Martingales in Finance

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Workshop on Martingales in Finance and Physics

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Why Martingales in Finance?

- Efficient Markets Hypothesis (EMH): prices in financial markets should incorporate all available information
- Crucial for EMH: the prices at which financial securities trade must not allow for arbitrage opportunities
  - it must not be possible to trade in such a way that you never "lose" and you "win" with positive probability
- Fundamental Theorem of Finance (FTF): no arbitrage holds if and only if "suitably normalized" securities prices are martingales under a "suitable" probability
- The "suitable" probability in the FTF takes the name of Risk-Neutral Probability/Equivalent Martingale measure
  - it is different from the physical probability, i.e. the probability that governs the actual law of motion of prices

To be on the same page.....

- $\mathcal{T} \subset \Re$ a set of time-indexes
- $\left(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t \in \mathcal{T}}\right)$ a filtered probability space
- $\{X(t)\}_{t \in \mathcal{T}}$ a Stochastic Process i.e.
  - $X(t)\ \mathcal{F}_t - \textit{measurable}$ (plus some integrability condition....)
- $\mathbf{E}\left[\bullet / \mathcal{F}_t\right]$ the conditional expectation operator

Definition

$\{X(t)\}_{t \in \mathcal{T}}$ is a martingale if

$$X(t) = \mathbf{E}\left[X(s) / \mathcal{F}_t\right], \qquad \forall s, t \in \mathcal{T},\ s \geq t$$

Plan of the Talk

- A very simple one-period model to grasp the basic intuition
- Expanding on the simple model: the discrete-time case
- The continuous-time model of Black and Scholes
- The general continuous-time cases: a primer

A simple one-period model

- Dates: $t = 0, 1$ (today, tomorrow)
- States: $\Omega = \{\omega_{1,\cdots,}\omega_K\}$, Probabilities: $\mathbf{P}(\omega_k) > 0$
- N risky investments (e.g. shares of a risky business) plus 1 riskless investment (e.g. money in the bank)
  - $S_j(0)$ share price today of risky investment j
  - $S_j(1)(\omega_k)$ share value tomorrow of risky investment j in state k
  - $r =$ interest rate: 1\$ in the bank at time 0 becomes $(1 + r)$ \$ at time 1

Investment strategies and trading

- $\vartheta_1, \ldots, \vartheta_N$ units held of N risky investments
- ϑ_0 money in the bank today
- Total money invested today

$$V_\vartheta(0) = \vartheta_0 + \sum_{j=1}^{N} \vartheta_j S_j(0)$$

- Total value generated tomorrow in state k

$$V_\vartheta(1)(\omega_k) = \vartheta_0(1+r) + \sum_{j=1}^{N} \vartheta_j S_j(1)(\omega_k)$$

Arbitrage

Definition (Arbitrage Opportunity)

An investment strategy ϑ such that $V_{\vartheta}(0) \leq 0$, $V_{\vartheta}(1)(\omega_k) \geq 0$, for all k and

$$V_{\vartheta}(1)(\omega_{\bar{k}}) > 0, \quad \text{for some } \bar{k}$$

- In words: an investment strategy whose cost today is non positive, whose revenue tomorrow is non-negative, and the revenue tomorrow is positive in at least one state (i.e. with positive probability)
- When arbitrages exist markets unravel

The Fundamental Theorem of Finance (FTF)

Theorem

The following are equivalent:

1. *no-arbitrage holds;*
2. *there exists* $\mathbf{Q}(\omega_k) > 0$ *for all* k *such that for all* j

$$\begin{aligned} S_j(0) &= \frac{1}{1+r}\mathbf{E}^Q\left[S_j(1)\right] \\ &\triangleq \frac{1}{1+r}\sum_{k=1}^{K}\mathbf{Q}(\omega_k)S_j(1)(\omega_k) \end{aligned}$$

- In words: arbitrage opportunities disappear if and only if there is some probability $\mathbf{Q}$ that makes the price today of each security equal to the discounted expected value tomorrow

- Where are the martingales?

Martingales and Finance, act 1

- Define the Discounted Price as follows: $\widetilde{S}_j(0) \triangleq S_j(0)$ while

$$\widetilde{S}_j(1)(\omega_k) \triangleq \frac{1}{1+r} S_j(1)(\omega_k), \qquad k = 1, ..., K$$

- Statement 2 in the FTF becomes then

$$\widetilde{S}_j(0) = \mathbf{E}^Q \left[\widetilde{S}_j(1) \right]$$

  a (Mickey Mouse......) martingale!

- The jargon for **Q**:
  - Risk-Neutral probability in Finance: only averages matter, variance/risk is irrelevant
  - Equivalent Martingale Measure in Math: **Q** and the physical probability **P** are equivalent measures (but $\mathbf{Q} \neq \mathbf{P}$ in general!!)

The multi-period framework

- Dates: $t = 0, 1, \ldots.., T$
- A filtered probability space $\left(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t=0}^{T}\right)$
- $S_j(t)$ the price at time t of risky investment j
  - $S_j(t)$ an $\mathcal{F}_t - measurable$, $square - integrable$ random variable
  - 1 in the bank at time 0 becomes $(1+r)^t$ at time t
- Discounted prices

$$\widetilde{S}_j(t) \triangleq \frac{1}{1+r} S_j(t), \qquad t = 0, 1, \ldots, T$$

Equivalent Martingale Measures (EMMs)

Definition

An Equivalent Martingale Measure (EMM) is a probability measure $\mathcal{Q} \backsim \mathcal{P}$ such that

$i)$ $\quad L = \frac{dQ}{dP} > 0, \ \frac{L}{1+r} \in \mathcal{L}^2$

$ii)$ $\left\{ \widetilde{S}_j(t) \right\}_{t=0}^{T}$ is a $\mathcal{Q}-$martingale $\forall j$ that is

$$\widetilde{S}_j(t) = \mathbf{E}^Q \left[\widetilde{S}_j(s) \Big/ \mathcal{F}_t \right], \qquad \forall s \geq t$$

- EMMs extend the notion seen in the very simple one-period case: for $t = 0$, $s = 1$

$$\widetilde{S}_j(0) = \mathbf{E}^Q \left[\widetilde{S}_j(1) \Big/ \mathcal{F}_0 \right] = \mathbf{E}^Q \left[\widetilde{S}_j(1) \right]$$

The multi-period FTF

Theorem

The following are equivalent in a multiperiod market:

1. *(a suitably extended notion of) no-arbitrage holds*
2. *there exist* **EMMs**

- How many EMMs?
  - One and only one if and only if markets are **complete!**
- What's their use (besides characterizing No-Arbitrage)?
  - To price new securities (stocks, bonds, options, other derivative securities....) constantly added to the market by the finance industry. More on this later

The Continuous-time Black-Scholes (BS) Model: the primitives

- Dates: $t \in [0, T]$
- A Standard Brownian Motion $\{W_t\}_{t \in [0,T]}$
- A filtered probability space $\left(\Omega, \mathcal{F}, P, \{\mathcal{F}_t^W\}_{t \in [0,T]}\right)$
  - $\left\{\mathcal{F}_t^W\right\}_{t \in [0,T]}$ the filtration generated by $\{W_t\}_{t \in [0,T]}$
- Only two investment opportunities: a share of common stock and a bank account

The stock and the bank account

- The stock price $S(t)$ follows a Geometric Brownian Motion under the physical probability P

$$S(t) = S(0)e^{\left(\mu-\frac{1}{2}\sigma^2\right)t+\sigma W(t)}$$

Ito's Lemma yields

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$$

- Letting $\delta = \ln(1+r)$, 1 Euro in the bank at time 0 becomes $B(t) = (1+r)^t \equiv e^{\delta t}$, i.e.

$$dB(t) = \delta B(t)dt$$

- Discounted stock price: $\widetilde{S}(t) = e^{-\delta t}S(t)$, so that

$$d\widetilde{S}(t) = (\mu - \delta)\widetilde{S}(t)dt + \sigma\widetilde{S}(t)dW(t)$$

Economic interpretation and properties

- The stock has a lognormal distribution:
  - therefore stock price never falls below zero, satisfying the economic condition of limited liability
- Basic economic assumption: $\mu > \delta$
  - the average instantaneous return on the stock μ is greater than the instantaneous return δ from keeping money in the bank
  - $\mu - \delta >$ is called the risk premium: compensation to stockholders for the risk from holding stocks
- Both $S(t)$ and $\widetilde{S}(t)$ display a drift:
  - neither one is a martingale!
- Where are the martingales in the BS model?

The EMM in the BS model: existence

Theorem (Girsanov)

Under suitable integrability conditions on $\nu(t)$ there exists a probability $Q \sim P$ s.t.

$$dW^Q(t) = \nu(t)dt + dW(t)$$

is a Standard Brownian Motion

- Therefore, in the BS model there exists $Q \sim P$ s.t.

$$\begin{aligned} d\widetilde{S}(t) &= \sigma\widetilde{S}(t)\left[\underbrace{\frac{(\mu-\delta)}{\sigma}}_{\nu(t)}dt + dW(t)\right] \\ &= \sigma\widetilde{S}(t)dW^Q(t) \end{aligned}$$

i.e. there exists $Q \sim P$ such that $\widetilde{S}(t)$ under Q is a driftless diffusion: a Martingale!

The EMM in the BS model: properties

- By Ito's Lemma

$$\widetilde{S}(t) = S(0)e^{-\frac{1}{2}\sigma^2 t + \sigma W^Q(t)}$$

- Therefore, since

$$E^Q\left[\widetilde{S}(t)\right] = S(0)$$

and $S(t) = e^{\delta t}\widetilde{S}(t)$, then

$$E^Q\left[S(t)\right] = e^{\delta t}S(0)$$

- Under Q the average instantaneous return on the stock is δ, the same as the bank account:
  - the notion of Risk-Neutral Probability!

Trading in the BS model

- $\vartheta_0(t)$, $\vartheta_1(t)$
  - money in the bank, stock shares held at time t
- $V_\vartheta(t)$ value invested at time t :

$$V_\vartheta(t) = \vartheta_0(t)B(t) + \vartheta_1(t)S(t)$$

Definition (Self-financing trading)

A trading strategy is self-financing if

$$dV_\vartheta(t) = \vartheta_0(t)dB(t) + \vartheta_1(t)dS(t)$$

equivalently if the discounted value $\widetilde{V}_\vartheta(t) = e^{-\delta t}V_\vartheta(t)$ satisfies

$$d\widetilde{V}_\vartheta(t) = \vartheta_1(t)d\widetilde{S}(t)$$

Self-financing trading and arbitrage

- A self-financing trading strategy $\vartheta_0(t)$, $\vartheta_1(t)$ is an arbitrage opportunity if
  1. $V_\vartheta(0) \leq 0$
  2. $V_\vartheta(T) \geq 0 \quad$ *P-almost surely*
  3. $P[V_\vartheta(T) > 0] > 0$
- The same economic intuition as in the simple one-period case (technicalities aside)

No-Arbitrage and Martingales in the BS model

- The BS *EMM* implies no-arbitrage (modulo integrability conditions....)

$$\left\{ \begin{array}{c} \widetilde{S}(t) \\ Q - martingale \end{array} \right\} \quad \vee \quad d\widetilde{V}_{\vartheta}(t) = \vartheta_1(t) d\widetilde{S}(t)$$

$$\Downarrow$$

$$\widetilde{V}_{\vartheta}(t) \;\; Q - martingale$$

$$\Downarrow$$

$$E^{Q}\left[\widetilde{V}_{\vartheta}(T)\right] = \widetilde{V}_{\vartheta}(0) = V_{\vartheta}(0)$$

- Since $Q \sim P$

$$V_{\vartheta}(T) \geq 0 \quad \vee \quad P\left[V_{\vartheta}(T) > 0\right] > 0 \qquad \Longleftrightarrow$$

$$\widetilde{V}_{\vartheta}(T) \geq 0 \quad \vee \quad Q\left[\widetilde{V}_{\vartheta}(T) > 0\right] > 0 \quad \Longrightarrow \quad V_{\vartheta}(0) > 0 \;\blacksquare$$

Pricing and Hedging in the BS model: the problem

- European call option: at $t < T$ a subject (the owner) buys from another subject (the seller) the right to buy from the seller the stock at the future time T at a fixed price K
- Therefore at maturity T the owner receives the random payoff

$$\max\left(S(T) - K, 0\right)$$

- Problem: determine the option price $c(t, S(t))$ that prevents from arbitrage opportunities to emerge in the market
- Solution: take the perspective of a trader that sells the option and wants to hedge the risk

The setup

- A trader sells one option at the price $c(t, S(t))$, and wants to hedge the risk by holding $h(t)$ shares of the stock
- The value of the trader's position is therefore

$$V(t) = h(t)S(t) - c(t, S(t))$$

- The hedging strategy must be self-financing, i.e.

$$dV(t) = h(t)dS(t) - dc(t, S(t))$$

- At maturity assets and liabilities must balance

Computing the law of motion of the value

- Recall that

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$$

- By Ito's Lemma

$$dc(t, S(t)) = \left[\frac{\partial c}{\partial t} + \frac{\partial c}{\partial S}\mu S + \frac{1}{2}\frac{\partial^2 c}{\partial S^2}\sigma^2 S^2\right] dt + \frac{\partial c}{\partial S}\sigma S dW(t)$$

- Therefore

$$\begin{aligned} dV = & \left(-\tfrac{\partial c}{\partial t} + \left(h - \tfrac{\partial c}{\partial S}\right)\mu S + \tfrac{1}{2}\left(-\tfrac{\partial^2 c}{\partial S^2}\right)\sigma^2 S^2\right) dt \\ & + \left(h - \tfrac{\partial c}{\partial S}\right)\sigma S dW(t) \end{aligned}$$

Computing the optimal hedging strategy

- Objective of the trader: eliminate risk, that is eliminate the diffusion term in the value dynamics

$$h(t) - \frac{\partial c(t,S(t))}{\partial S} = 0 \quad \Longrightarrow \quad h(t) = \frac{\partial c(t,S(t))}{\partial S}$$

- But then the law of motion of value reduces to

$$dV = \left(-\frac{\partial c}{\partial t} - \frac{1}{2}\frac{\partial^2 c}{\partial S^2}\sigma^2 S^2 \right) dt$$

- Recall now that the value of cash in the bank evolves as

$$dB(t) = \delta B(t) dt$$

- Both instantaneously risk-free (no diffusion term!): what does no-arbitrage imply?

No-Arbitrage and the BS PDE

- No-Arbitrage implies that the optimal trading strategy and cash in the bank must earn the same return δ per unit of time

$$\frac{1}{dt}\frac{dV(t)}{V(t)} = \delta = \frac{1}{dt}\frac{dB(t)}{B(t)}$$

- Recalling the expressions for $V(t)$ and $dV(t)$ under optimal hedging, the first equality rewrites as

$$\begin{cases} \delta c(t,S) = \frac{\partial}{\partial t}c(t,S) + \frac{\partial}{\partial S}c(t,S)\cdot\delta S + \frac{1}{2}\frac{\partial^2}{\partial S^2}c(t,S)\cdot\sigma^2 S^2 \\ c(T,S) = \max\left(S-K,0\right) \end{cases}$$

which is the celebrated PDE for the option price of F. Black and M. Scholes (1973)

The Black-Scholes formula

- The solution of the BS PDE is the celebrated Black-Scholes formula:

$$c\left(t, S(t)\right) = S\left(t\right) N\left(d_1\right) - Ke^{-\delta(T-t)} N\left(d_2\right)$$

where

$$N\left(y\right) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz,$$

while

$$d_1 = \frac{1}{\sigma\sqrt{(T-t)}}\left(\ln\left(\frac{S\left(t\right)}{K}\right) + \left(\delta + \frac{1}{2}\sigma^2\right)\left(T-t\right)\right)$$

and

$$d_2 = d_1 - \sigma\sqrt{(T-t)}$$

Extension to the general diffusion case

- The law of motion of the stock is now a general diffusion process

$$dS(t) = \mu(t, S(t)) \cdot S(t)\ dt + b(t, S(t)) \cdot S(t)\ dW(t)$$

- Problem: hedge and price an asset that pays $F(S(T))$ Euro at time T, with F regular enough

- Replicating the same arguments above, the price $f(t, S(t))$ of the asset must satisfy the following PDE $\forall t \in (0, T)$, $S > 0$

$$\begin{cases} \delta f(t, S) = \frac{\partial}{\partial t} f(t, S) + \frac{\partial}{\partial S} f(t, S) \cdot \delta S + \frac{1}{2} \frac{\partial^2}{\partial S^2} f(t, S) \cdot b^2(t, S) \cdot S^2 \\ f(T, S) = F(S) \end{cases}$$

Coming up full circle.....

Theorem (Corollary from the Feyman-Kac Formula)

If f solves the PDE

$$\begin{cases} \delta f(t,S) = \frac{\partial}{\partial t} f(t,S) + \frac{\partial}{\partial S} f(t,S) \cdot \delta S + \frac{1}{2} \frac{\partial^2}{\partial S^2} f(t,S) \cdot b^2(t,S) \cdot S^2 \\ f(T,S) = F(S) \end{cases}$$

then under suitable regularity conditions

$$f(t, S(t)) = e^{-\delta(T-t)} E^{\mathbf{Q}} \left[F(S(T)) | \mathcal{F}_t \right]$$

where $S(t)$ satisfies

$$dS(t) = \delta \cdot S(t)\ ds + b(t, S(t)) \cdot S(t)\ d\widetilde{W}(t)$$

with $\widetilde{W}$ a Standard Brownian Motion under $\mathbf{Q}$

Conclusions

- The results seen so far extend in many various directions
  - several stocks driven by a vector-valued SBM
  - stochastic volatility
  - jump-diffusion dynamics
  - more generally, semimartingales
- Technicalities aside, the unifying theme is the powerful connection between the economic notion of No-Arbitrage and the mathematical tool of Martingales

Some essential references

1. Black, F. and M. Scholes, (1973), *The Pricing of Options and Corporate Liabilities,* Journal of Political Economy
2. Merton, R. (1973), *Theory of Rational Option Pricing,* Bell Journal of Economics and Management Science
3. Harrison, J.M. and D. Kreps, (1979), *Martingales and Arbitrage in Multiperiod Securities Markets,* Journal of Economic Theory
4. Harrison, J.M. and S.R. Pliska, (1981), *Martingales and Stochastic Integrals in the Theory of Continuous Trading,* Stochastic Processes and Their Applications
5. F. Delbaen and W. Schachermayer, (1994), *A general version of the fundamental theorem of asset pricing,* Mathematische Annalen

Ito's Lemma

Given a diffusion process

$$dX(t) = a(t, X(t))dt + b(t, X(t))dW(t)$$

and a function $\varphi : [0; T] \times \Re \rightarrow \Re$ continuously differentiable, once with respect to the first variable, twice with respect to the second, let

$$Y(t) = \varphi(t; X(t))$$

Then $Y(t)$ is itself a diffusion process with

$$Y(t) = \left[\frac{\partial\varphi(t;X(t))}{\partial t} + \frac{\partial\varphi(t;X(t))}{\partial x} \cdot a(t, X(t)) + \frac{1}{2}\frac{\partial^2\varphi(t;X(t))}{\partial x^2} \cdot b^2(t, X(t))\right] dt$$

$$+\frac{\partial\varphi(t;X(t))}{\partial x} \cdot b(t, X(t))dW(t)$$

back