Theory of Magnetoresistance in Three-Dimensional Dirac Materials

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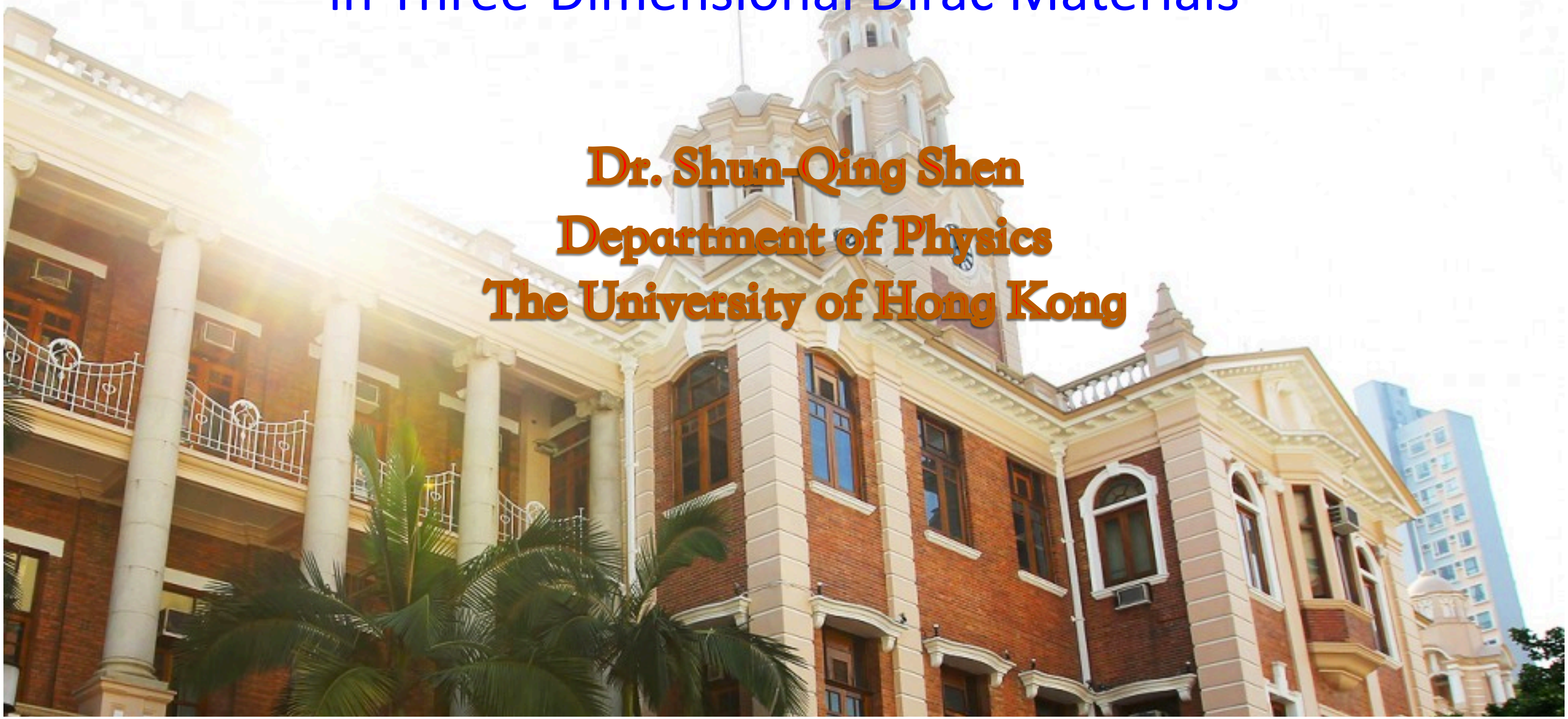


Magnetoresistance

Magnetoresistance is the tendency of a material to change the value of its electrical resistance in an externally-applied magnetic field.

-- Wikipedia

- Geometrical magnetoresistance
- Shubnikov de Haas oscillations
- Anisotropic magnetoresistance (AMR)
- Giant magnetoresistance (GMR)
- Tunnel magnetoresistance (TMR)
- Colossal magnetoresistance (CMR) in manganites
- Negative longitudinal magnetoresistance in Weyl/Dirac semimetals
- ………………

$$\delta\rho = \frac{\rho(B) - \rho(B=0)}{\rho(B=0)}$$

Lorentz Force Induced MC

$$m\frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Relaxation time approximation

$$\frac{d\mathbf{v}}{dt} \to \frac{\mathbf{v}}{\tau}$$

$$en_e v \to j$$

$$\mathbf{j} = \sigma\mathbf{E} + \mu\mathbf{j} \times \mathbf{B}$$

$$\boxed{\mathbf{E} = \frac{1}{\sigma}\mathbf{j} - \frac{\mu}{\sigma}\mathbf{j} \times \mathbf{B}}$$

$$\rho = \frac{1}{\sigma}$$

μ is the electric mobility.

No magnetoresistance

$$\boxed{\mathbf{j} = \frac{\sigma}{1+\mu^2 B^2}\mathbf{E} + \frac{\mu^2\sigma}{1+\mu^2 B^2}(\mathbf{E}\cdot\mathbf{B})\,\mathbf{B} + \frac{\mu\sigma}{1+\mu^2 B^2}\mathbf{E} \times \mathbf{B}}$$

E perpendicular to B:

$$\mathbf{j} = \frac{\sigma}{1+\mu^2 B^2}\mathbf{E}$$

E parallel with B:

$$\mathbf{j} = \sigma\mathbf{E}$$

Usually the transverse magnetoconductivity is negative.

Magnetoresistivity

Perpendicular magnetoconductivity tensor

$$\sigma(B) = \frac{\varrho e^2 \tau}{m^*} \frac{1}{1+(\omega_c\tau)^2} \begin{pmatrix} 1 & -\omega_c\tau \\ +\omega_c\tau & 1 \end{pmatrix}$$

Perpendicular magnetoresistivity tensor

$$\rho(B) = \frac{m^*}{\varrho e^2 \tau} \begin{pmatrix} 1 & \omega_c\tau \\ -\omega_c\tau & 1 \end{pmatrix}$$

In the spherical one-band model with a single relaxation time, the diagonal magnetoresistivity turns out to be independent from the magnetic field
The Hall coefficient is independent both from the effective mass and the relaxation time.

Several mechanisms to produce MR:

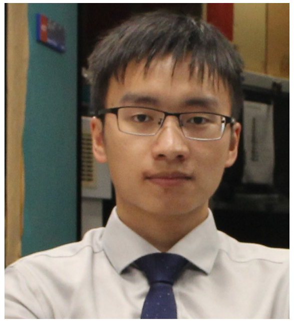
- Energy dependence of the relaxation time
- Anisotropy of the band structure
- Multiple bands
- ....

Two-band model: one charge carrier is of electron while the other is of hole

In this talk

1. Intrinsic Magnetoresistance in 3D topological materials
2. Theory of weak localization and anti-localization in 3D topological materials
3. Anomaly-induced magnetoresistivity in massive Dirac materials

Dr. Bo Fu
Postdoc., HKU



Mr. Huan-Wen Wang
Ph.D candidate., HKU

B. Fu, H. W. Wang & S. Q. Shen, arXiv: 1909.09297
B. Fu, H. W. Wang & S. Q. Shen, PRL 122, 246601 (2019)
H. W. Wang, B. Fu, and S. Q. Shen, PRB 98, 081202(R) (2018)

From Dirac to Weyl

$$i\hbar\frac{\partial}{\partial t}\psi = \left(c\alpha\cdot\mathbf{p}+\beta mc^2\right)\psi$$

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$$

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Electron spin
- Anti-particle
- Dirac sea

When the mass m=0, the Dirac equation is reduced into

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = c\begin{pmatrix} 0 & \mathbf{p}\cdot\sigma \\ \mathbf{p}\cdot\sigma & 0\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$$

$$i\hbar\frac{\partial}{\partial t}\psi_1 = c\mathbf{p}\cdot\sigma\psi_2$$

$$i\hbar\frac{\partial}{\partial t}\psi_2 = c\mathbf{p}\cdot\sigma\psi_1$$

$$i\hbar\frac{\partial}{\partial t}(\psi_1\pm\psi_2) = \pm c\mathbf{p}\cdot\sigma(\psi_1\pm\psi_2)$$

$$i\hbar\frac{\partial}{\partial t}\chi_\pm = \pm c\mathbf{p}\cdot\sigma\chi_\pm$$

Dirac Equation and Topological Materials

$$H = vp \cdot \alpha + (mv^2 - Bp^2)\beta$$

1D Dimerized Polymer

2D Quantum Spin Hall Effect/Quantum Anomalous Hall Effect

3D Topological Insulator

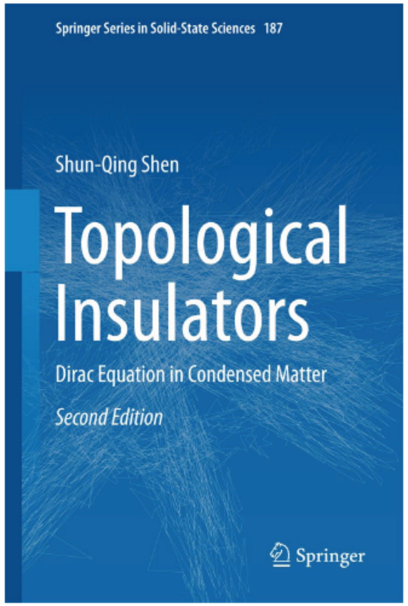
P-wave Superconductor

He_3 Superfluidity

Topological Superconductors

Topological Weyl Semimetals

1^{st} Ed., 2012; 2^{nd} Ed., 2017



Part I: Intrinsic Magnetoresistance

Intrinsic Magnetoresistivity
Quantum Oscillation and the Phase Shift
Magnetoresistivity in Quantum Limit

- H. W. Wang, B. Fu & S. Q. Shen, *Intrinsic magnetoresistance in three-dimensional Dirac materials*, Phys. Rev. B 98, 081202(R) (2018)

Exact Solutions in a Magnetic Field

$$\mathcal{H}_D = v\hbar\mathbf{k}\cdot\alpha + \Delta\beta$$

With magnetic field $\mathbf{A} = (-By, 0, 0)$ $\mathbf{k} \to (k_x - y/l_B^2, -i\partial_y, k_z)$ $\quad l_B \equiv \sqrt{\hbar/eB}$

$$\mathcal{H}_m = \begin{bmatrix} \Delta & 0 & v\hbar k_z & \frac{\sqrt{2}\hbar v}{l_B}a \\ 0 & \Delta & \frac{\sqrt{2}\hbar v}{l_B}a^\dagger & -v\hbar k_z \\ v\hbar k_z & \frac{\sqrt{2}\hbar v}{l_B}a & -\Delta & 0 \\ \frac{\sqrt{2}\hbar v}{l_B}a^\dagger & -v\hbar k_z & 0 & -\Delta \end{bmatrix}$$

Two variables: k_z and the Landau index n

$$|\psi_{ns}^\zeta\rangle = \begin{bmatrix} \cos\frac{\phi_{n\zeta}}{2}\cos\frac{\theta_{ns}}{2}|n-1\rangle \\ s\cos\frac{\phi_{n\zeta}}{2}\sin\frac{\theta_{ns}}{2}|n\rangle \\ s\zeta\sin\frac{\phi_{n\zeta}}{2}\cos\frac{\theta_{ns}}{2}|n-1\rangle \\ \zeta\sin\frac{\phi_{n\zeta}}{2}\sin\frac{\theta_{ns}}{2}|n\rangle \end{bmatrix}$$

$$\varepsilon_{ns}^\zeta = \zeta\sqrt{v^2\hbar^2k_z^2 + 2n\left(\hbar v/l_B\right)^2 + \Delta^2},$$

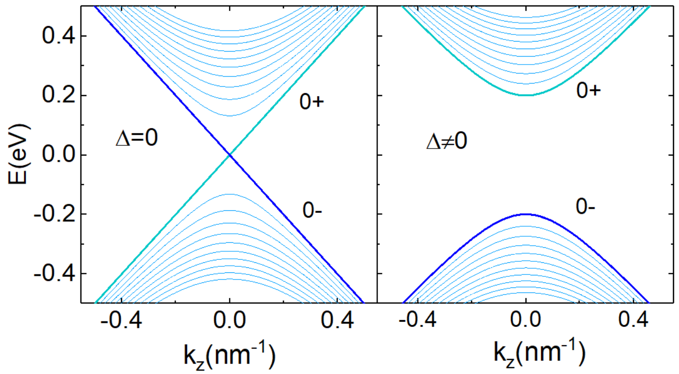
$$n \geq 1, \zeta = \pm, s = \pm$$

$$\varepsilon_0^\zeta = \zeta\sqrt{v^2\hbar^2k_z^2 + \Delta^2}, n = 0, \zeta = \pm$$

$$|\psi_0^\zeta\rangle = [0, \cos\frac{\phi_{0\zeta}}{2}|0\rangle, 0, -\zeta\mathrm{sign}(k_z)\sin\frac{\phi_{0\zeta}}{2}|0\rangle]^T$$

$$\cos\phi_{n\zeta} = \Delta/\varepsilon_n^\zeta \qquad \cos\phi_{0\zeta} = \Delta/\varepsilon_0^\zeta$$

$$\cos\theta_{ns} = sv\hbar k_z/\sqrt{(\varepsilon_n^\zeta)^2 - \Delta^2}$$



Linear Response Theory

dc conductivity

$$\mathrm{Re}\sigma_{\alpha\beta} = -\lim_{\omega\to 0}\frac{1}{\omega}\lim_{q\to 0}\mathrm{Im}[\Pi_{\alpha\beta}(\mathbf{q},\omega)], \alpha,\beta = x,y,z$$

Current-current correlation function

$$\Pi_{\alpha\beta}(i\omega) = \frac{e^2\hbar}{\beta V}\sum_l\sum_k \mathrm{Tr}[\hat{v}_\alpha(\mathbf{k})\mathcal{G}(\mathbf{k},i\omega_l+i\omega)\hat{v}_\beta(\mathbf{k})\mathcal{G}(\mathbf{k},i\omega_l)]$$

The Matsubara Green function can be represent in the Lehmann's representation in the terms of spectral function as

$$\mathcal{G}(\mathbf{k},i\omega_l) = \int_{-\infty}^{\infty}\frac{d\epsilon}{2\pi}\frac{A(\mathbf{k},\epsilon)}{i\omega_l+\mu-\epsilon}$$

$$A(\mathbf{k},\epsilon) = -2\mathrm{Im}G^R(\mathbf{k},\epsilon)$$

Conductivity Formula

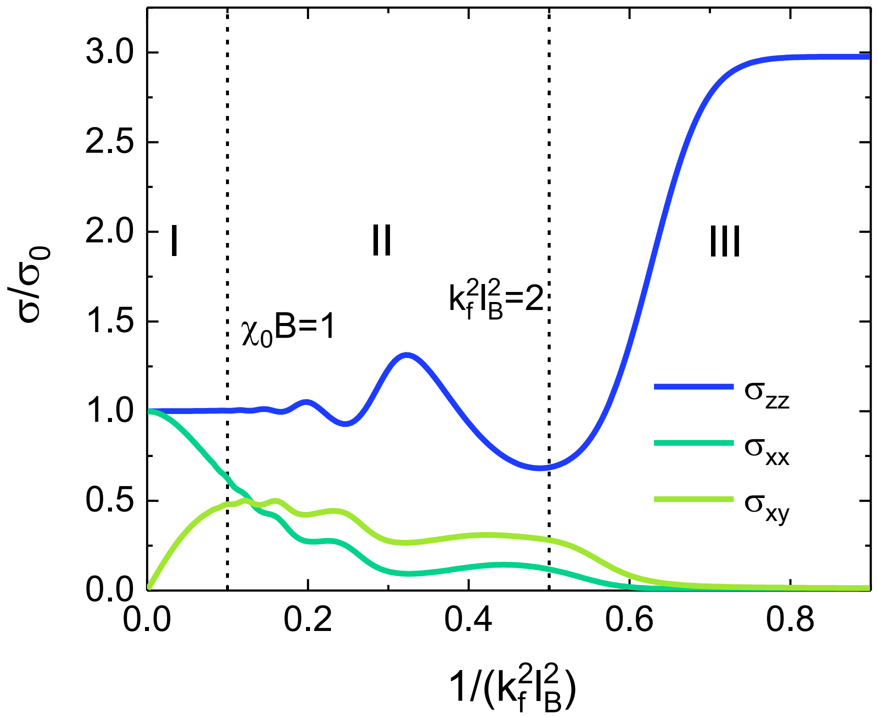
$$\sigma_{zz} = \frac{e^2 v}{2\pi^2 l_B^2} \sum_{n=0}^{\infty} \alpha_n \frac{b \sin\frac{\theta_n}{2} + (2a_n + 2\gamma^2)\cos\frac{\theta_n}{2}}{2b\sqrt{\rho_n}},$$

$$\sigma_{xx} = \frac{e^2\gamma^2 v}{2\pi^2 l_B^2} \sum_{n=0}^{\infty} \{ \frac{\cos\frac{\theta_n}{2}}{b\sqrt{\rho_n}} + \frac{\cos\frac{\theta_{n+1}}{2}}{b\sqrt{\rho_{n+1}}} + \frac{[(2n+1)\mu^2 - \frac{h^2v^2}{2l_B^2}]}{\frac{h^4v^4}{l_B^4} + b^2} \times [\frac{(3a_n - a_{n+1} - 2\rho_n)\cos\frac{\theta_n}{2}}{b\sqrt{\rho_n}} + \frac{(2\rho_{n+1} - 3a_{n+1} + a_n)\cos\frac{\theta_{n+1}}{2}}{b\sqrt{\rho_{n+1}}}]\},$$

$$\sigma_{xy}^I = \frac{e^2 v\gamma\mu}{2\pi^2 l_B^2} \sum_{n=0}^{\infty} [\frac{\cos\frac{\theta_{n+1}}{2}}{b\sqrt{\rho_{n+1}}} - \frac{\cos\frac{\theta_n}{2}}{b\sqrt{\rho_n}} + \frac{[\gamma^2 + (n+\frac{1}{2})\frac{h^2v^2}{l_B^2}]}{\frac{h^4v^4}{l_B^4} + b^2} \times [\frac{(3a_n - a_{n+1} - 2\rho_n)\cos\frac{\theta_n}{2}}{b\sqrt{\rho_n}} + \frac{(2\rho_{n+1} - 3a_{n+1} + a_n)\cos\frac{\theta_{n+1}}{2}}{b\sqrt{\rho_{n+1}}}]\}.$$

where $\alpha_n = 2 - \delta_{n0}$, $a_n \equiv \mu^2 - \Delta^2 - \gamma^2 - 2n\left(\hbar v/l_B\right)^2$, $b \equiv 2\mu\gamma$,
$\rho_n \equiv \sqrt{a_n^2 + b^2}$ and $\cos\theta_n \equiv \frac{a_n}{\rho_n}$ with $\theta_n \in [0, \pi]$

Three Regimes



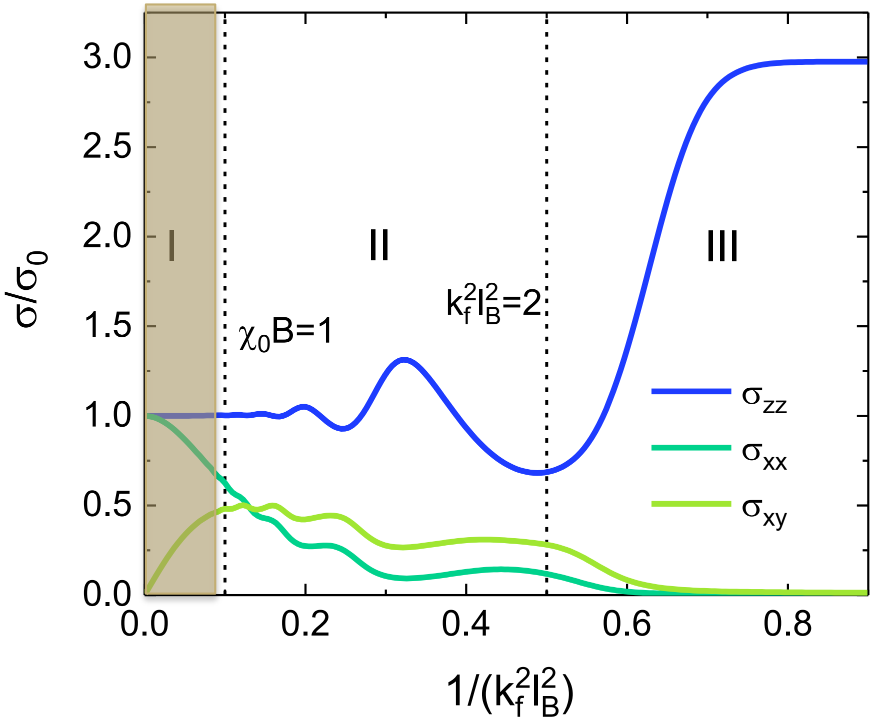
I. semiclassical regime overlapping LLs with dominating background
II. Separated LLs with quantum oscillations
III. 0^{th} LL and quantum limit

$$l_B \equiv \sqrt{\hbar/eB}$$

$$k_f = (3\pi^2\varrho)^{1/3}$$

$$(k_f l_B)^{-2} = \frac{B}{2B_F} \text{ with } B_F = \frac{\hbar}{2e}k_f^2$$

Regime I: Semiclassical



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Special Functions and Their Expansions

$$G^{R/A}_{ns\zeta} = \frac{1}{\mu - \varepsilon^{\zeta}_{ns} \pm i\gamma}$$

In the basis of relativistic Landau levels

$$\sigma_{\alpha\alpha} = \frac{e^2\hbar}{2\pi^2 l_B^2}\int_{-\infty}^{\infty}\frac{dk_z}{2\pi}\sum_{ss'}\sum_{\zeta\zeta'}\sum_{nn'}|v^{\alpha}_{n\zeta s,n'\zeta' s'}|^2 \mathrm{Im}G^R_{n\zeta s}\mathrm{Im}G^R_{n'\zeta' s'}$$

$$\sigma^I_{xy} = -i\frac{\hbar e^2}{2\pi V}\int_{-\infty}^{\infty}\frac{dk_z}{2\pi}\sum_{ss'}\sum_{\zeta\zeta'}\sum_{nn'} v^x_{n\zeta s,n'\zeta' s'} v^y_{n'\zeta' s',n\zeta s} \times [\mathrm{Re}G^R_{n'\zeta' s'}\mathrm{Im}G^A_{n\zeta s} + \mathrm{Im}G^R_{n'\zeta' s'}\mathrm{Re}G^A_{n\zeta s}]$$

With the help of Hurwitz zeta function and digamma function we can do the series summation in weak magnetic field regime

$$\zeta(s,q) = \sum_{n=0}^{\infty}\frac{1}{(q+n)^s},$$

$$\frac{\psi(b)-\psi(a)}{b-a} = \sum_{n=0}^{\infty}\frac{1}{(n+a)(n+b)},$$

Using the asymptotic expansion at large z, we can do the integral over k_z

$$\psi(z) = \log z - \frac{1}{2z} - \frac{1}{12z^2} + \cdots$$

$$\zeta(2,z) = \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + \cdots$$

Intrinsic Magnetoresistivity

$$\sigma_{xx} = \frac{\sigma_D}{1+\chi^2 B^2}$$

$$\sigma_{xy} = \frac{\chi B \sigma_D}{1+\chi^2 B^2}$$

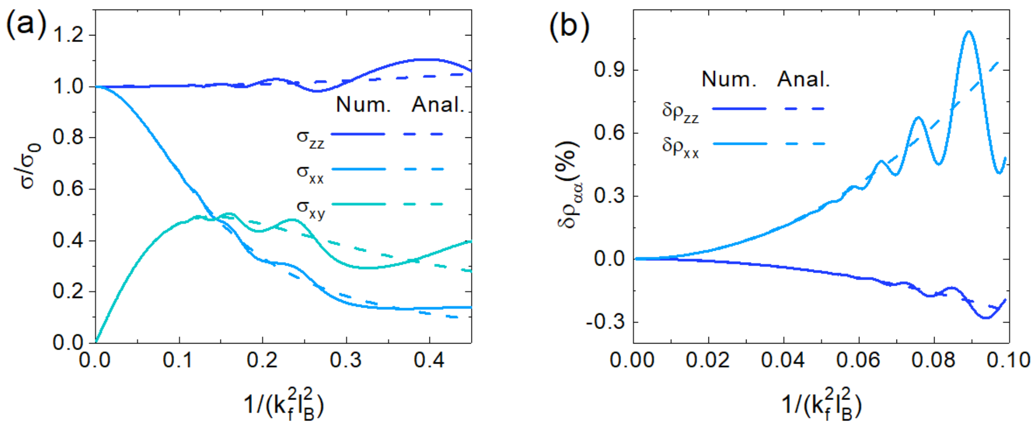
The mobility and effective Drude conductivity are defined as

$$\chi = \sigma_{xy}/\sigma_{xx}B$$

$$\sigma_D = \sigma_{xx}(1+\chi^2 B^2)$$

Relative magnetoresistivity

$$\delta\rho_{\alpha\alpha}(B) = \frac{\rho_{\alpha\alpha}(B)}{\rho(0)} - 1 = \frac{c_\alpha}{(l_B k_f)^4} = c_\alpha \frac{B^2}{\left(\frac{\hbar}{e}k_f^2\right)^2}.$$



For the weak disorder scattering limit, irrelevant to the band gap and external scattering, the amplitude of relative magnetoresistivity is determined by k_f only.

$$k_f = (3\pi^2\varrho)^{1/3}$$

The Coefficients

For a weak band broadening width,

$$c_x = c_y \approx 1 + \frac{3}{4}\left(1 - \frac{8\mu^2}{v^2\hbar^2 k_f^2}\right)\frac{\gamma^2}{v^2\hbar^2 k_f^2} \Rightarrow 1$$

$$c_z \approx -\frac{1}{4} + \frac{1}{2}\frac{\gamma^2}{v^2\hbar^2 k_f^2} \Rightarrow -1/4$$

$$c_\chi \approx -\frac{3}{4} + \frac{3}{4}\left(1 + \frac{2\mu^2}{v^2\hbar^2 k_f^2}\right)\frac{\gamma^2}{v^2\hbar^2 k_f^2} \Rightarrow -3/4$$

Estimation

This intrinsic magnetoresistivity is expected to be measurable in the system with low carrier density and high mobility

$$\delta\rho_{\alpha\alpha}(B) = c_\alpha \left(\frac{B}{2.92\varrho_0^{2/3}}\right)^2, \varrho = \varrho_0 \times 10^{16}/\mathrm{cm}^3$$

In the weak scattering limit,

$$\rho_{zz} = \frac{1}{\sigma_{zz}} \simeq \frac{1}{\sigma_0}\left[1 - \frac{1}{4}\frac{1}{(k_f l_B)^4}\right]$$

Negative longitudinal MR

$$\rho_{xx} = \frac{1}{\sigma_D} \simeq \frac{1}{\sigma_0}\left[1 + \frac{1}{(k_f l_B)^4}\right]$$

Positive transverse MR

Intrinsic Magnetoresistance

Non-Abelian Berry Curvature

The intrinsic magnetoresistance is attributed to the presence of Berry curvature for the Dirac particles [See Y. Gao et al, PRB 95, 165135(2017)]. Due to the correlation between four bands, the Berry curvature becomes non-Abelian in the presence of a finite mass. For example, focusing on the two positive bands,

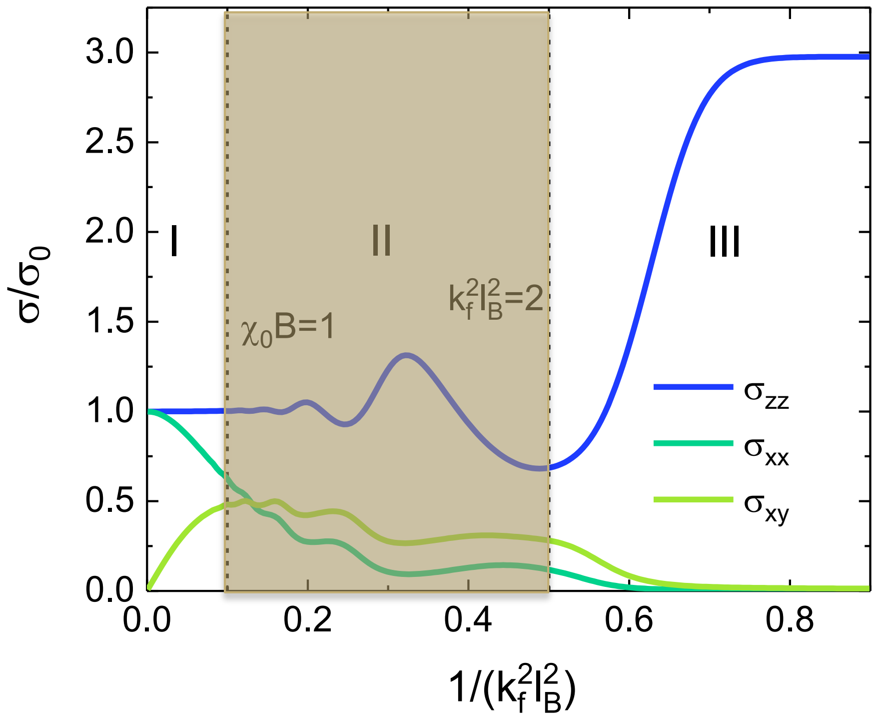
$$\mathcal{A}_{ss'}(\mathbf{p}) = iu^{s\dagger}(\mathbf{p})\nabla_{\mathbf{p}}u^{s'}(\mathbf{p})$$

$$\mathcal{A}(\mathbf{p}) = \begin{bmatrix} \frac{1+\cos\theta}{2|\mathbf{p}|\sin\theta}\hat{\phi} & -\frac{m(\hat{\boldsymbol{\phi}}+i\hat{\boldsymbol{\theta}})}{2|\mathbf{p}|E_{\mathbf{p}}} \\ -\frac{m(\hat{\boldsymbol{\phi}}-i\hat{\boldsymbol{\theta}})}{2|\mathbf{p}|E_{\mathbf{p}}} & \frac{1-\cos\theta}{2|\mathbf{p}|\sin\theta}\hat{\phi} \end{bmatrix}$$

$$\boldsymbol{\Omega}^a \equiv \nabla_{\mathbf{p}} \times \mathcal{A}^a - \frac{1}{2}\epsilon_{abc}\mathcal{A}^b \times \mathcal{A}^c.$$

$$\Omega = -\frac{m}{E_{\mathbf{p}}^3}\sigma_x\hat{\phi} - \frac{m}{E_{\mathbf{p}}^3}\sigma_y\hat{\theta} - \frac{1}{E_{\mathbf{p}}^2}\sigma_z\hat{\mathbf{p}}$$

Regime II: Quantum Oscillation



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Quantum Oscillation

In quantum oscillation regime, the broadening γ ➔0

$$\sigma_{zz} = \frac{e^2 v\tau}{2\mu\pi^2\hbar l_B^2}\sum_{n=0}^{\infty}\alpha_n\sqrt{\mu^2-\Delta^2-2n\frac{\hbar^2v^2}{l_B^2}}$$

Using the Poisson summation formula

$$2\frac{\hbar^2v^2}{l_B^2}\sum_{n=0}^{\infty}\alpha_n\sqrt{\mu^2-\Delta^2-2n\frac{\hbar^2v^2}{l_B^2}} = \frac{4}{3}\hbar^3v^3k_f^3\{1-\frac{3}{2\pi k_f^3 l_B^3}\sum_{p=1}^{\infty}\frac{1}{p\sqrt{p}}\cos[\pi p k_f^2 l_B^2+\frac{\pi}{4}]\}$$

To consider the dephase effect caused the band broadening, we can introduce the Dingle factor λ_D for Lorentz distribution function in the series summation, then

$$\sigma_{zz} = \frac{e^2v^2k_f^3\tau}{3\pi^2\mu}\{1-\frac{3}{2\pi k_f^3 l_B^3}\sum_{p=1}^{\infty}\frac{e^{-p\lambda_D}}{p\sqrt{p}}\cos[\pi p k_f^2 l_B^2+\frac{\pi}{4}]\},$$

The Dingle factor $\lambda_D = \dfrac{\pi}{\chi_0 B}$

Quantum Oscillation

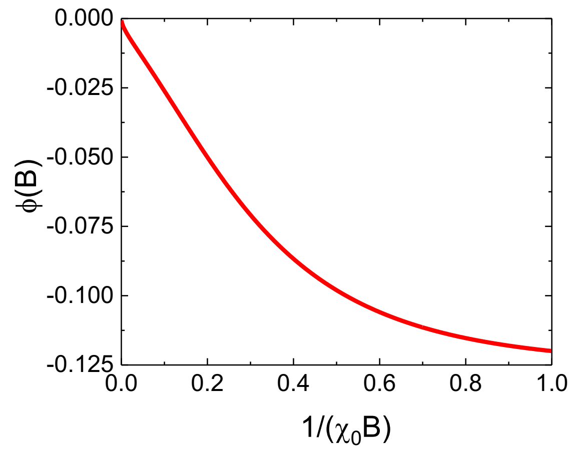
The quantum oscillation can be summarized as

$$\delta\rho_{\alpha\alpha}^{os} = \frac{d_\alpha}{k_f l_B \cos 2\pi\phi_B} \mathrm{Li}_{\frac{1}{2}}\left(e^{-\frac{\pi}{\chi_0 B}}\right)\cos[2\pi(\frac{B_F}{B}+\phi_B)]$$

Lifshitz-Kosevich formula

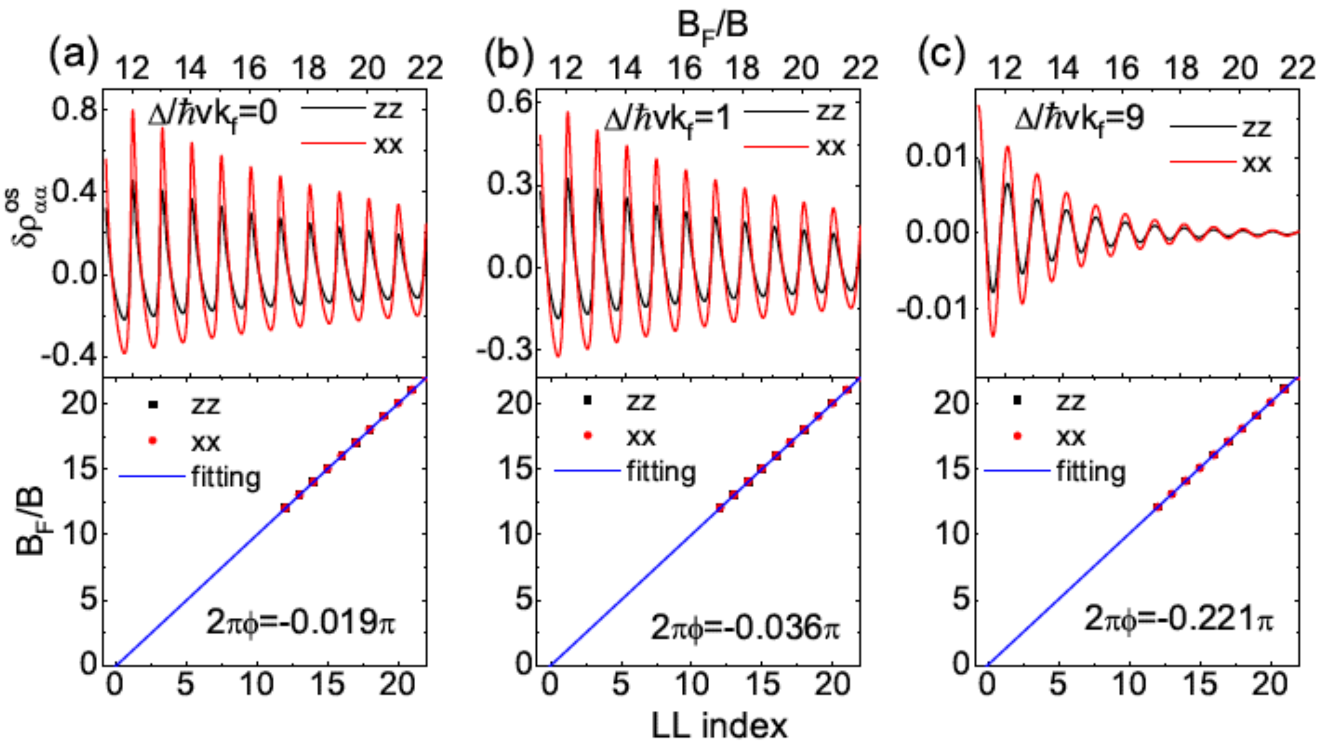
where $\mathrm{Li}_s(x)$ is the polylogarithm function of order s and argument x .

$$2\pi\phi_B = \arctan\left\{\frac{Re\left[\sqrt{2}\exp(i\frac{3\pi}{4})\mathrm{Li}_{\frac{1}{2}}\left(ie^{-\frac{\pi}{\chi_0 B}}\right)\right]}{\mathrm{Li}_{\frac{1}{2}}\left(e^{-\frac{\pi}{\chi_0 B}}\right)}\right\}$$



The phase shift here is a function of the dingle factor. In the previous literatures, the phase shift is usually regarded as a constant, which is only valid in the semiclassical regime.

Landau Level Fan Diagram and Phase Shift

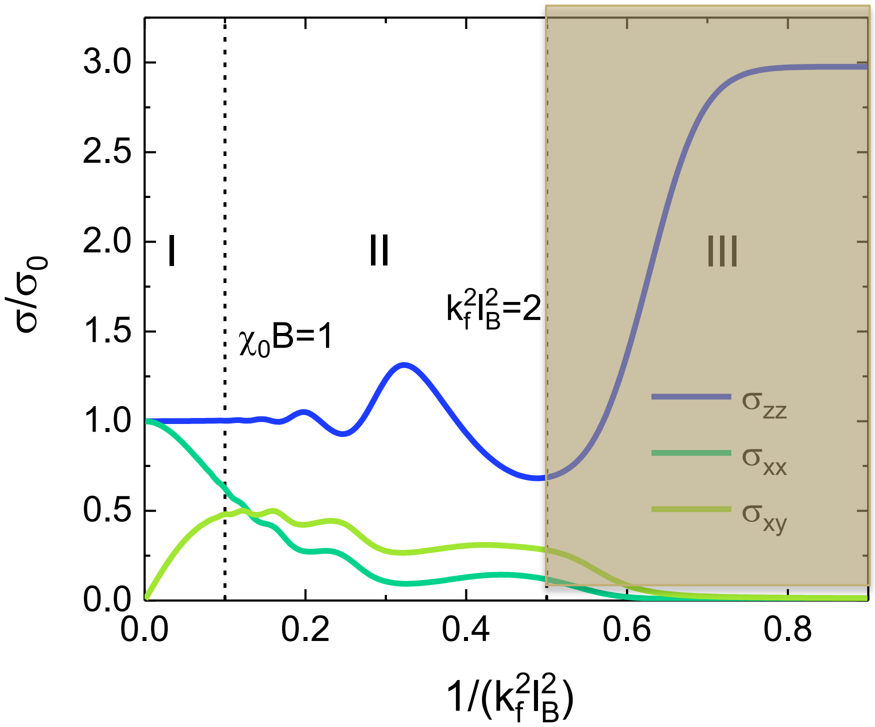


$$\Delta\sigma_{xx} \propto \cos\left[2\pi\left(\frac{B_F}{B}+\frac{1}{2}-\frac{\gamma}{2\pi}\right)\right]$$

$$\frac{\gamma}{2\pi}=\frac{1}{2}-\phi_F$$

Figure 1. The numerical calculated oscillation part of magnetoresistivity ρ^{os}_{xx} (red lines) and ρ^{os}_{zz} (blue lines) for (a) $\Delta = 0$, (b)$\Delta/\hbar v k_f = 1$ and (c) $\Delta/\hbar v k_f$ as a function of B_F/B. The corresponding bottom panel is the Landau-level fan diagram for the quantum oscillation, when a linear fit to the Landau-level fan diagram is extrapolated to $B_F/B \to 0$, the intercept on the LL index axis gives the phase factor ϕ. Since ρ_{zz} and ρ_{xx} share the same phase, the Landau-level fan diagram of them overlapped with each other, we use ϕ to represent the phase shift for both ρ_{xx} and ρ_{zz}. Broadening width at zero magnetic field has been chosen as $\gamma/\hbar v k_f = 0.003$ for calculation here.

Regime III: Quantum Limit



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Quantum Limit

$$\mu = \sqrt{(2\pi^2 l_B^2 \hbar v \varrho)^2 + \Delta^2}$$

$$\tau = \frac{2\pi^3 \ell_B^4 \hbar^3 v^2 \varrho}{n_i u_0^2 \sqrt{(2\pi^2 \ell_B^2 \hbar v \varrho)^2 + \Delta^2}}$$

$$\sigma_{zz} = \frac{e^2 v^2 \varrho \tau}{\mu}$$

A relation between longitudinal and transverse conductivity

$$\sigma_{xx}\sigma_{zz} = \frac{e^4}{h^2}\frac{1}{2\pi^2 l_B^2} \sim B$$

$$\Delta = 0$$

$$\sigma_{zz} = \frac{e^2 \hbar}{2\pi}\frac{v^2}{n_i U_0^2} = \text{const}$$

$$\sigma_{xx} \propto B$$

$$\Delta \gg 2\pi^2 \ell_B^2 \hbar v \varrho$$

$$\sigma_{zz} \approx \frac{2\pi^3 e^2 \hbar^3 v^4 l_B^4 \varrho^2}{n_i U_0^2 \Delta^2} \propto \frac{1}{B^2}$$

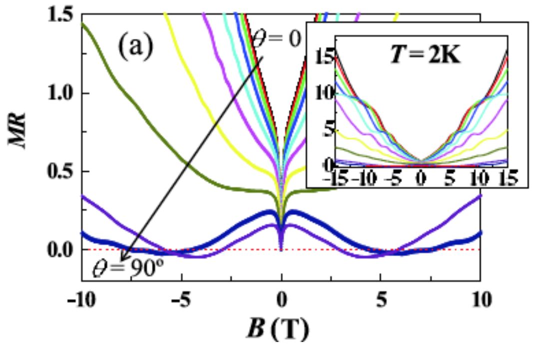
$$\sigma_{xx} \propto B^3$$

Lu, Zhang & Shen
PRB 92, 045203(2015)

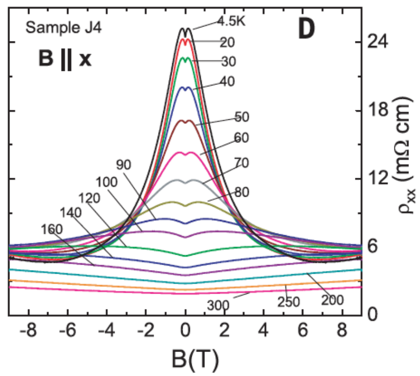
Quantum Interference Theory:
Weak localization and Weak antilocalization

- B. Fu, H. W. Wang & S. Q. Shen, *Quantum interference theory of magnetoresistance in Dirac materials*, Phys. Rev. Lett. 122, 246601 (2019)

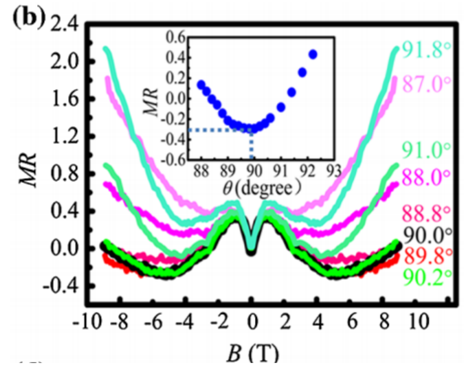
MR in 3D Topological Materials



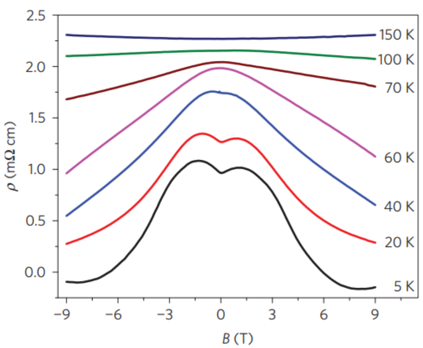
$Bi_{1-s}Sb_x$: Kim et al, PRL 111, 246603(2013)



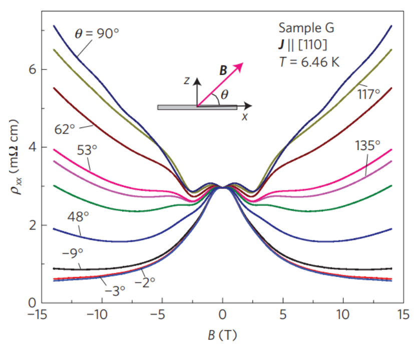
Na_3Bi: Xiong et al, Science 350, 6259(2015)



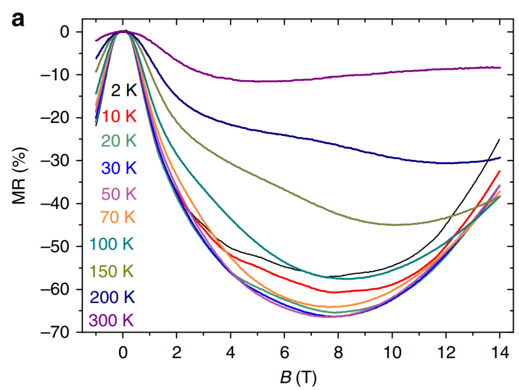
TaAs: Huang et al, PRX 5, 031023(2015); Zhang et al, Nat. Commun. 7, 10735(2016)



$ZrTe_5$: Li et al., Nat. Phys. 12, 550 (2016); Mutch et al., arXiv: 1808.07898

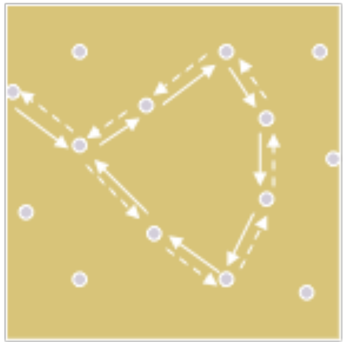


GdPtBi: Hirschberger et al., NM 15,1161(2016); Liang et al., PRX 8, 031002 (2018).



Cd_3As_2: H Li et al., NC 7,10301(2016); CZ Li et al., NC 6, 10137(2015)

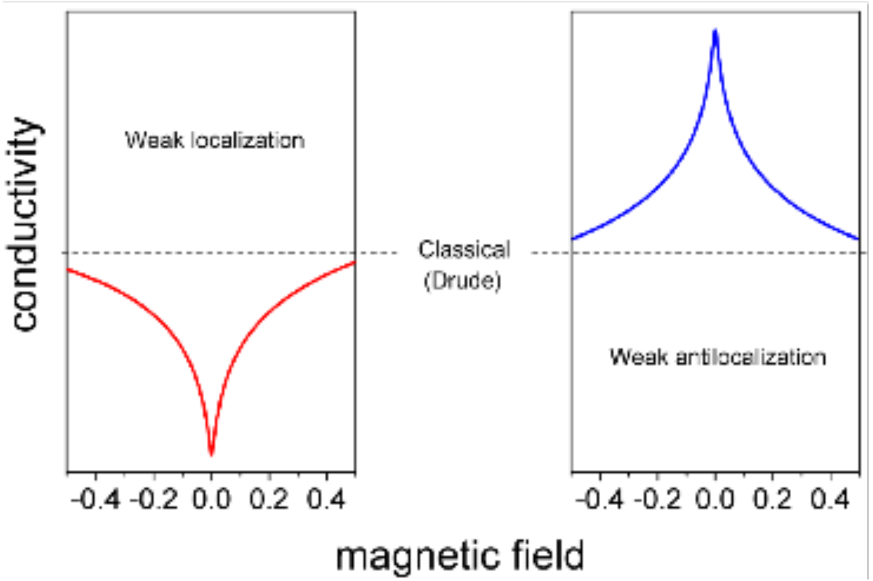
Quantum Interference Effect



quantum diffusion regime $\ell_e \ll \ell_\phi \sim L$

" weak localization results from the interference between a closed Feynman path and its time-reversed path "
Al'tershuler and Lee, Phys. Today 1988

Constructive interference
⇩
enhance backscattering
⇩
lower conductivity



Destructive interference
⇩
suppress backscattering
⇩
higher conductivity

Weak localization vs weak antilocalization

Hikami-Larkin-Nagaoka-Formula (2D)

Prog. Theo. Phys. 1980

$$\Delta\sigma = \sigma(H) - \sigma(0)$$

$$= -\frac{\alpha e^2}{2\pi^2\hbar}\left[\ln\frac{1}{\tau_\varepsilon a} - \psi\left(\frac{1}{2} + \frac{1}{\tau_\varepsilon a}\right)\right]$$

Wigner-Dyson ensemble of random matrices (Dyson, J. Math. Phys. 1962):
Time reversal invariant (no magnetic scattering):
Orthogonal (WL, α= 1, no spin-orbit scattering)
Symplectic (WAL, α = -1/2, spin-orbit scattering)
Time reversal breaking (strong magnetic scattering)
Unitary , small $\sigma(B) \sim B^2$

Impurity scattering	Symmetry	Time-reversal	Spin-rotational	Transport
Scalar	Orthogonal	✓	✓	WL
Spin-orbit	Symplectic	✓	✗	WAL
Magnetic	Unitary	✗		Semiclassica l

Model for 3D Topological Materials

Shen, Topological Insulators (Springer 2012)

$$H(\mathbf{k}) = v\hbar k_j \alpha_j + m(k)\beta \qquad m(k) = mv^2 - b\hbar^2 k^2$$

where the Dirac matrices are chosen as

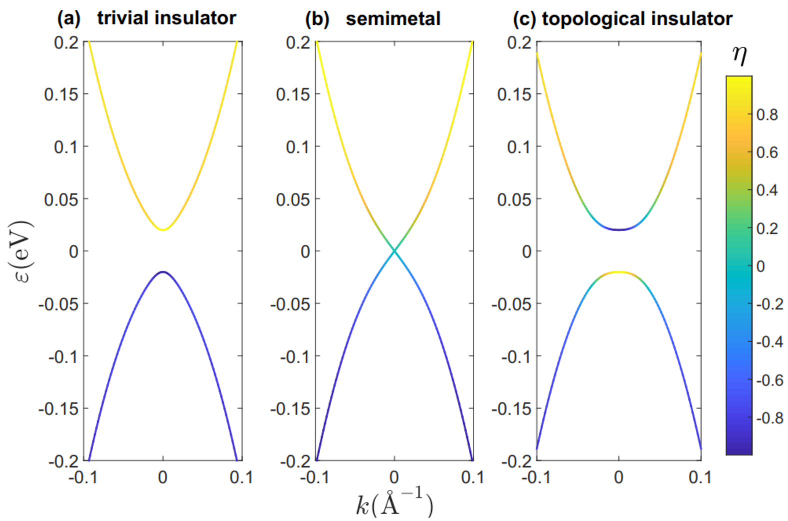
$$\alpha_j = \tau_x \otimes \sigma_j, \beta = \tau_z \otimes \sigma_0$$

The orbital polarization:
the expectation value of $\tau_z \otimes \sigma_0$,

$$\eta(k) \equiv \langle \tau_z \otimes \sigma_0 \rangle = \frac{m(k)}{\varepsilon(k)},$$

$\eta(k)$ plays a primary role in the magnetotransport properties of Dirac materials

(a). $mb > 0$: topologically non-trivial;
(b). $mb = 0$: Dirac semimetal
(c). $mb < 0$: topologically trivial

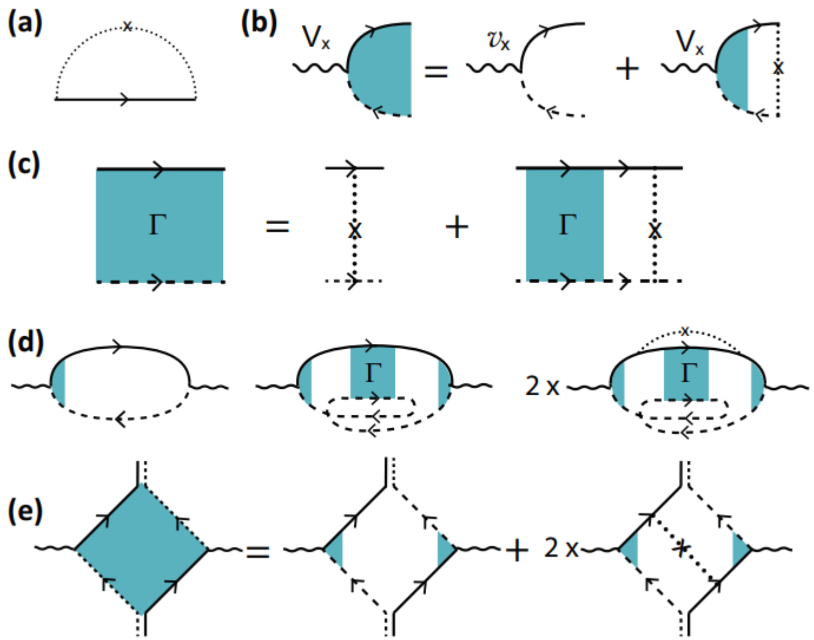


$$G^{R,A}_{\mathbf{k}} = [\mu - H(\mathbf{k}) \mp i\mathrm{Im}\Sigma^R(\mathbf{k})]^{-1}$$

Topological Invariant: $\nu = \frac{1}{2}(\mathrm{sgn}(m) + \mathrm{sgn}(b))$

16 (=4X4) Green's functions

Feynmann Diagram Techniques



Total conductivity includes two parts, the classical conductivity and its correction from the quantum interference

$$\sigma_{\alpha\alpha} = \frac{e^2\hbar}{2\pi\Omega}\sum_k \mathrm{Tr}\langle \hat{v}_\alpha \hat{G}^R \hat{v}_\alpha \hat{G}^A\rangle_{\mathrm{dis}} \approx \sigma_{cl} + \sigma_{qi}$$

Classical conductivity with vertex correction

$$\sigma_{cl} = \frac{e^2\hbar}{2\pi\Omega}\sum_k \mathrm{Tr}[\hat{V}_\alpha G^R \hat{v}_\alpha G^A]$$

Quantum correction due to the interference between a closed multiple scattering path and its time reversal counterpart

$$\sigma_{qi} = \frac{e^2\hbar}{2\pi}\sum_{\mu\mu'\nu\nu'}\sum_{\mathbf{q}}[W]^{\nu'\mu}_{\mu'\nu}[\Gamma(\mathbf{q})]^{\mu\mu'}_{\nu\nu'}$$

Γ is Cooperon structure factor and W is the corresponding Hikami box.

Effective Cooperon Channels

There are 16 Cooperon modes, and only the four effective channels listed below govern the quantum correction to the conductivity

i	Cooperon Channel in $\lvert C^{\tau,\tau_z}_{\sigma,\sigma_z}\rangle$	$\mathcal{F}_i$	z_i
0	$\lvert C^{1,1}_{0,0}\rangle, \lvert C^{1,-1}_{0,0}\rangle$	$\mathcal{F}_0 = 1$	$z_0 = 0$
s	$\lvert C^{1,0}_{0,0}\rangle, \lvert C^{1,1}_{1,0}\rangle, \lvert C^{1,-1}_{1,0}\rangle$	$\mathcal{F}_s$	z_s
$t_\pm$	$\lvert C^{1,1}_{1,\pm 1}\rangle, \lvert C^{1,-1}_{1,\pm 1}\rangle, \lvert C^{0,0}_{1,\pm 1}\rangle$	$\mathcal{F}_{t_\pm} = \mathcal{F}_t$	$z_{t_\pm} = z_t$

The full Hamiltonian is invariant under the symplectic time-reversal symmetry transformation $\mathcal{T}_{02}\colon \mathcal{O} \mapsto \tau_0\sigma_2\mathcal{O}^T\tau_0\sigma_2$, there is always one gapless Cooperon channel.

$$\sigma_{qi} = \frac{2e^2}{h} \sum_{i=0,s,t_\pm} \sum_{\mathbf{q}} \frac{\mathcal{F}_i}{q^2 + z_i/\ell_e^2}.$$

Parameters

$$\mathcal{F}_s = \frac{2\mathcal{I}(P_s,Q_s)}{3(1-\frac{1}{2}g)^2g^3}[z_s^2g^3(-2+\frac{3}{2}g)+z_sg(1+\frac{1}{2}g-\frac{7}{2}g^2+\frac{9}{4}g^3)-(2-4g-\frac{1}{2}g^2+3g^3-\frac{3}{2}g^4)]$$

$$z_s = -2\sqrt{-\frac{P_s}{3}}\cos\left[\frac{1}{3}\arccos(\frac{3Q_s}{2P_s}\sqrt{\frac{-3}{P_s}})\right]+\frac{2}{9}\frac{(3+g-3g^2)}{g^2}$$

$$P_s = -\frac{4}{3g^4}(1-\frac{1}{3}g-\frac{8}{9}g^2+\frac{1}{3}g^3-\frac{1}{2}g^4)$$

$$Q_s = \frac{1}{g^6}(\frac{16}{27}-\frac{8}{27}g-\frac{80}{81}g^2+\frac{664}{729}g^3-\frac{28}{81}g^4-\frac{2}{27}g^5-\frac{7}{27}g^6)$$

$$\mathcal{F}_t = \frac{2\mathcal{I}(P_t,Q_t)}{3(1-\frac{1}{2}g)^2g^3}[z_t^2g^3(-2+\frac{3}{2}g)+z_tg(1+\frac{3}{2}g-\frac{35}{8}g^2+\frac{39}{16}g^3)-(3-\frac{9}{2}g-\frac{3}{2}g^2+\frac{7}{2}g^3-\frac{3}{2}g^4)]$$

$$z_t = -2\sqrt{-\frac{P_t}{3}}\cos\left[\frac{1}{3}\arccos(\frac{3Q_t}{2P_t}\sqrt{\frac{-3}{P_t}})\right]+\frac{2}{9g^2}(3+\frac{3g}{2}-\frac{27}{8}g^2)$$

$$P_t = \frac{4}{3g^4}(-1+\frac{1}{2}g+\frac{5}{4}g^2-\frac{3}{8}g^3+\frac{27}{64}g^4)$$

$$Q_t = \frac{1}{g^6}(\frac{16}{27}-\frac{4}{9}g-\frac{14}{9}g^2+\frac{67}{54}g^3+\frac{1}{6}g^4-\frac{1}{8}g^5-\frac{5}{32}g^6)$$

All these parameters are functions of the orbital polarization

$$\eta(k) \equiv \langle \tau_z \otimes \sigma_0 \rangle = \frac{m(k)}{\varepsilon(k)}$$

Magnetoconductivity Formula

In the presence of magnetic field, the q integration in the transverse direction is replaced with a summation over the effective Landau levels of the Cooperon as

$$q_x^2 + q_y^2 \mapsto (n + \frac{1}{2})\ell_B^{-2} \qquad \sum_{\mathbf{q}} \mapsto \frac{1}{4\pi\ell_B^2}\sum_n \int \frac{dq_z}{2\pi}$$

$$\boxed{\sigma_{qi}(B) = \frac{e^2}{4\pi h \ell_B} \sum_{i=0,s,t_\pm} \mathcal{F}_i \zeta[\frac{1}{2}, \frac{1}{2} + (z_i + x^2)\frac{\ell_B^2}{\ell_e^2}]\Big|_1^{\ell_e/\ell_\phi}}$$

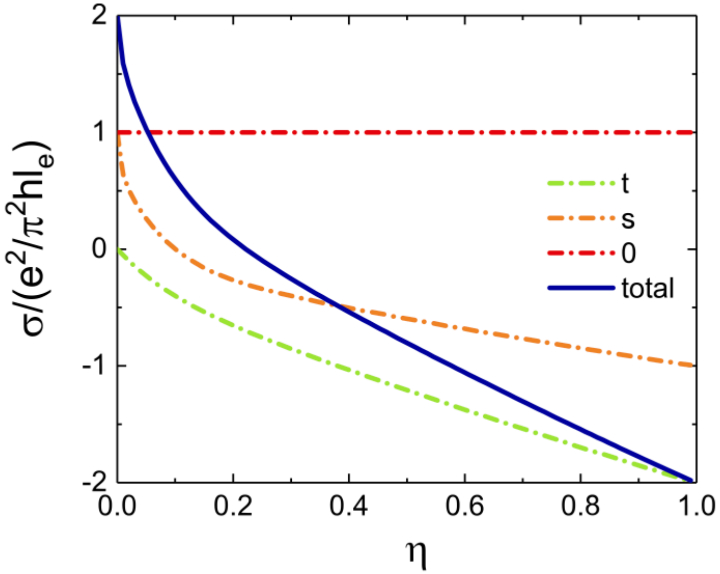
The magnetoresistance

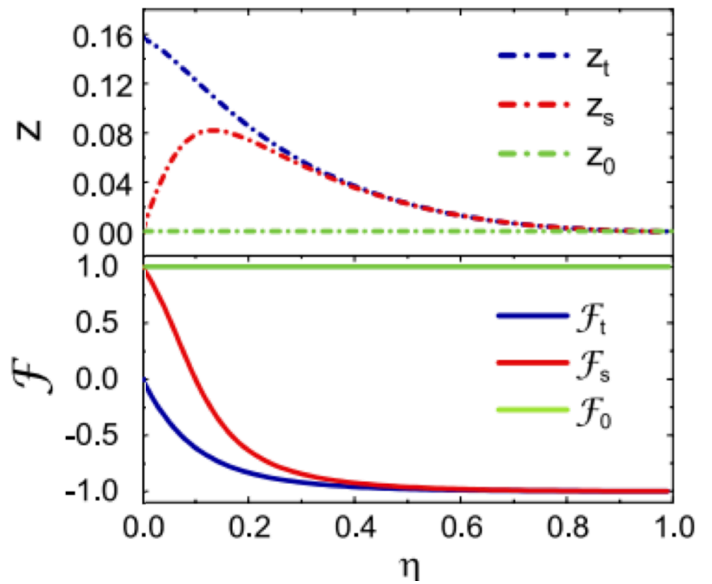
$$\delta\sigma(B) = \sigma_{qi}(B) - \sigma_{qi}(0)$$

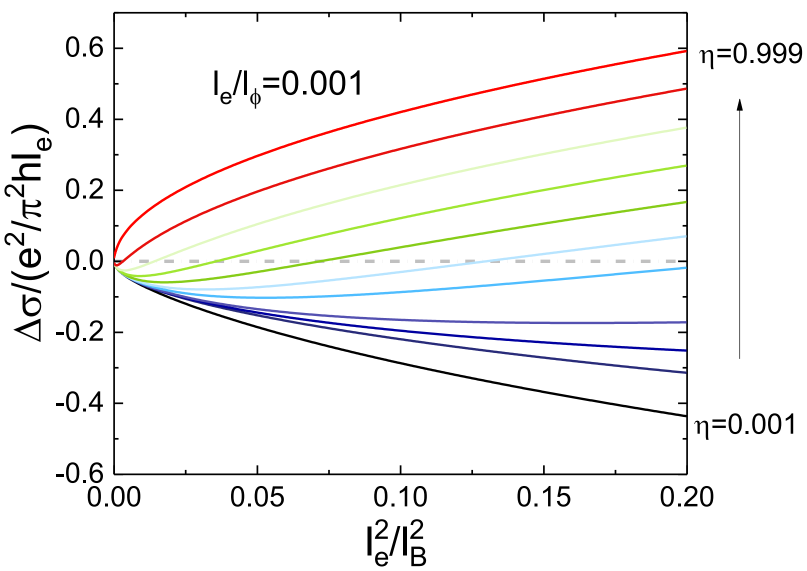
The Hurwitz zeta function

$$\zeta(s,t) = \sum_{n=0}^{\infty}(n+t)^{-s}$$

Orbital Polarization







$$\eta(k) \equiv \langle \tau_z \otimes \sigma_0 \rangle = \frac{m(k)}{\varepsilon(k)}$$

acts as a momentum-dependent effective magnetic field that polarizes the orbital pseudo-spin τ along the z direction

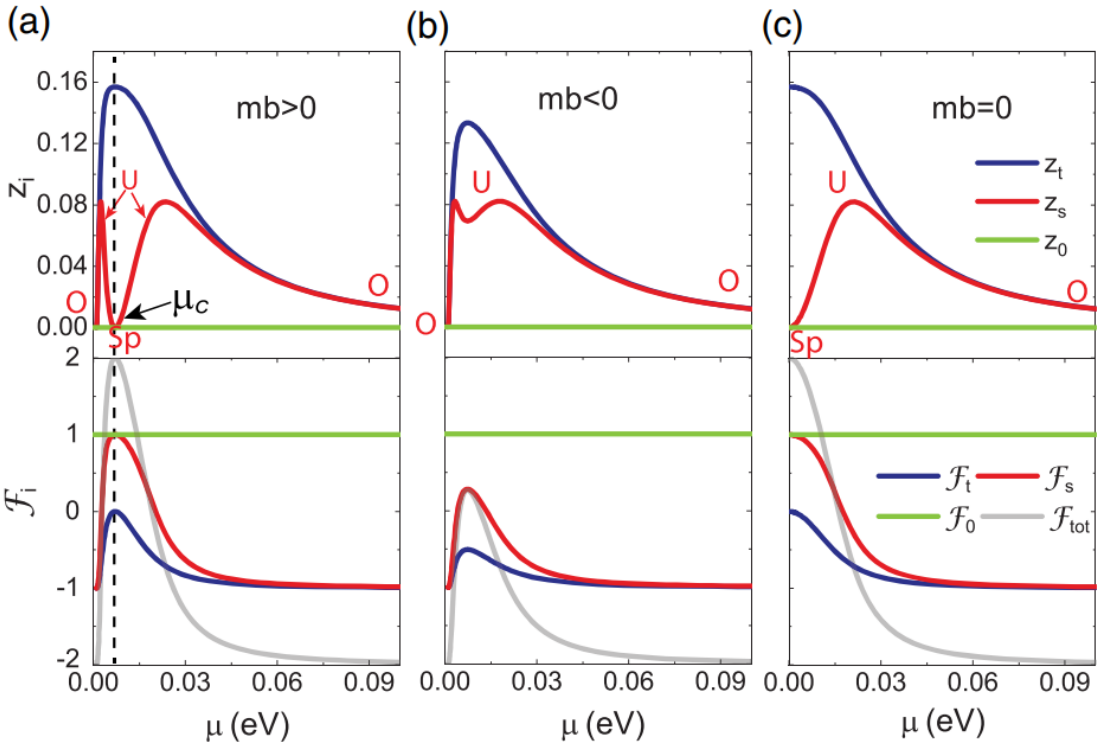
$$\varepsilon(k) = \sqrt{v^2\hbar^2k^2 + m^2(k)}$$

$$m(k) = mv^2 - b\hbar^2k^2$$

All F_i and z_i are functions of eta!

μ Dependence of z_i and $\mathcal{F}_i$

z_i and $\mathcal{F}_i$ are function of chemical potential and exhibit different behavior for different band topology



$$\mathcal{F}_{\text{tot}} = \sum_i \mathcal{F}_i$$

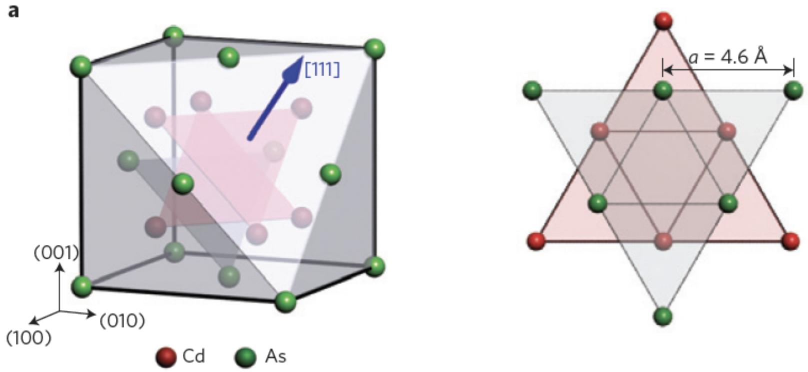
Topological insulator   Trivial insulator   Semimetal

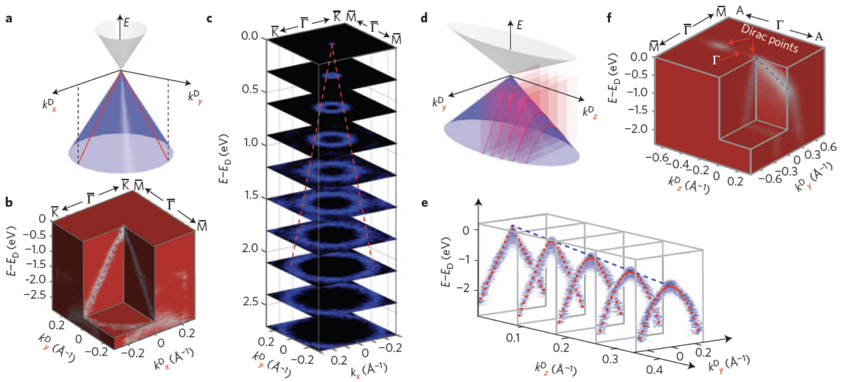
O: orthogonal

U: unitary

Sp: symplectic

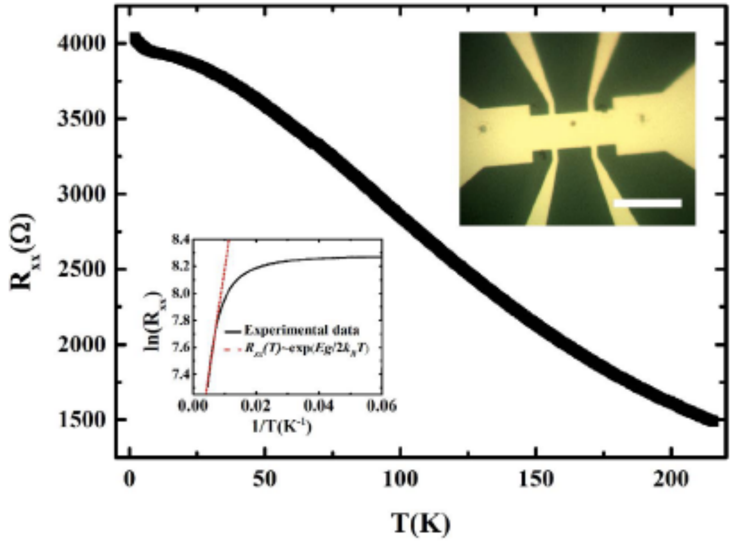
Application: Cd_3As_2 Thin Film

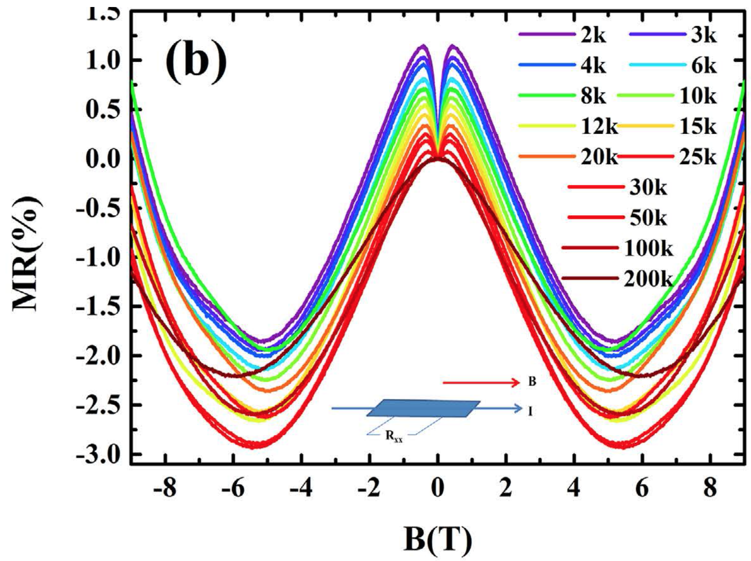




Cd_3As_2: a Dirac semimetal

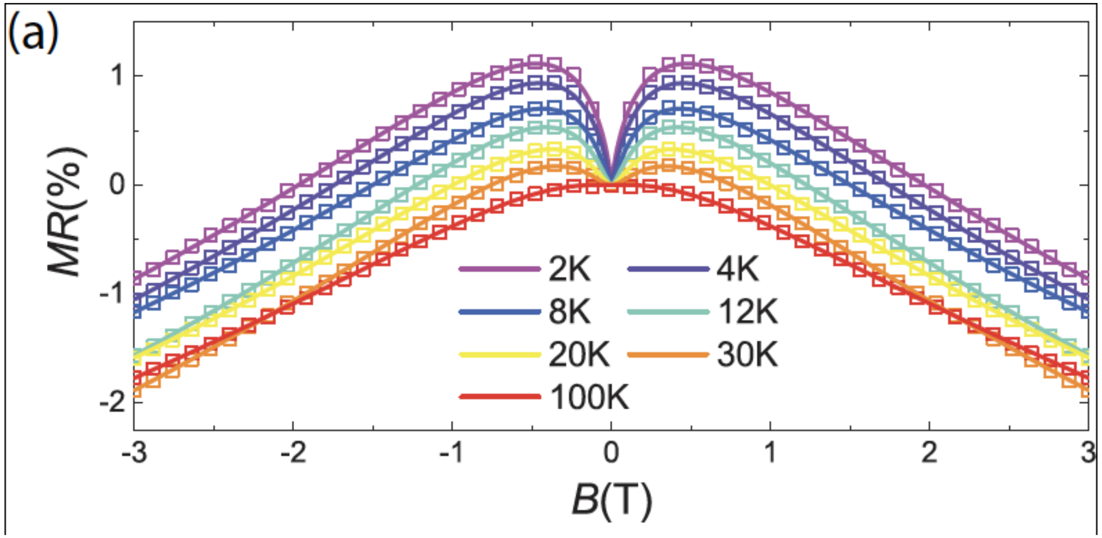
Liu et al, Nat. Mater. 13, 677(2014)





Zhao et al, Sci. Rep. 6, 22377(2017)

Data Fitting

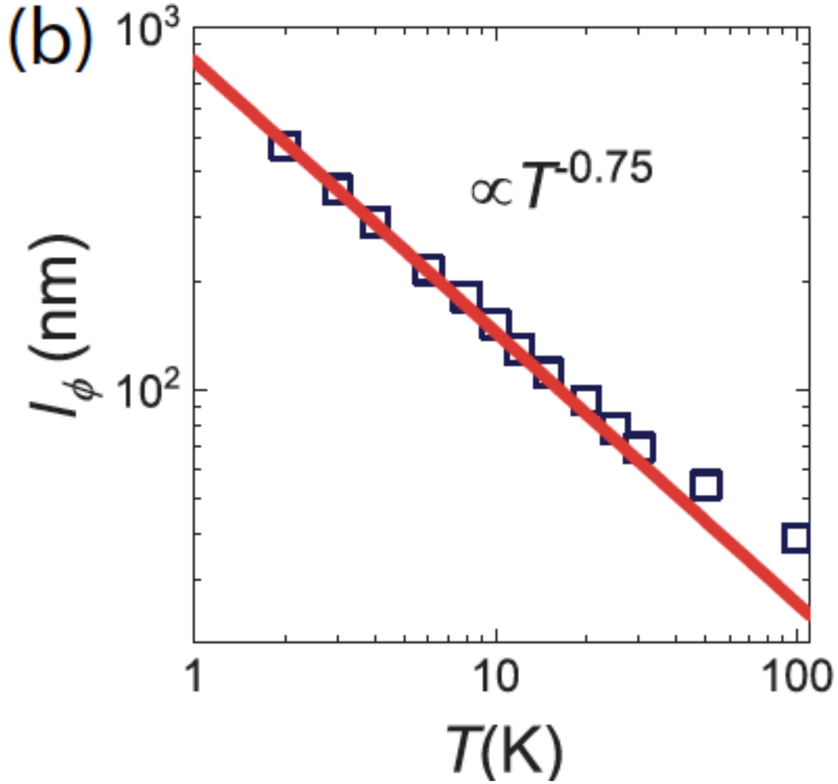


$T(\mathrm{K})$	$\ell_e(\mathrm{nm})$	$l_\phi(10^2\mathrm{nm})$	η^2
2	8.5 ± 0.2	4.7 ± 0.5	0.268 ± 0.009
3	8.6 ± 0.2	3.6 ± 0.3	0.268 ± 0.008
4	8.7 ± 0.2	2.9 ± 0.2	0.266 ± 0.017
6	8.8 ± 0.2	2.15 ± 0.11	0.266 ± 0.007
8	9.4 ± 0.2	1.81 ± 0.08	0.244 ± 0.006
10	9.1 ± 0.2	1.52 ± 0.06	0.262 ± 0.007
12	8.4 ± 0.2	1.29 ± 0.07	0.288 ± 0.009
15	8.8 ± 0.2	1.11 ± 0.05	0.277 ± 0.008
20	9.4 ± 0.2	0.93 ± 0.04	0.254 ± 0.008
25	8.9 ± 0.2	0.78 ± 0.04	0.27 ± 0.01
30	8.3 ± 0.3	0.69 ± 0.04	0.30 ± 0.01
50	8.1 ± 0.3	0.54 ± 0.04	0.32 ± 0.02
100	6.9 ± 0.6	0.39 ± 0.04	0.41 ± 0.04

$$\mathrm{MR} = -\delta\sigma(B)/[\rho_0^{-1} + \delta\sigma(B)]$$

ρ_0 is the experimentally measured resistivity at $B = 0\ T$.

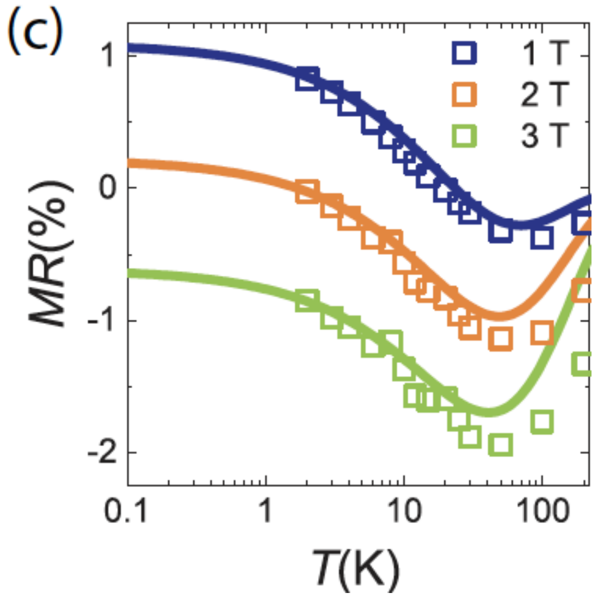
Phase Coherence Length



$$l_\phi \propto T^{-3/4}$$

The temperature dependence Indicates that the decoherence mechanism is dominated by electron-electron interaction. [Lee and Ramakrishnan, RMP 57, 287 (1985).

Temperature Dependence of MR



At low temperatures, for 2D

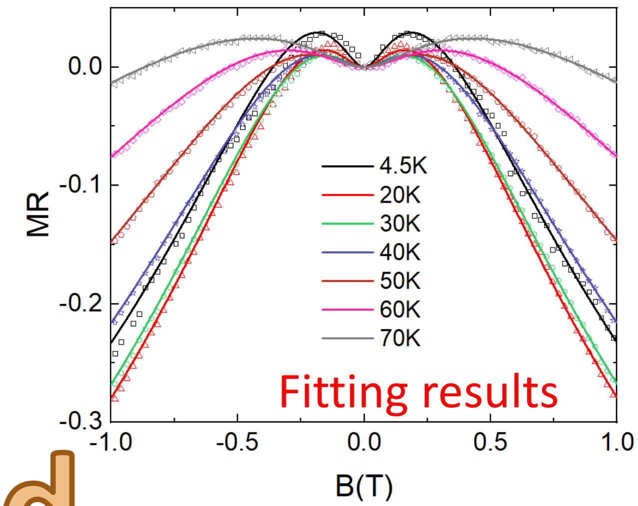
$$\delta\sigma \rightarrow \ln l_{\phi}$$

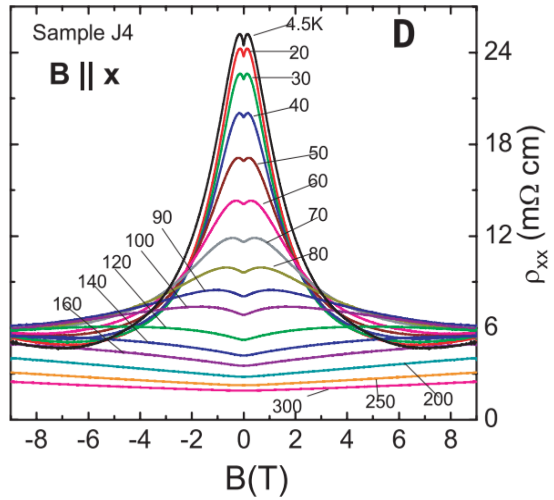
For 3D

$$\delta\sigma \rightarrow const$$

The MR is nonlinear and approaches to a constant at low temperatures. This is distinct from the behaviors of 2D systems

MR in Dirac Semimetal Na_3Bi





Na_3Bi:
Xiong et al, Science 350, 6259(2015)

Failed

$T(K)$	4.5	20	30	40	50	60	70
$\rho_0(\mathrm{m\Omega\,cm})$	24.37	23.67	22.19	19.67	16.82	14.02	11.51
α	64.4	38.6	43.4	41.3	38.1	38.7	28.8
$\ell_e(\mathrm{nm})$	14.6	1.53	1.7	1.8	2.5	4.7	6.0
η	0.18	0.85	0.82	0.81	0.73	0.50	0.37
tt	0.25	0.05	0.03	0.03	0.05	0.10	0.11

Theory:

$$\alpha = 1$$

The large discrepancy between the theoretical prediction and the fitting parameters indicates other mechanisms may exist in this experiment.

"Quantum" Diffusive Theory: Anomaly-Induced Magnetoresistance

- B. Fu, H. W. Wang & S. Q. Shen, *Quantum Diffusive Magneto-transport in massive Dirac materials with chiral symmetry breaking*, arXiv.: 1909.09297

Chiral or ABJ Anomaly

S. L. Adler, PR 177, 2426(1969)
J. S. Bell & R. Jackiw, Nuovo Cinmento A 60, 47(1969)

The Weyl fermions satisfy a relation in the presence of electromagnetic fields E and B,

$$\frac{d\rho_\chi}{dt} + \nabla \cdot \mathbf{j}_\chi = -\chi \frac{e^3}{4\pi^2\hbar^2}\mathbf{E}\cdot\mathbf{B}$$

$$\partial_\mu j^\mu_\chi = -\chi \frac{e^3}{4\pi^2\hbar^2}\mathbf{E}\cdot\mathbf{B} \qquad \chi = \pm$$

These equations demonstrate that the charges are not conserved for Weyl fermions with a single chirality, which is also called Adler-Bell-Jackiw (ABJ) anomaly. Thus in reality, Weyl nodes always come in pairs of opposite charities such that the total currents are conserved.

$$\partial_\mu \left(j^\mu_+ + j^\mu_-\right) = 0 \qquad \partial_\mu \left(j^\mu_+ - j^\mu_-\right) \neq 0$$

Nielsen-Ninomiya Theory

PHYSICS LETTERS | Volume 130B, number 6 | 389 | 3 November 1983

THE ADLER–BELL–JACKIW ANOMALY AND WEYL FERMIONS IN A CRYSTAL

H.B. NIELSEN
Niels Bohr Institute and Nordita, 17 Blegdamsvej, DK2100, Copenhagen Ø, Denmark

Masao NINOMIYA [1]
Department of Physics, Brown University, Providence, RI 02912, USA

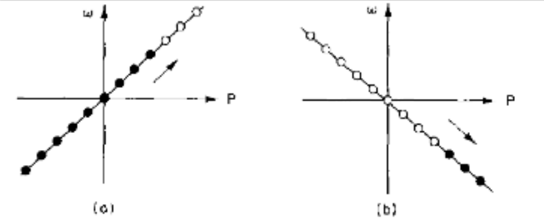


Fig. 1. Dispersion laws for the RH (a) and LH (b) Weyl fermions in 1 + 1 dimensions. The black and white points denote the filled and unfilled levels and the arrows indicate the direction of the movement of the Fermi surface when E in on.

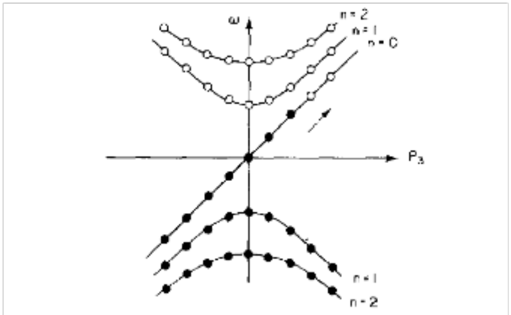


Fig. 2. Dispersion law for the RH Weyl fermion in 3 + 1 dimensions in the presence of a magnetic field in the x^3-direction.

2. Let us start with a (1 + 1)-dimensional right-handed (RH) Weyl fermion theory coupled to a uniform electric field $\dot{A}^1 = E$ in the temporal gauge. The one component RH Weyl equation for $\psi_R = \frac{1}{2}(1 + \gamma_5)\psi$ reads

$$i\dot{\psi}_R(x) = (-i\partial_x - A^1)\psi_R(x) . \qquad (1)$$

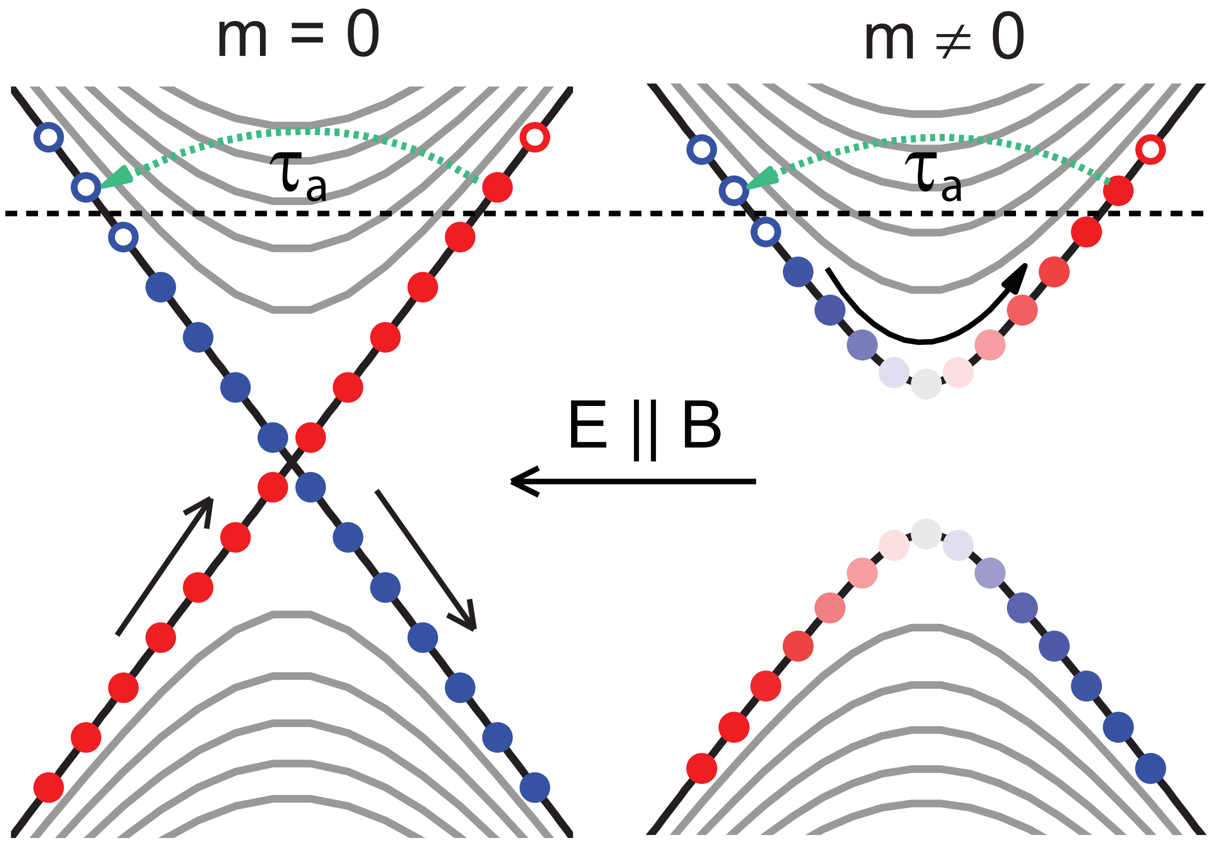
The dispersion law is $\omega(P) = P$. Corresponding to the classical equation of a charged particle in the presence of an electric field where $\dot{P} = eE$, the acceleration of the RH particles in quantum theory is given by $\dot{\omega} = \dot{P} = eE$. The creation rate of the RH particles per unit time and unit length is determined by a change of the Fermi surface, which distinguishes the filled and unfilled states as illustrated in fig. 1a. Let the quantization length be L; the density of states per length L is $L/2\pi$ and the rate of change of the RH particle number N_R is

$$\dot{N}_R = L^{-1}(L/2\pi)\,\dot{\omega}_{fs} = (e/2\pi)E . \qquad (2)$$

$$\frac{dN_R}{dt} = \frac{dk/dt}{2\pi} = \frac{e}{2\pi\hbar}E \qquad N_B = \frac{eB}{h}$$

To end the present paper we suggest possible candidates for the zero-gap semiconductors for our purpose of simulating the relativistic Weyl equation. The type of compounds with a zinc-blend structure [9, 16] which do not possess parity symmetry such as $Hg_{1-x}Cd_xTe$ or $Pb_{1-x}Sn_xTe$ with an appropriate range of composition x may have the generic type of Weyl degeneracy points.

Spectra of Massless and Massive Dirac fermions



$$\mathscr{H}_0 = \int d^3\mathbf{x}\bar{\Psi}(\mathbf{x})\left(v\hat{\mathbf{p}}\cdot\boldsymbol{\gamma} + \varepsilon_F\gamma^0 + mv^2\right)\Psi(\mathbf{x})$$

bilinear (γ^A)	physical quantity	$\mathcal{T}$	$\mathcal{I}$	$\mathcal{C}$	disorder
$\bar{\Psi}\gamma^0\Psi$	total charge (J^0)	✓	✓	✓	Δ
$\bar{\Psi}\gamma^0\gamma^5\Psi$	axial charge (J^{a0})	✓	×	✓	Δ_a
$\bar{\Psi}\Psi$	scalar mass ($\mathfrak{n}_\beta$)	✓	✓	×	Δ_m
$\bar{\Psi}i\gamma^5\Psi$	pseudo-scalar density ($\mathfrak{n}_P$)	×	×	×	Δ_P
$\bar{\Psi}\gamma^i\Psi$	current (J^i)	×	×	✓	Δ_c
$\bar{\Psi}\gamma^5\gamma^i\Psi$	axial current (J^{ai})	×	✓	✓	Δ_{ac}
$\bar{\Psi}i\gamma^0\gamma^i\Psi$	electric polarization ($\mathfrak{p}_i$)	✓	×	×	Δ_p
$\bar{\Psi}\gamma^5\gamma^0\gamma^i\Psi$	magnetization ($\mathfrak{m}_i$)	×	✓	×	Δ_M

The continuity equation for axial charge:

$$\partial_\mu \hat{J}^{a\mu}(x) = 2mv^2\hat{\mathfrak{n}}_P + \frac{e^3}{2\pi^2\hbar^2}\mathbf{E}\cdot\mathbf{B}$$

Green's functions in a finite field

$$G(\mathbf{x},\mathbf{x}';i\omega_m) = \exp\left[i\Phi(\mathbf{x}_\perp,\mathbf{x}'_\perp)\right]\widetilde{G}(\mathbf{x}-\mathbf{x}';i\omega_m)$$

The Schwinger phase:

$$\Phi(\mathbf{x}_\perp,\mathbf{x}'_\perp) = e\int_{\mathbf{x}'_\perp}^{\mathbf{x}_\perp} d\mathbf{x}''_\perp \cdot \mathbf{A}(\mathbf{x}''_\perp) = -\frac{(x_1-x'_1)(x_2+x'_2)}{2\ell_B^2}$$

$$\widetilde{G}^{R/A}(\boldsymbol{k},\omega) = ie^{-k_\perp^2\ell_B^2}\sum_{n=0}^{\infty}\frac{(-1)^n D_n[\omega+\varepsilon\pm i\hbar/2\tau,\boldsymbol{k}]}{[\omega+\varepsilon\pm i\hbar/2\tau]-\varepsilon_n^2(k^3)}$$

$$\varepsilon_n(k^3) = \sqrt{m^2v^4+2nv^2\hbar|eB|+v^2(k^3)^2}$$

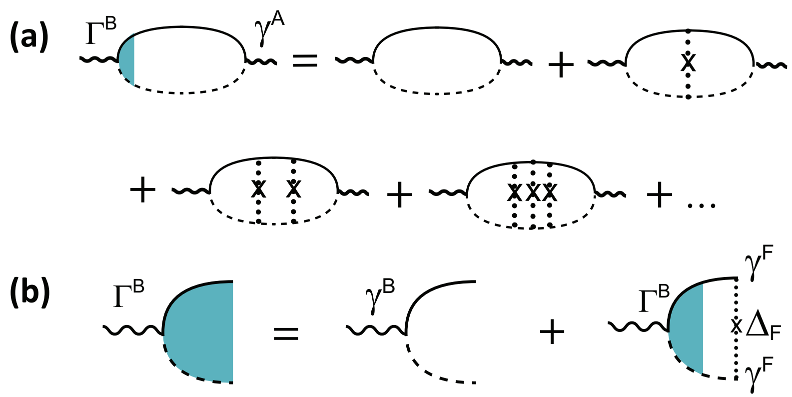
Expanding the Green's function in B field

$$\widetilde{G}^{R/A}(\boldsymbol{k},\omega) = \widetilde{G}_0^{R/A}(\boldsymbol{k},\omega)+\widetilde{G}_1^{R/A}(\boldsymbol{k},\omega)+\ldots$$

$$\widetilde{G}_0^{R/A}(\boldsymbol{k},\omega) = i\frac{(\omega+\varepsilon\pm\frac{i\hbar}{2\tau})\gamma^0 - v\hbar\mathbf{k}\cdot\gamma+mv^2}{(\omega+\varepsilon\pm\frac{i\hbar}{2\tau})^2-(v^2\hbar^2|\mathbf{k}|^2+m^2v^4)}$$

$$\widetilde{G}_1^{R/A}(\boldsymbol{k},\omega) = -\gamma^1\gamma^2 ev^2\hbar B\frac{(\omega+\varepsilon\pm\frac{i\hbar}{2\tau})\gamma^0 - v\hbar k^3\gamma^3+mv^2}{\left[(\omega+\varepsilon\pm\frac{i\hbar}{2\tau})^2-(v^2\hbar^2|\mathbf{k}|^2+m^2v^4)\right]^2}$$

Quantum Diffusive Equations



The linear response theory: $\mathcal{S}(\mathbf{x},\Omega) = \int d\mathbf{x}' \chi(\mathbf{x},\mathbf{x}';\Omega)\mathcal{A}(\mathbf{x}',\Omega)$

$$\mathscr{D}_{\mathbf{x}}^{-1}\mathcal{S}^{(1)}(\mathbf{x},\omega) = -\frac{2}{\pi}\left[\mathcal{W}^{-1} - \mathscr{D}_{\mathbf{x}}^{-1}\right]\mathcal{W}^{-1} i\omega\mathcal{A}(\mathbf{x},\omega)$$

$$\mathscr{D}_{\mathbf{x}}^{-1} = \mathcal{W}^{-1} - \Omega\int d\mathbf{x}_1 \partial_\Omega \mathcal{M}^T(\mathbf{x},\mathbf{x}_1;\Omega)|_{\Omega=0} - \int d\mathbf{x}_1 \mathcal{M}^T(\mathbf{x},\mathbf{x}_1;0)\left[1 + (\mathbf{x}_1-\mathbf{x})\cdot\nabla_{\mathbf{x}} + \frac{1}{6}(\mathbf{x}_1-\mathbf{x})^2\nabla_{\mathbf{x}}^2\right]$$

$$\mathcal{M}^{\mathrm{DC}}(\mathbf{x},\mathbf{x}',\Omega) = \frac{1}{4}\mathrm{Tr}\left[\gamma^{\mathrm{C}} G^R(\mathbf{x},\mathbf{x}';\epsilon+\Omega)\gamma^{\mathrm{D}} G^A(\mathbf{x}',\mathbf{x};\epsilon)\right]$$

The impurity related diagonal matrix: $\mathcal{W}_{\mathrm{CC}} = \sum_{\mathrm{F}} \kappa_{\mathrm{FC}}\Delta_{\mathrm{F}}$

Quantum Diffusive Equation

$$\mathscr{D}_{\mathbf{x}}^{-1}\mathcal{S}^{(1)}(\mathbf{x},\omega) = -\frac{2}{\pi}\left[\mathcal{W}^{-1} - \mathscr{D}_{\mathbf{x}}^{-1}\right]\mathcal{W}^{-1} i\omega\mathcal{A}(\mathbf{x},\omega)$$

In the case of the parallel electric and magnetic fields

$$\frac{\hbar}{\pi\rho\tau^2}\mathscr{D}_{4\times4}^{-1} = \begin{pmatrix} i\omega + \frac{\Lambda_0}{\tau} - \mathcal{D}\partial_z^2 & \Upsilon v\partial_z & \eta(i\omega - \frac{1}{\tau} - \mathcal{D}\partial_z^2) & \frac{1}{3}(1-\eta^2)v\partial_z \\ \Upsilon v\partial_z & (1-\eta^2)(i\omega + \frac{\Lambda_{a0}}{\tau} - \mathcal{D}\partial_z^2) & \frac{\eta}{2}\Upsilon v\partial_z & -\frac{1}{\tau}\Upsilon \\ \eta(i\omega - \frac{1}{\tau} - \mathcal{D}\partial_z^2) & \frac{\eta}{2}\Upsilon v\partial_z & \eta^2(i\omega + \frac{\Lambda_\beta}{\tau} - \mathcal{D}\partial_z^2) & \frac{1}{3}\eta(1-\eta^2)v\partial_z \\ \frac{1}{3}(1-\eta^2)v\partial_z & -\frac{1}{\tau}\Upsilon & \frac{1}{3}\eta(1-\eta^2)v\partial_z & \frac{1}{3}(1-\eta^2)(i\omega + \frac{\Lambda_3}{\tau} - \mathcal{D}\partial_z^2) \end{pmatrix}$$

$$\mathcal{W} = \mathbf{\Delta 1}_4 + \mathbf{\Delta}_m \mathrm{diag}(1,-1,1,-1)$$

$$\delta J^{a0} = \frac{\varepsilon_F}{v\hbar k_F}\frac{i\omega}{i\omega + \mathcal{D}^* q_z^2}\frac{e^3}{\hbar^2}\frac{EB\tau_a}{2\pi^2};$$

$$\delta J^3 = \frac{i\omega\sigma_D E}{i\omega + \mathcal{D}^* q_z^2}\left(1 + \frac{3}{4}\frac{\tau_a}{\tau^*}\frac{1}{k_F^4\ell_B^4}\frac{i\omega}{i\omega + q_z^2\mathcal{D}^*}\right);$$

$$\delta J^0 = \frac{iq_z\sigma_D E}{i\omega + \mathcal{D}^* q_z^2}\left(1 + \frac{3}{4}\frac{\tau_a}{\tau^*}\frac{1}{k_F^4\ell_B^4}\frac{i\omega}{q_z^2\mathcal{D}^* + i\omega}\right);$$

$$\delta\mathfrak{n}_\beta = \eta\delta J^0,$$

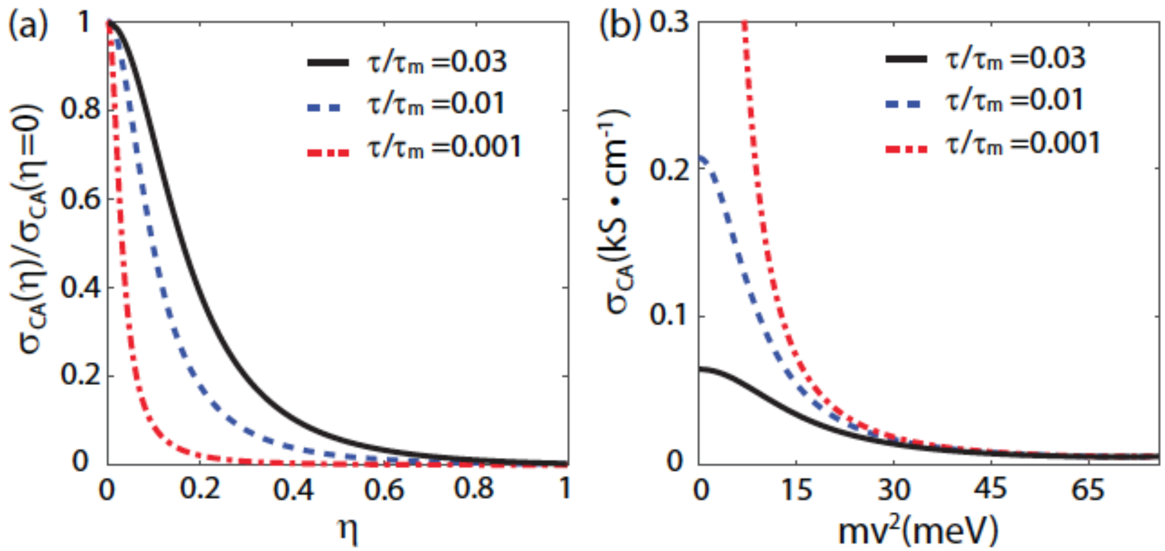
The dc conductivity:

$$\lim_{\omega\to0}\lim_{q\to0}\sigma_{zz}(\omega,\mathbf{q},B) = \sigma_D + \sigma_{CA}(\eta,B)$$

$$\sigma_{CA}(\eta,B) = \frac{3}{4}\frac{\tau_a}{\tau^*}\frac{1}{k_F^4\ell_B^4}\sigma_D = \frac{3}{16}\frac{\tau_a}{\tau^*}\left(\frac{B}{B_F}\right)^2\sigma_D$$

Anomaly-induced Magnetoconductivity

$$\sigma_{CA}(\eta, B) = \frac{3}{4}\frac{\tau_a}{\tau^*}\frac{1}{k_F^4 \ell_B^4}\sigma_D = \frac{3}{16}\frac{\tau_a}{\tau^*}\left(\frac{B}{B_F}\right)^2 \sigma_D$$



$$\frac{\tau_a}{\tau^*} = \frac{1}{\Lambda_{a0}}\frac{\mathcal{D}^*}{\mathcal{D}}$$

$$\mathcal{D}^* = \frac{3}{2}\frac{(1+\eta^2)(\Delta+\Delta_m)}{(\Delta+2\Delta_m)+\eta^2(2\Delta+\Delta_m)}\mathcal{D}$$

The axial relaxation rate

$$\Lambda_{a0} = 2\frac{\Delta_m+\Delta\eta^2}{(\Delta-\Delta_m)(1-\eta^2)}$$

The Orbital Polarization:

$$\eta \equiv \langle \gamma^0 \rangle = mv^2/\varepsilon_F$$

Modified Anomaly Equation

$$\partial_\mu \hat{J}^{a\mu}(x) = 2mv^2 \hat{\mathfrak{n}}_P + \frac{e^3}{2\pi^2\hbar^2}\mathbf{E}\cdot\mathbf{B}$$

The expectation value of the pseudo-scalar density

$$\langle \hat{\mathfrak{n}}_P \rangle = \frac{1}{2mv^2}\left(\frac{\varepsilon_F}{\hbar v k_F} - 1\right)\frac{e^3}{2\pi^2\hbar^2}\mathbf{E}\cdot\mathbf{B}$$

$$\partial_\mu J^{a\mu} = \frac{\varepsilon_F}{\hbar v k_F}\frac{e^3}{2\pi^2\hbar^2}\mathbf{E}\cdot\mathbf{B}$$

$$\frac{\varepsilon_F}{\hbar v k_F} = \sqrt{1 + \left(\frac{mv}{\hbar k_F}\right)^2}$$

Chiral separation effect:

$$\mathbf{J}^a = -e\mathbf{B} v\hbar k_F \frac{e^2}{2\pi^2\hbar^2}$$

Chiral magnetic effect:

$$\mathbf{J} = \frac{\varepsilon_F}{\hbar v k_E}\frac{e^2}{2\pi^2\hbar^2}\mu_5\mathbf{B}$$

Summary

1. Anomaly-induced negative magnetoresistance
2. Intrinsic magentoresistivity for massless and massive Dirac fermions: quadratic positive transverse and negative longitudinal MR;
3. Quantum osicillation: the phase shift is a function of the dingle factor;
4. Quantum Interference Effect: weak localization and antilocalization

$$\sigma_{qi}(B) = \frac{e^2}{4\pi h \ell_B} \sum_{i=0,s,t_\pm} \mathcal{F}_i \zeta[\frac{1}{2}, \frac{1}{2} + (z_i + x^2)\frac{\ell_B^2}{\ell_e^2}]\Big|_1^{\ell_e/\ell_\phi}$$

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Thank you for your attention!