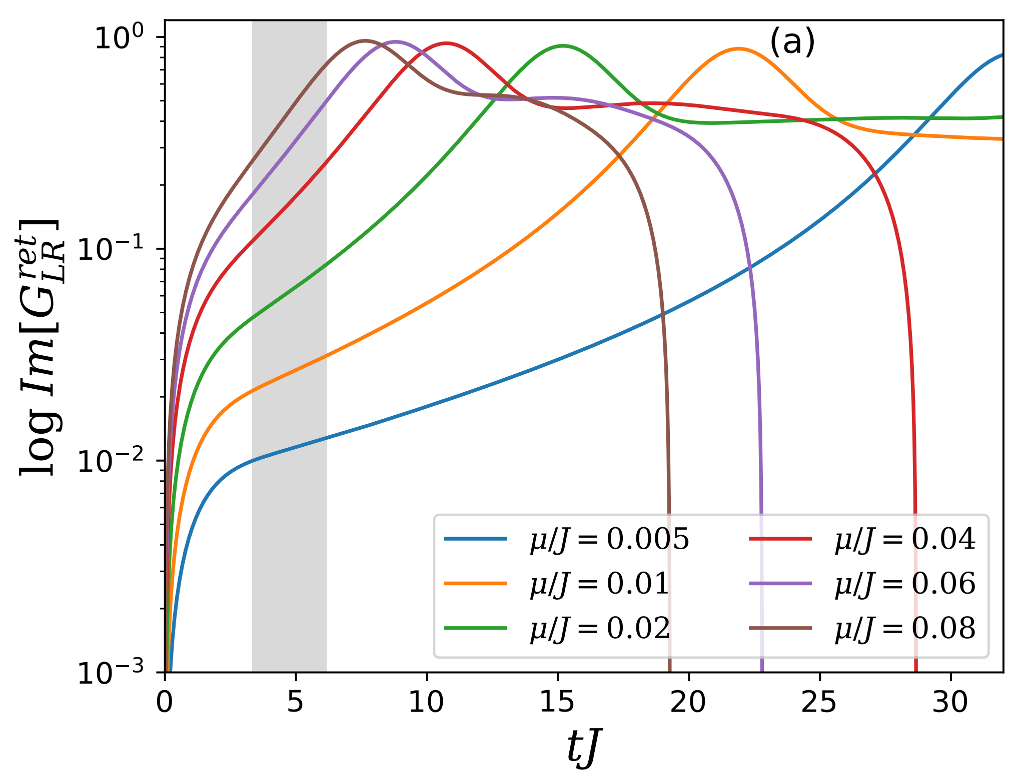
Results of numerical exact diagonalization arXiv:1907.01628

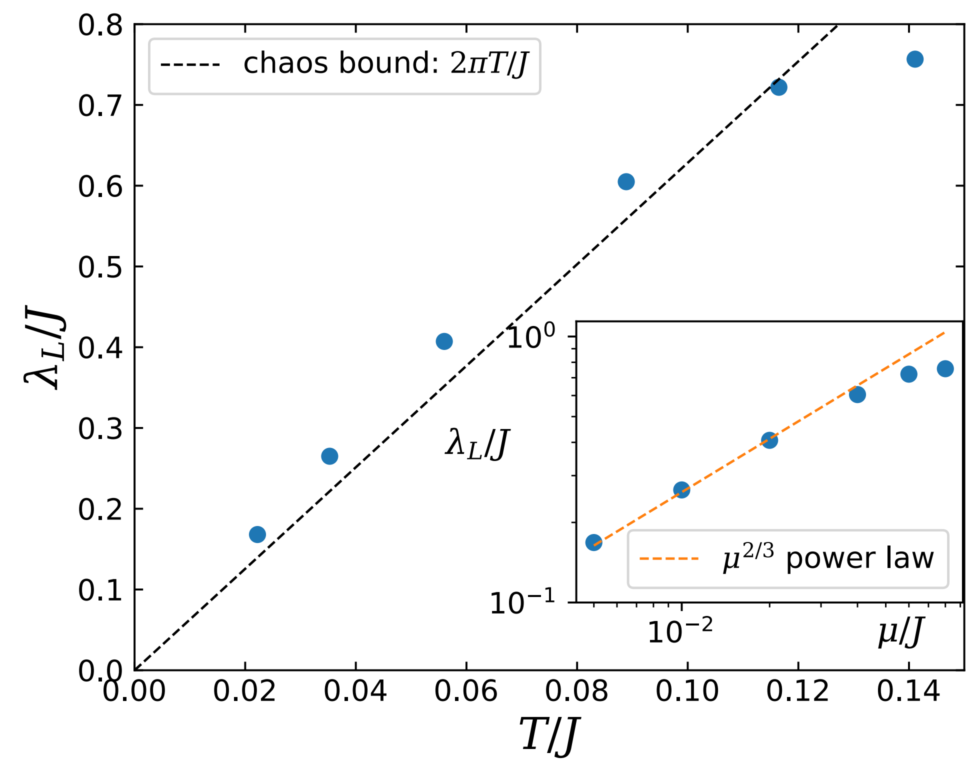
$$H = H_L^{\mathrm{SYK}} + H_R^{\mathrm{SYK}} + i\mu \sum_j \chi_L^j \chi_R^j$$

OTOC and Lyapunov exponent

We argued previously that LR correlator in TFD ground state contains OTOC contribution, i.e. $iG^{\text{ret}}_{LR}(t,-t) \simeq A + Be^{2\lambda_L t}$. For Maldacena-Qi we find:

$$iG^{\text{ret}}_{LR}(t,-t) = \frac{\theta(t)}{N}\sum_j \langle\{\chi^j_L(t), \chi^j_R(-t)\}\rangle$$
$$\simeq \frac{4\mu}{N}\sum_{j,k}\int_0^t ds \ \mathrm{Re}\ \mathrm{tr}[\chi^j(t+s)\chi^k(0)y^2\chi^j(t-s)\chi^k(0)y^2] \ + \ \text{NTOC}$$





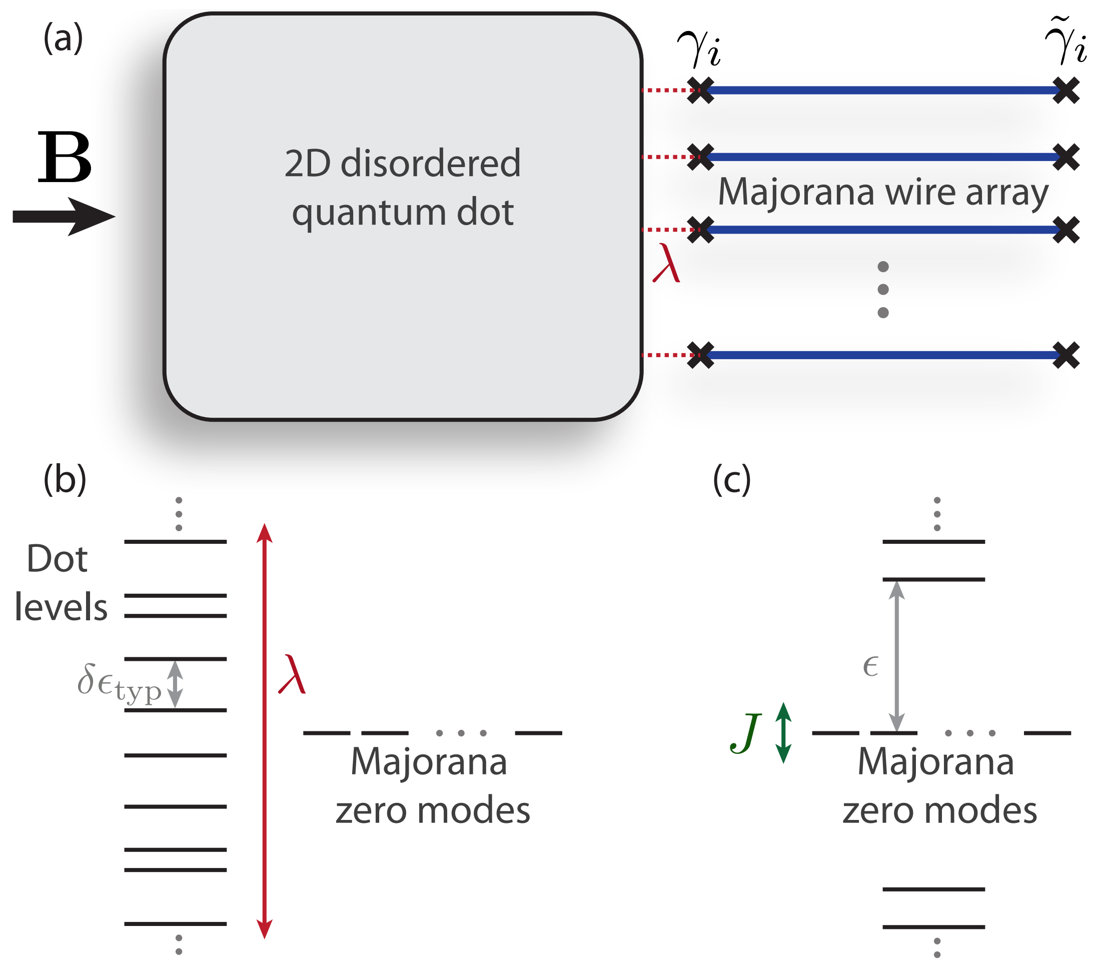
Results of large-N calculation arXiv:1907.01628

Consistent with maximal chaos $\lambda_L = 2\pi T$

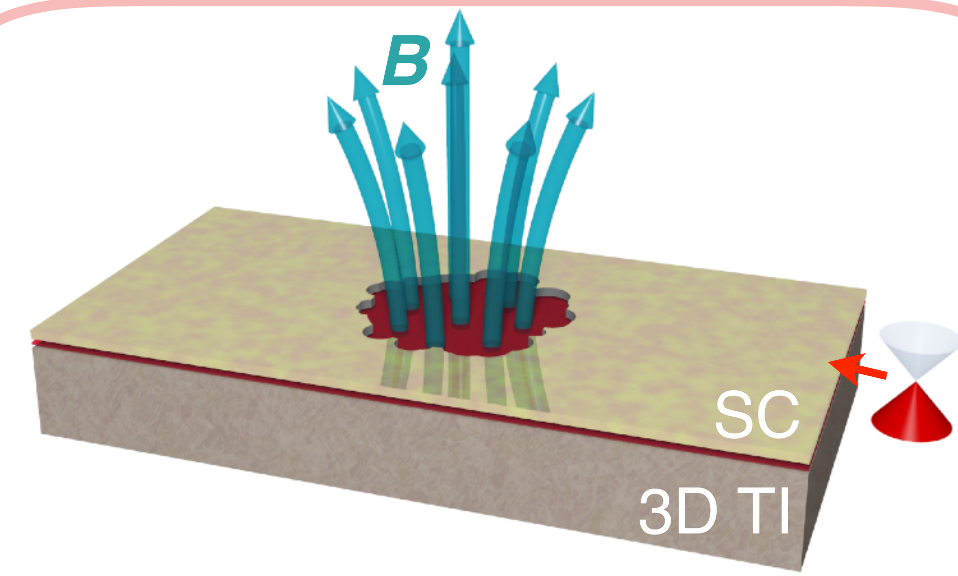
Possible experimental realizations

We want to realize the Maldacena-Qi model $H = H_L^{\mathrm{SYK}} + H_R^{\mathrm{SYK}} + i\mu \sum_j \chi_L^j \chi_R^j$

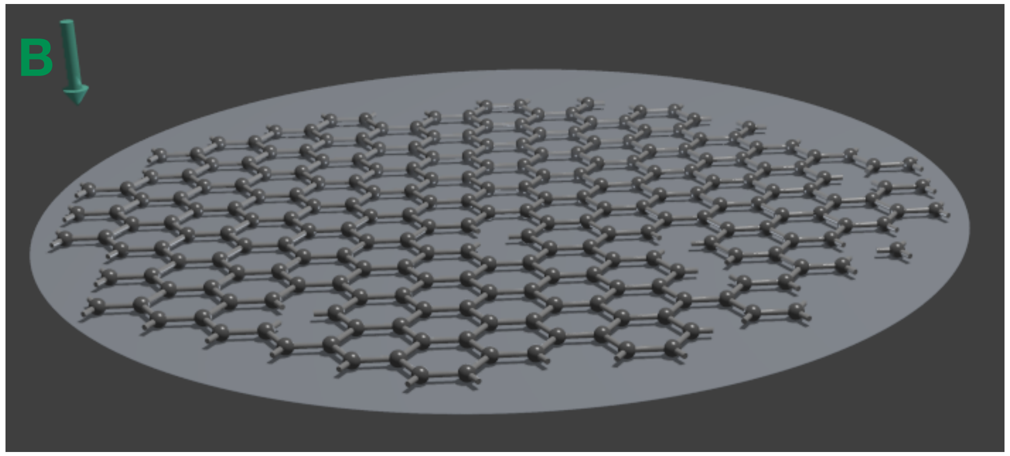
SYK model proposed realizations:



[Chew, Essin and Alicea, PRB 2017]



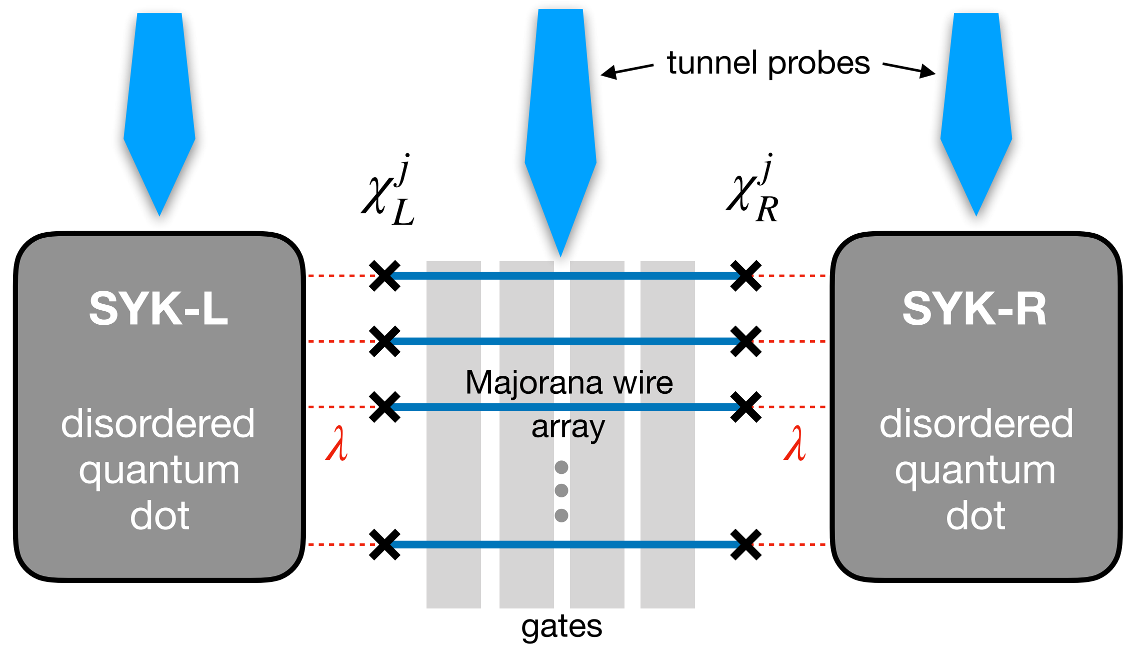
[Pikulin and Franz, PRX 2017]



[Chen, Ilan, de Juan, Pikulin and Franz, PRL 2018]

Two identical quantum dots bridged by Majorana wires could approximate the Maldacena-Qi model

$$H = H_L^{\mathrm{SYK}} + H_R^{\mathrm{SYK}} + i\mu \sum_j \chi_L^j \chi_R^j$$



- Tunnel probes can be used to probe electron spectral function $\rho_x(\omega)$ in each wire which is related to the LR Majorana correlator. We find:

$$iG_{LR}^{\mathrm{ret}}(t) \simeq K_x \theta(t) \int_{-\infty}^{\infty} d\omega \; \rho_x(\omega) \sin \omega t \, .$$

Tunneling conductance experiment in this setup therefore gives access to the Lyapunov chaos exponent through $iG_{LR}^{\mathrm{ret}}(t) \simeq A + Be^{\lambda_L t}$

Conclusions

- Diagnosing quantum chaos traditionally requires backward time evolution which is hard or impossible in complex many-body systems
- We proposed a new protocol for chaos detection which replaces a complicated measurement scheme by a simple measurement on a specific entangled state
- The challenge now is to fabricate two identical copies of an interesting system that are weakly coupled to one another
- A simple spectroscopic measurement then yields the chaos exponent λ_L

arXiv:1907.01628