Stochastic Thermodynamics with Martingales

Izaak Neri, Workshop on Martingales
in Finance and Physics, 24th of May 2019

KING'S College LONDON

Contributions

Statistical physics

Edgar Roldán (Trieste)

Frank Jülicher (Dresden)

Simone Pigolotti (Okinawa)

Shamik Gupta (Calcutta)

Raphaël Chétrite (Nice)

Information theory

Meik Dörpinghaus (Dresden)

Heinrich Meyr (Aachen)

Structure of the talk

1. Introduction to stochastic thermodynamics

2.Exponential **martingale** structure of **entropy production**

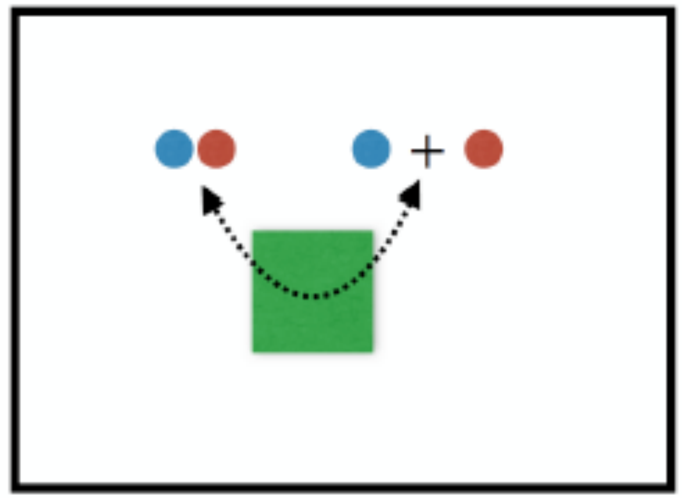
3.Thermodynamic **laws** at **stopping times**

4. **Universal** properties of entropy production

5. Example: overdamped **Langevin processes**

Introduction to stochastic thermodynamics

Thermodynamics of mesoscopic systems or stochastic thermodynamics



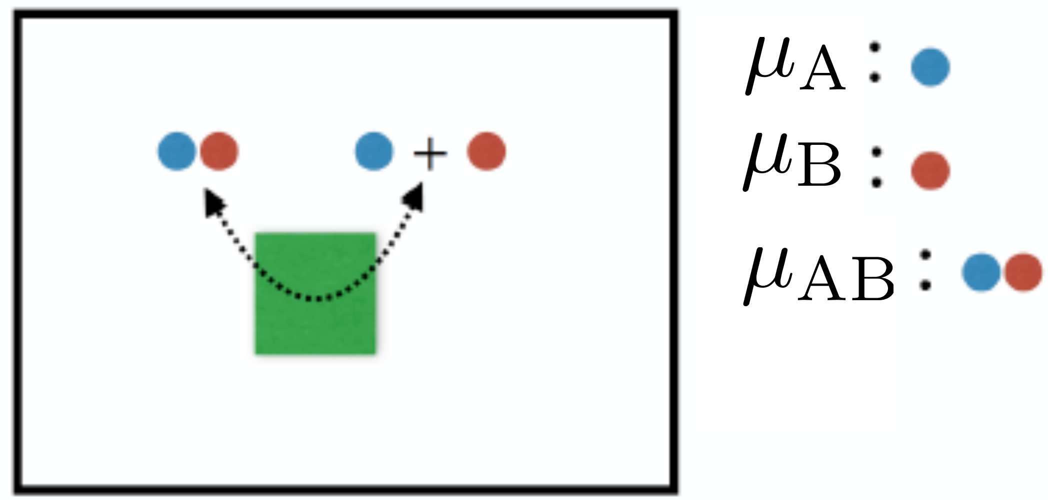
μ_{A} :

μ_{B} :

μ_{AB} :

Thermodynamics: $J(t)\,(\mu_{\mathrm{AB}} - \mu_A - \mu_B) \geq 0$

Thermodynamics of mesoscopic systems or stochastic thermodynamics



Thermodynamics: $J(t)\,(\mu_{\mathrm{AB}} - \mu_A - \mu_B) \geq 0$

Stochastic thermodynamics:

$$\frac{k_{{\color{blue}\bullet} + {\color{red}\bullet} \to {\color{blue}\bullet}{\color{red}\bullet}}}{k_{{\color{blue}\bullet}{\color{red}\bullet} \to {\color{blue}\bullet} + {\color{red}\bullet}}} = e^{S_{\mathrm{env}}({\color{blue}\bullet} + {\color{red}\bullet} \to {\color{blue}\bullet}{\color{red}\bullet})} = e^{\frac{\mu_{{\color{blue}\bullet}{\color{red}\bullet}} - \mu_{\color{blue}\bullet} - \mu_{\color{red}\bullet}}{T_{\mathrm{env}}}}$$

Local detailed balance and stochastic entropy production

$$\frac{p\left(X(2), X(2), \ldots, X(t) | X(1)\right)}{p\left(X(t-1), X(t-2), \ldots, X(1) | X(t)\right)} = e^{S_{\mathrm{env}}}$$

U Seifert, Rep. Prog. Phys. (2012)

Local detailed balance and stochastic entropy production

$$\frac{p\left(X(2), X(2), \ldots, X(t) | X(1)\right)}{p\left(X(t-1), X(t-2), \ldots, X(1) | X(t)\right)} = e^{S_{\mathrm{env}}}$$

$$S_{\mathrm{tot}}(t) = \Delta S_{\mathrm{sys}}(t) + S_{\mathrm{env}}(t) = \log \frac{p\left(X(1), \ldots X(t)\right)}{p\left(X(t), \ldots X(1)\right)}$$

where $S_{\mathrm{sys}}(t) \equiv -\log p_{\mathrm{ss}}(X(t))$

U Seifert, Rep. Prog. Phys. (2012)

Thermodynamic laws for mesoscopic processes

Integral fluctuation relation: $\langle e^{-S_{\mathrm{tot}}(t)} \rangle = 1$

$$\sum_{X(1),\ldots,X(t)} p(X(1),\ldots,X(t)) \frac{p(X(t),\ldots,X(1))}{p(X(1),\ldots,X(t))} = 1$$

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Implications

$\Rightarrow \quad \langle S_{\rm tot}(t) \rangle \geq 0$

$\Rightarrow$ Events of negative entropy production must exist

$\Rightarrow \quad P\left(S_{\rm tot}(t) \leq -s\right) \leq e^{-s}$

U Seifert, Rep. Prog. Phys. (2012)

Exponential martingale structure of entropy production

Martingales

M(t) is a martingale with respect to *X(t)* if:

- *M(t)* is a real-valued function on *X(0…t)*
- $\langle |M(t)| \rangle < \infty$
- $\langle M(t) | X(0 \ldots s) \rangle = M(s)$, for all $s<t$

For stationary processes the exponential of the negative entropy production is a martingale

$$\langle e^{-S_{\rm tot}(t)} | X(0 \ldots s) \rangle =$$

IN, Edgar Roldan, Frank Julicher, Phys. Rev. X (2017)

For stationary processes the exponential of the negative entropy production is a martingale

$$\langle e^{-S_{\rm tot}(t)} | X(0\ldots s)\rangle = \sum_{X(s^+\ldots t)} p\left(X(0\ldots t)|X(0\ldots s)\right)\, e^{-S_{\rm tot}(t)}$$

IN, Edgar Roldan, Frank Julicher, Phys. Rev. X (2017)

For stationary processes the exponential of the negative entropy production is a martingale

$$\langle e^{-S_{\rm tot}(t)} | X(0\ldots s)\rangle = \sum_{X(s^+\ldots t)} p\left(X(0\ldots t)|X(0\ldots s)\right)\; e^{-S_{\rm tot}(t)}$$

$$= \sum_{X(s^+\ldots t)} \frac{p\left(X(0\ldots t)\right)}{p\left(X(0\ldots s)\right)} \; \frac{\tilde{p}\left(X(0\ldots t)\right)}{p\left(X(0\ldots t)\right)}$$

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$$= \sum_{X(s^+\dots t)} \frac{p\left(X(\cancel{0}\dots t)\right)}{p\left(X(0\dots s)\right)} \frac{\tilde{p}\left(X(0\dots t)\right)}{p\left(X(\cancel{0}\dots t)\right)}$$

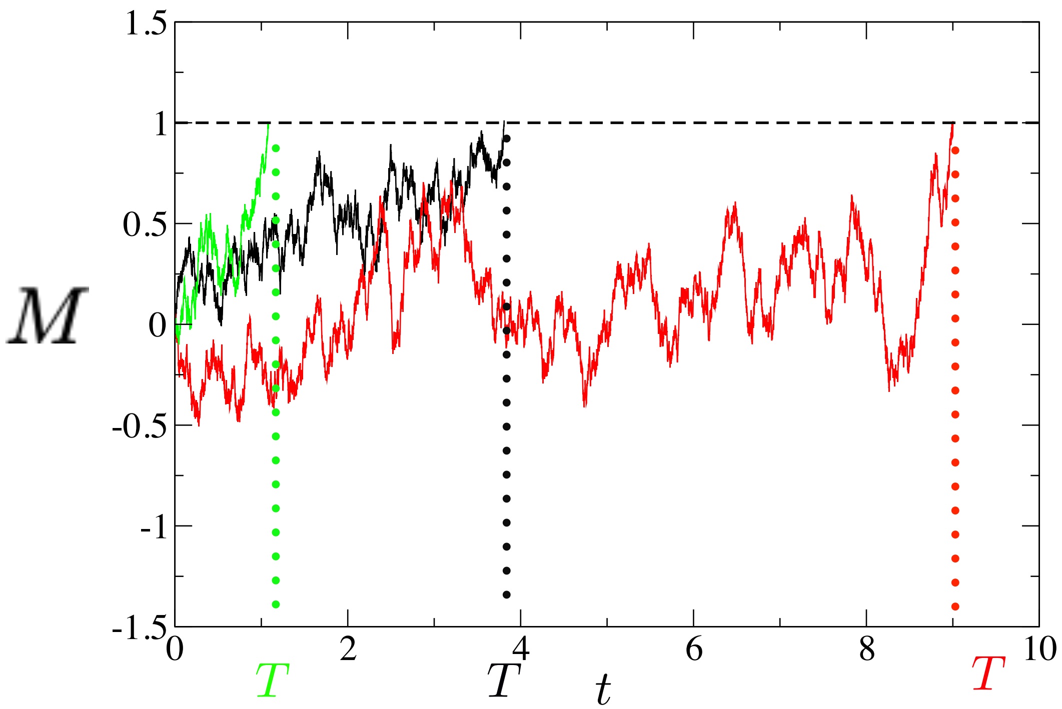
$$= \frac{\tilde{p}\left(X(0\dots s)\right)}{p\left(X(0\dots s)\right)}$$

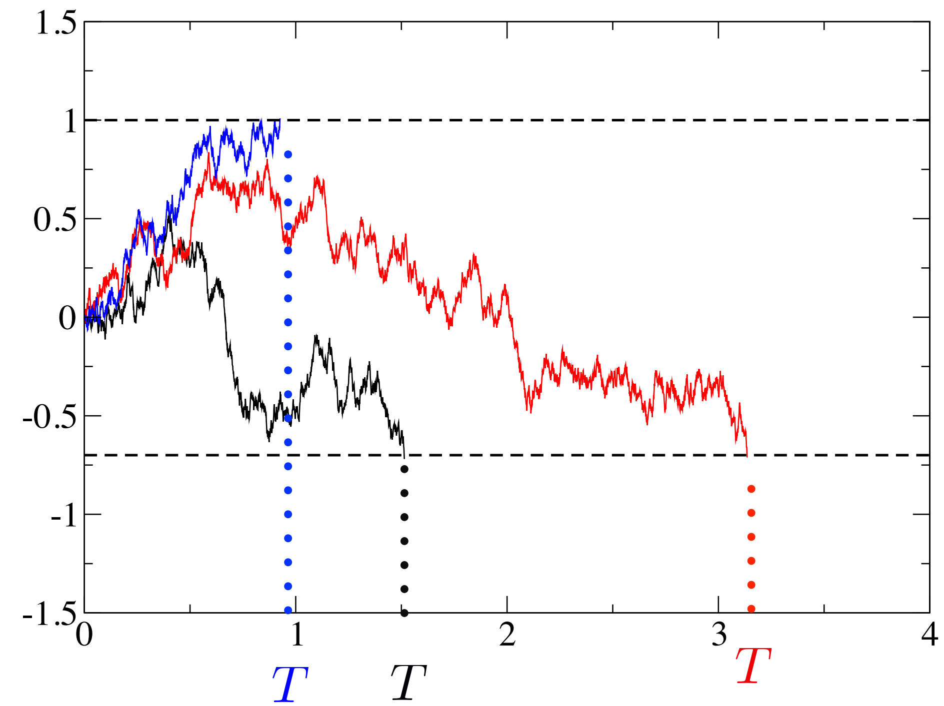
$$= e^{-S_{\mathrm{tot}}(s)}$$

IN, Edgar Roldan, Frank Julicher, Phys. Rev. X (2017)

Thermodynamic laws at stopping times

Can a gambler make fortune in a fair game by quitting at an intelligently chosen moment?





$$\langle M(T)\rangle = 1 \neq \langle M(0)\rangle = 0$$

$$\langle M(T)\rangle = \langle M(0)\rangle = 0$$

Gambler makes profit

Gambler on average makes no profit

Can a gambler make fortune in a fair game by quitting at an intelligently chosen moment?

No, if the gambler cannot foresee the future, cannot cheat, and has access to a finite budget

Can a gambler make fortune in a fair game by quitting at an intelligently chosen moment?

***T* is a stopping time**

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***M(t)* is uniformly integrable**

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No, if the gambler cannot foresee the future, cannot cheat, and has access to a finite budget

***M(t)* is uniformly integrable**

Doob's optional stopping theorem

$\langle M(T)|X(0)\rangle = M(0)$ if *M(t)* is uniformly integrable martingale and and *T* is a stopping time

R S Lipster and A N Shiryaev, Statistics of random processes: I General theory, 1977

Integral fluctuation relations for entropy production at stopping times

$$\langle e^{-S_{\mathrm{tot}}(T)} | X(0) \rangle = e^{-S_{\mathrm{tot}}(0)} = 1$$

IN, E Roldan, S Pigolotti, F Julicher, arXiv (2019)

Integral fluctuation relations for entropy production at stopping times

$$\langle e^{-S_{\rm tot}(T)} | X(0)\rangle = e^{-S_{\rm tot}(0)} = 1$$

Finite time windows

$$\langle e^{-S_{\rm tot}(T\wedge t)}\rangle = 1$$

IN, E Roldan, S Pigolotti, F Julicher, arXiv (2019)

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Finite time windows

$$\langle e^{-S_{\rm tot}(T \wedge t)} \rangle = 1$$

Infinite time windows

$$\langle e^{-S_{\rm tot}(T)} \rangle = 1$$

if $\quad e^{-S_{\rm tot}(t)} \to 0$

and $\; S_{\rm tot}(t) \in [-s_-, s_+]$

$\forall t \in [0, T]$

IN, E Roldan, S Pigolotti, F Julicher, arXiv (2019)

Second law of thermodynamics at stopping times

$$1 = \langle e^{-S_{\rm tot}(T)} \rangle \geq \; e^{-\langle S_{\rm tot}(T) \rangle}$$ Jensen's Inequality

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$$\downarrow$$

$$\langle S_{\rm tot}(T) \rangle \geq 0$$

Second law of thermodynamics at stopping times

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$$\langle S_{\rm tot}(T) \rangle \geq 0$$

For isothermal processes:

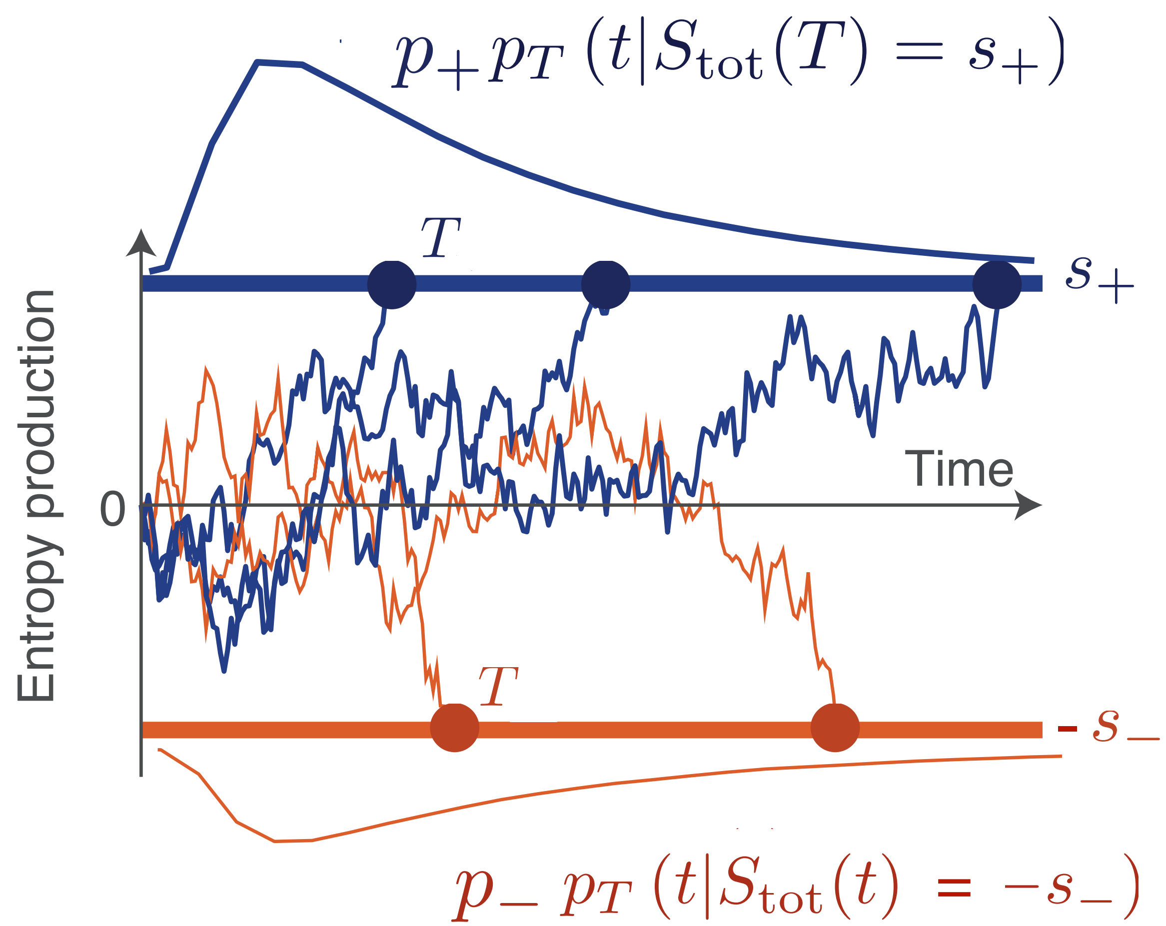
$$\langle Q(T) \rangle \leq \mathsf{T}_{\rm env} \langle \log \frac{p_{\rm ss}(X(0))}{p_{\rm ss}(X(T))} \rangle$$

Universal properties of entropy production

Universal properties of entropy production

(for continuous stochastic processes)

Splitting probabilities of the entropy production



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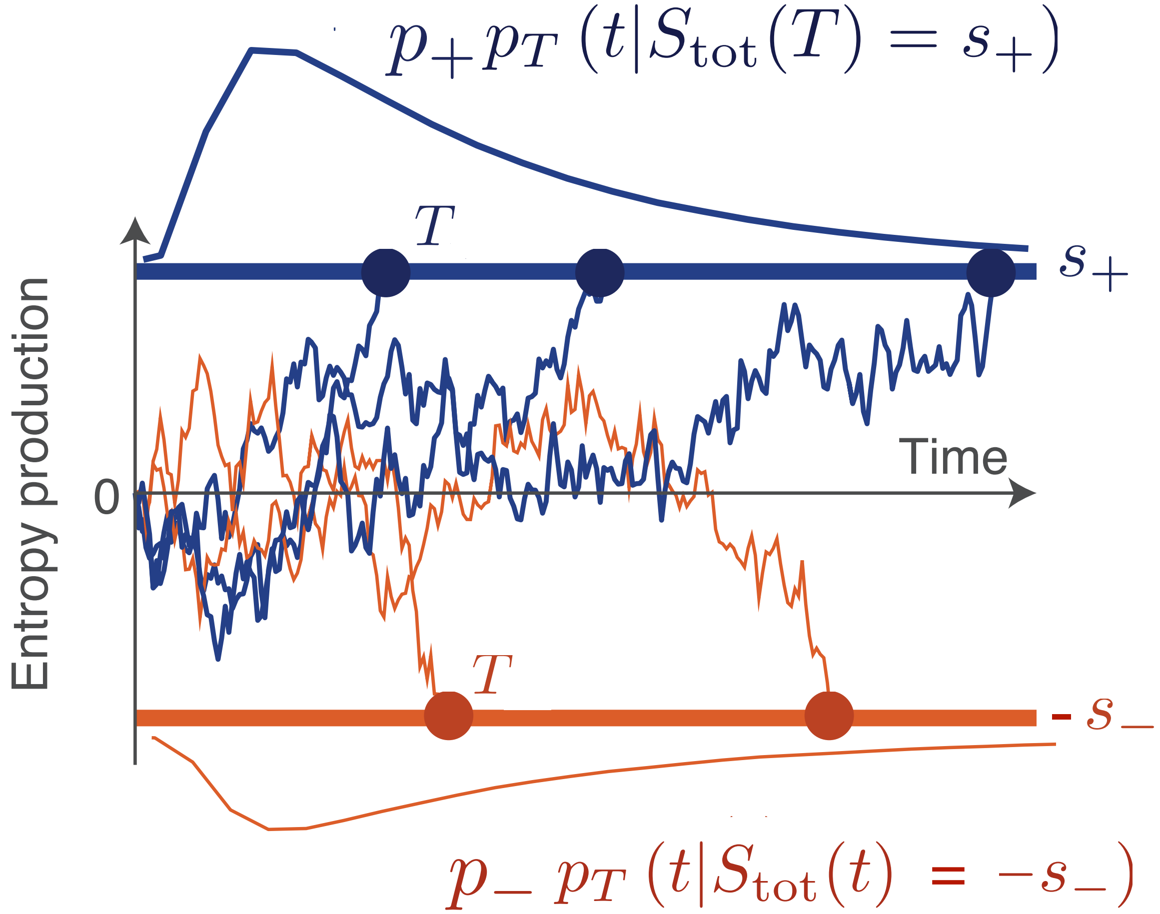
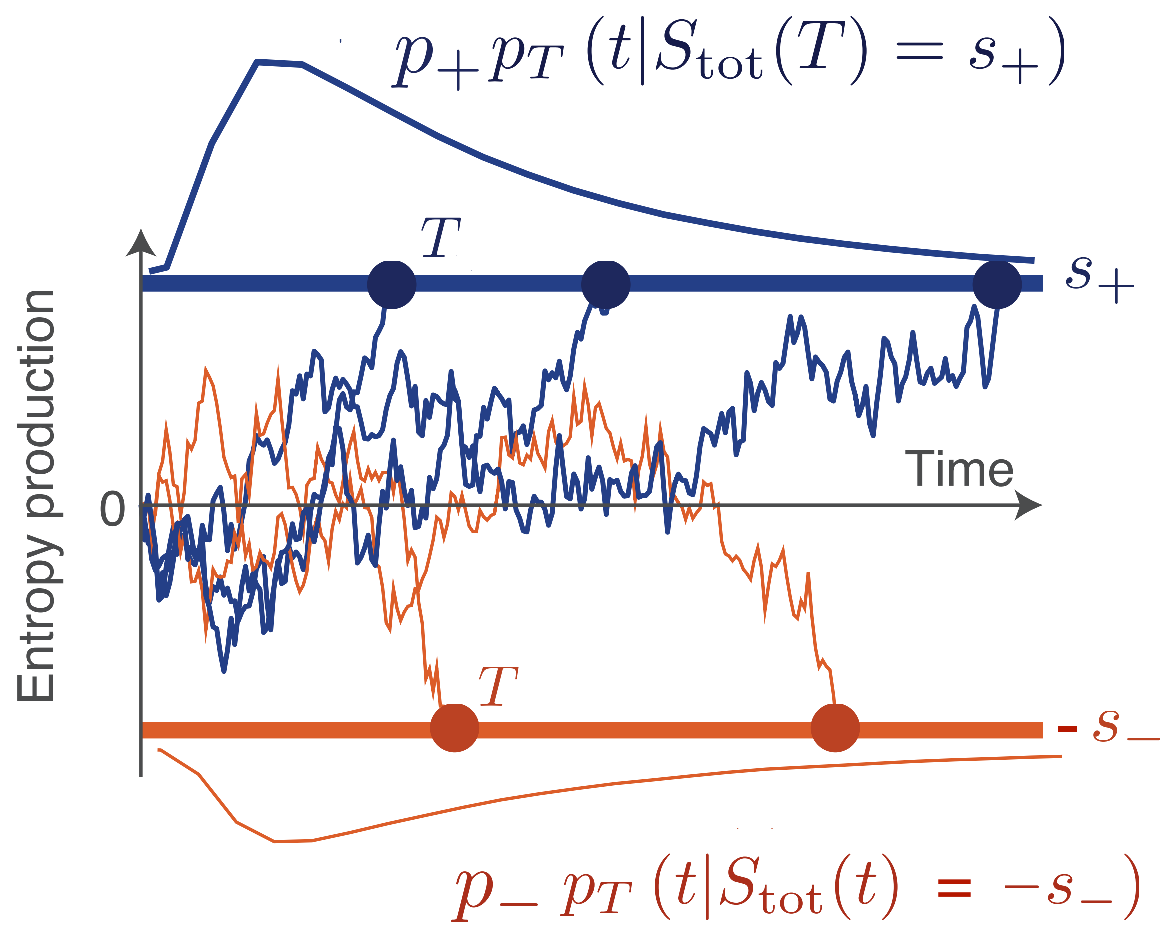


$$\langle e^{-S_{\rm tot}(T)}\rangle = 1$$

$$P\left[T < \infty\right] = 1$$

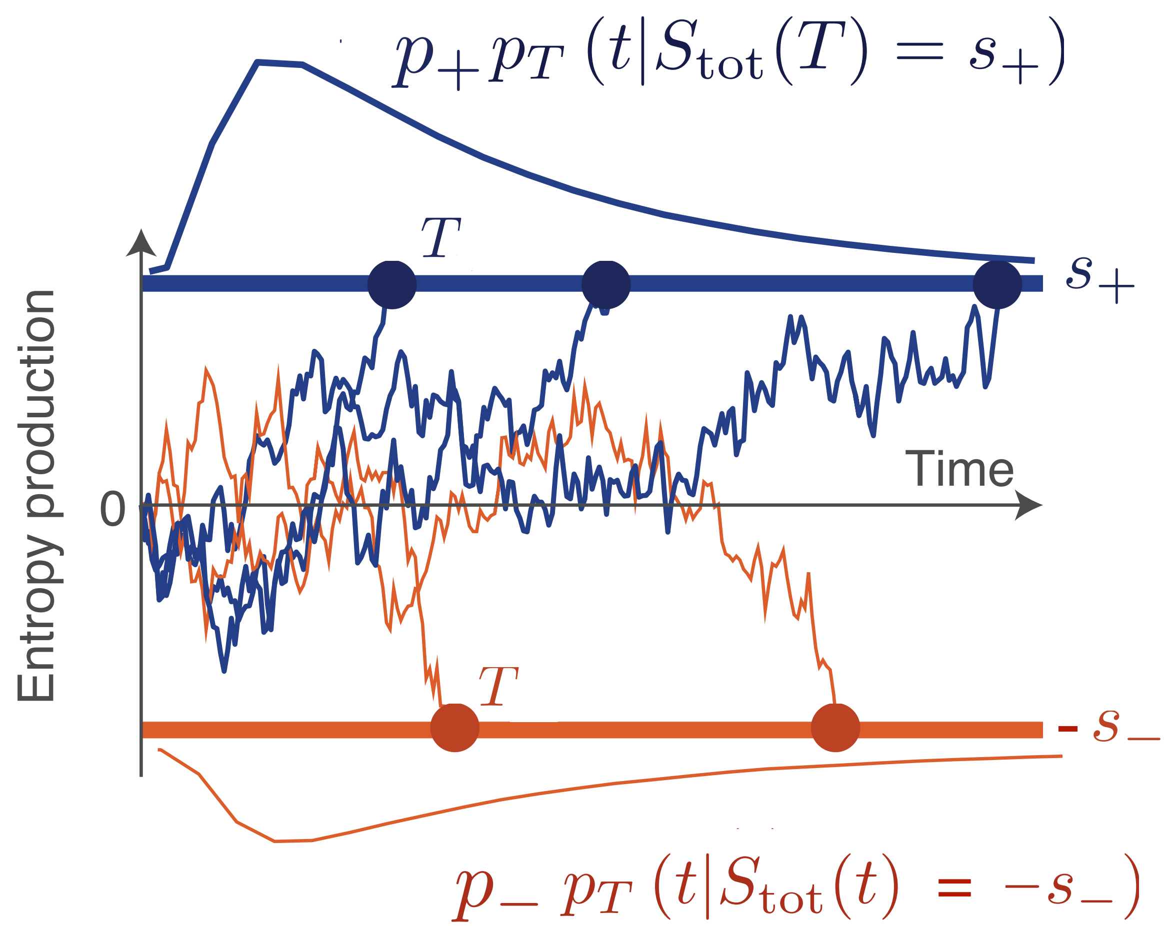
Splitting probabilities of the entropy production



$$\langle e^{-S_{\rm tot}(T)} \rangle = 1 \qquad \longrightarrow \qquad p_- e^{s_-} + p_+ e^{-s_+} = 1$$

$$P\left[T < \infty\right] = 1 \qquad\qquad p_- + p_+ = 1$$

Splitting probabilities of the entropy production



$$\langle e^{-S_{\rm tot}(T)}\rangle = 1$$

$$P\,[T < \infty] = 1$$

$\longrightarrow$

$$p_- e^{s_-} + p_+ e^{-s_+} = 1$$

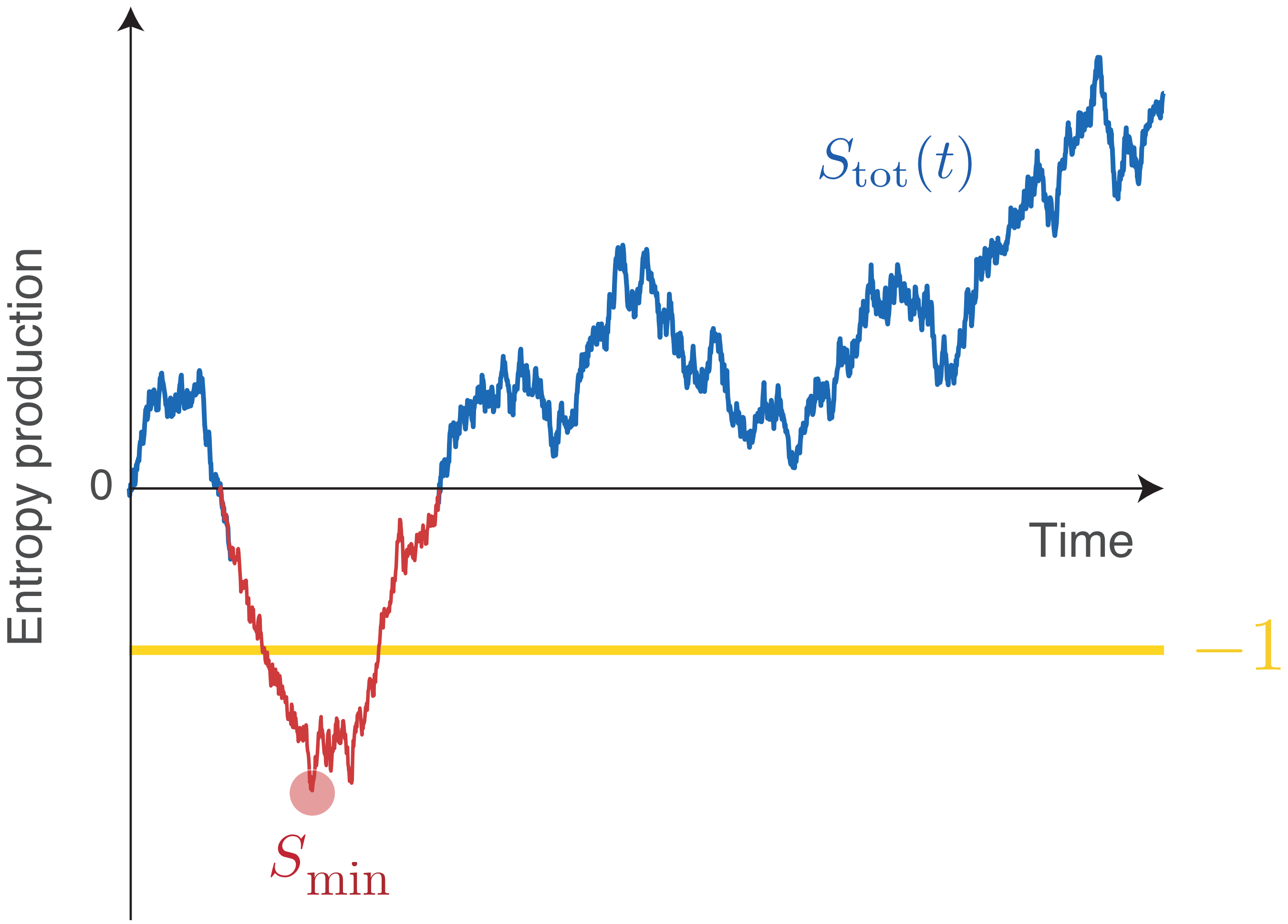
$$p_- + p_+ = 1$$

$\longrightarrow$

$$p_+ = \frac{e^{s_-} - 1}{e^{s_-} - e^{-s_+}}$$

$$p_- = \frac{1 - e^{-s_+}}{e^{s_-} - e^{-s_+}}$$

The statistics of minima of the entropy production of continuous stationary processes are universal



IN, Edgar Roldán, Frank Jülicher, Phys. Rev. X 7, 011019

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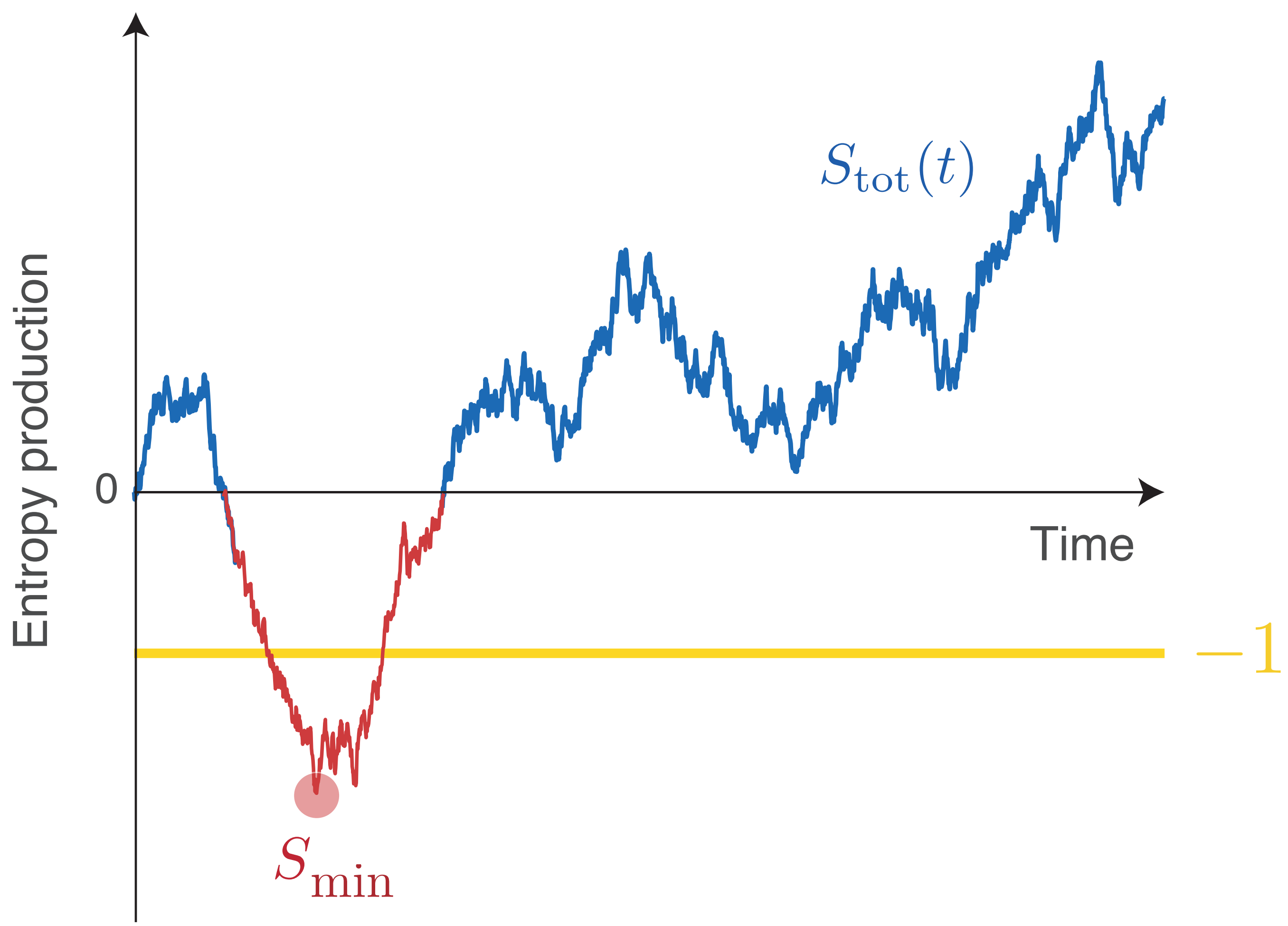

$$P\left(S_{\rm min} < -s_-\right) = \lim_{s_+ \to \infty} p_-$$
$$= \lim_{s_+ \to \infty} \frac{1 - e^{-s_+}}{e^{s_-} - e^{-s_+}}$$
$$= e^{-s_-}$$

IN, Edgar Roldán, Frank Jülicher, Phys. Rev. X 7, 011019

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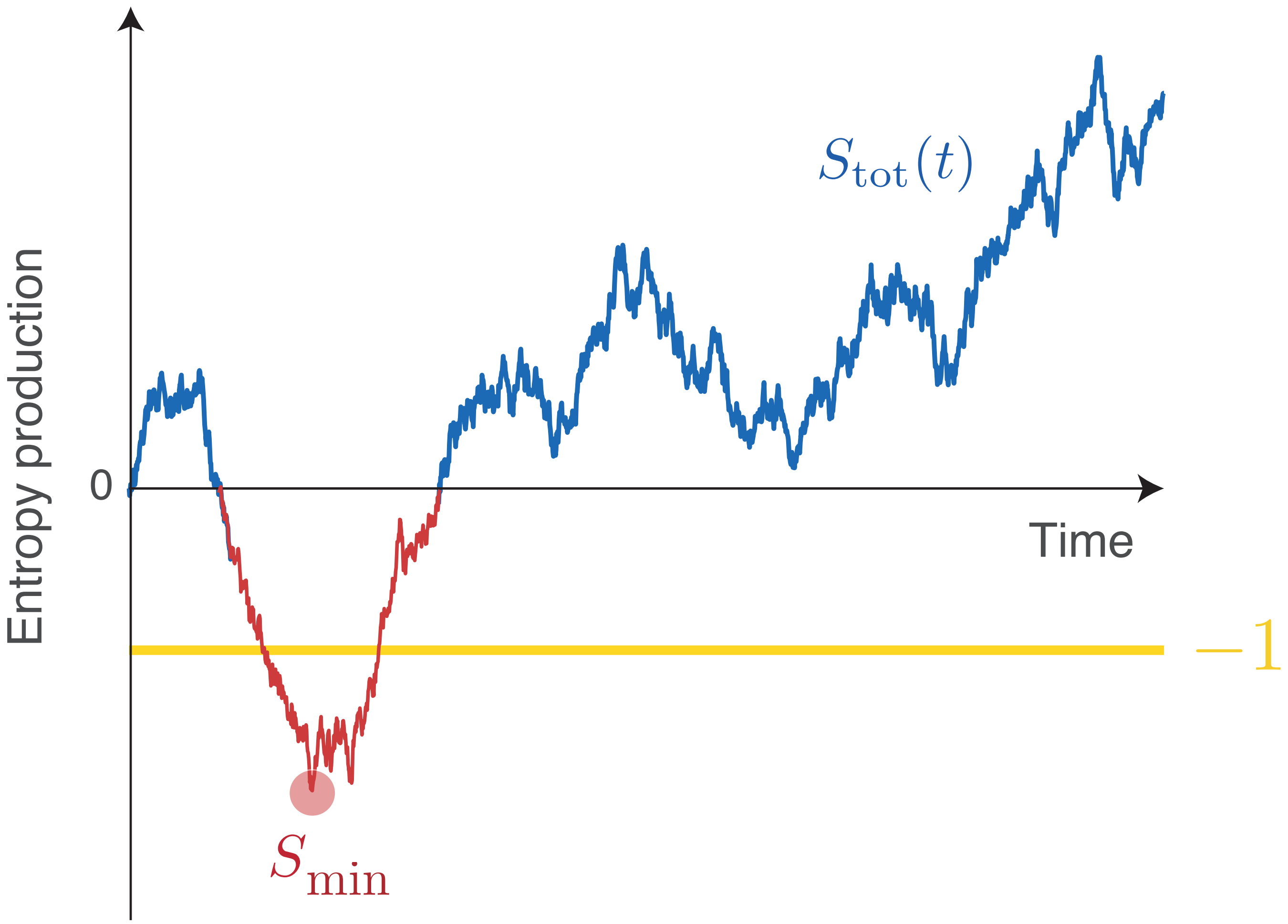

$$P\left(S_{\mathrm{min}} < -s_-\right) = \lim_{s_+ \to \infty} p_- = \lim_{s_+ \to \infty} \frac{1 - e^{-s_+}}{e^{s_-} - e^{-s_+}} = e^{-s_-}$$

$$P\left(S_{\mathrm{min}} > -s_-\right) = 1 - e^{-s_-}$$

$$\langle S_{\mathrm{min}} \rangle = -1$$

IN, Edgar Roldán, Frank Jülicher, Phys. Rev. X 7, 011019

Bounds on negative fluctuations of entropy production: "standard" thermodynamics vs martingale theory

$$\langle e^{-S_{\rm tot}(t)} \rangle = 1$$

$$\downarrow$$

$$P\left(S_{\rm tot}(t) \leq -s\right) \leq e^{-s}$$

U Seifert, Rep. Prog. Phys. (2012)

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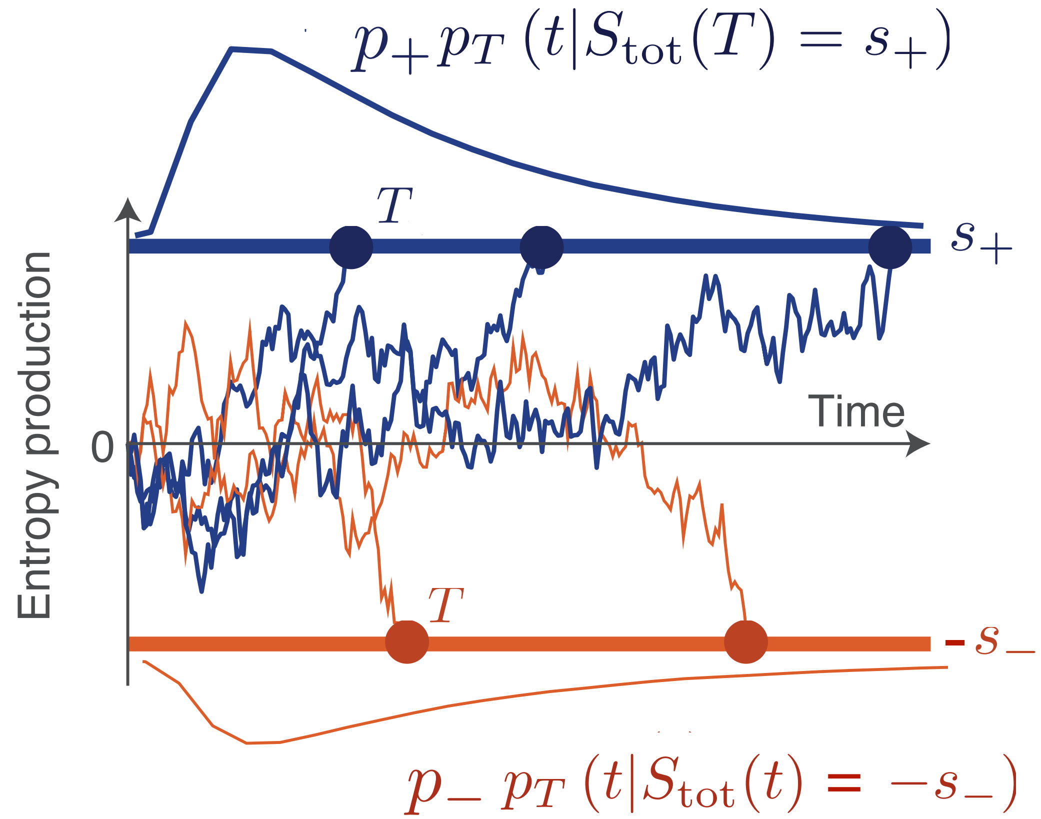
IN, E Roldan, S Pigolotti, F Julicher, arXiv (2019)

Symmetry relation in the conditional distributions of first-passage times for entropy production



$$p_T(t|S_{\rm tot} = s) = p_T(t|S_{\rm tot} = -s)$$

IN, Edgar Roldán, Frank Jülicher, Phys. Rev. X 7, 011019

Meik Dorpinghaus, IN, Edgar Roldan, Frank Julicher, Heinrich Meyer, arXiv

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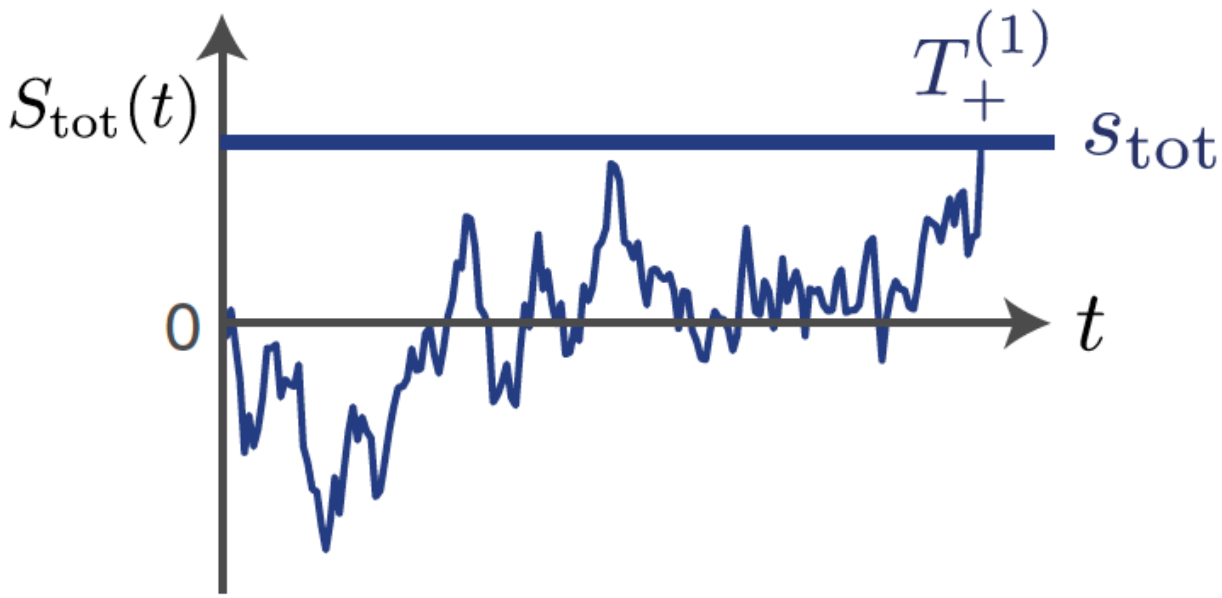

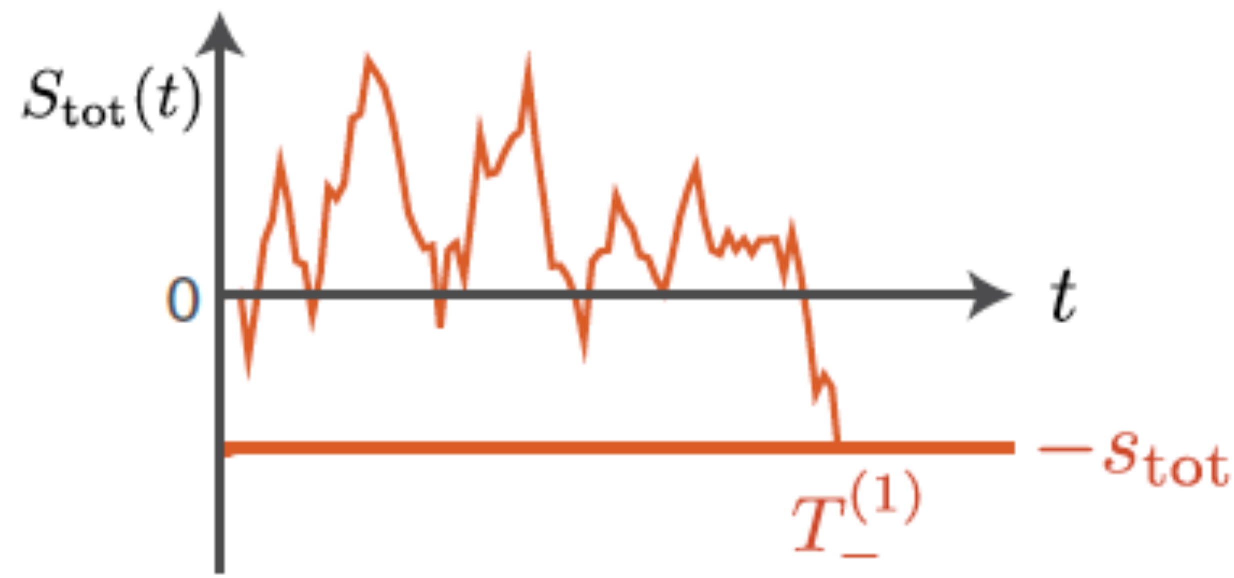

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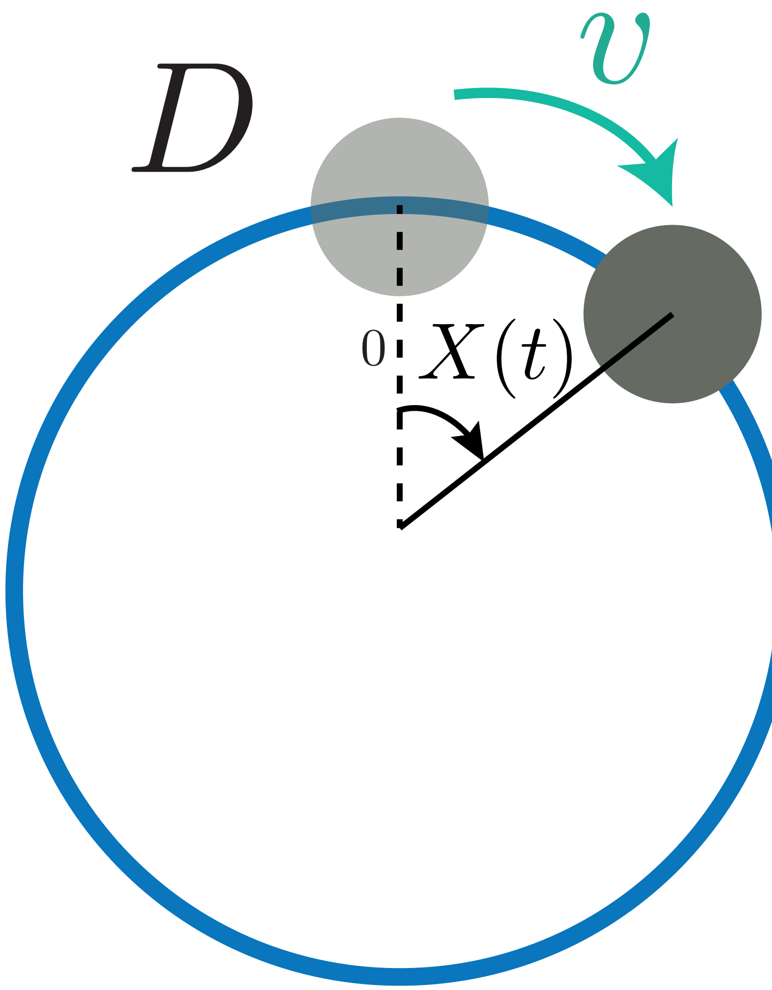
Duality in first-passage times $\frac{dX}{dt} = v + D\frac{dW}{dt}$



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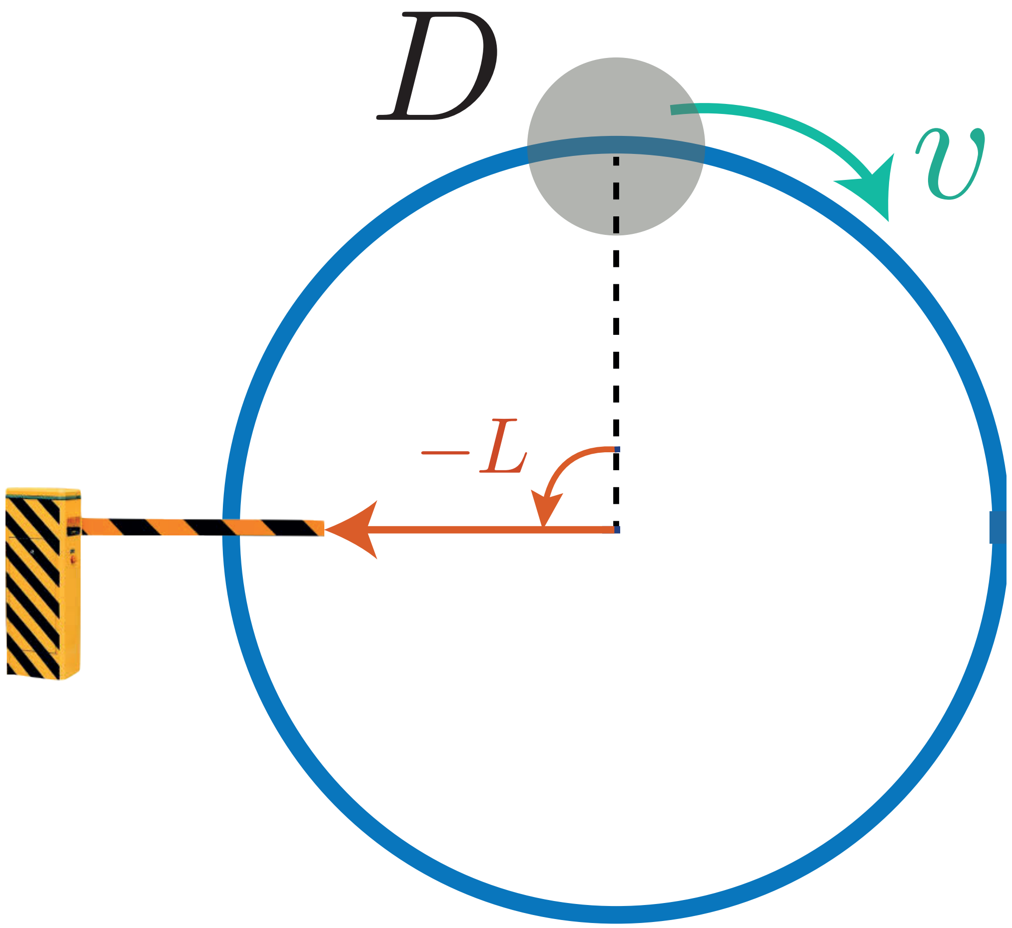
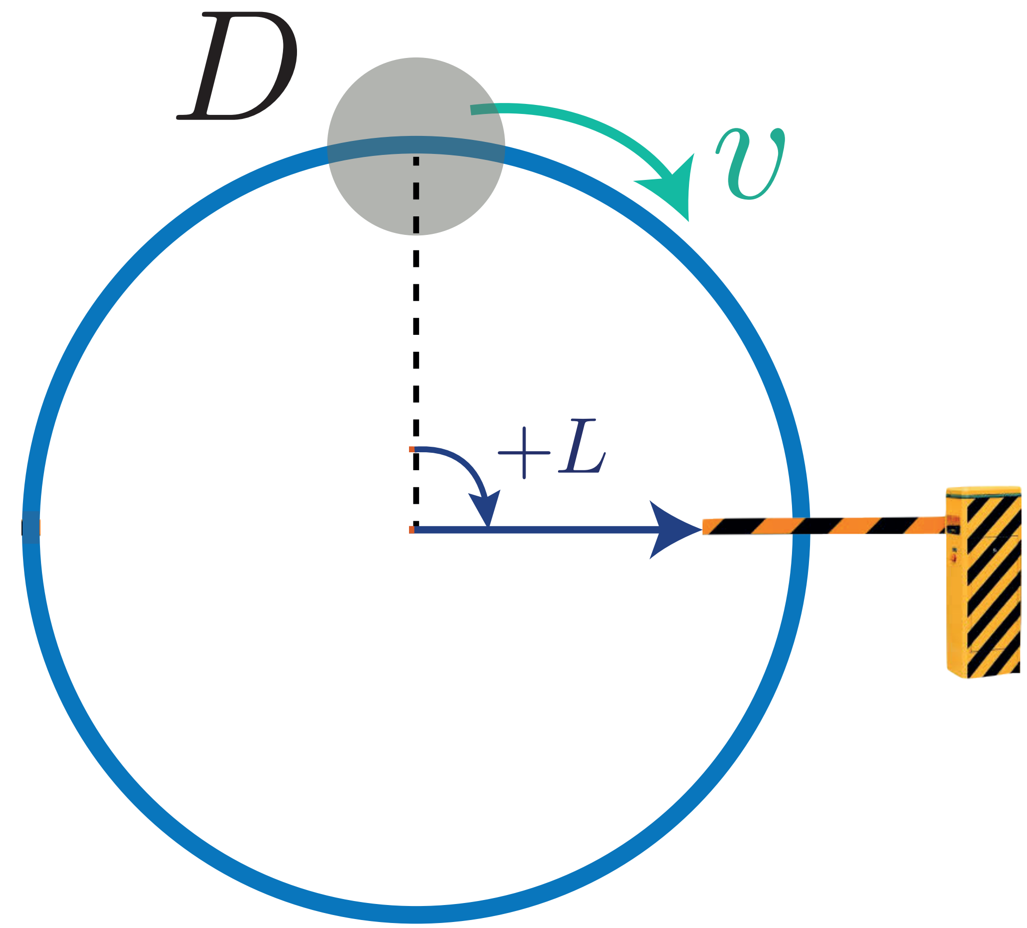


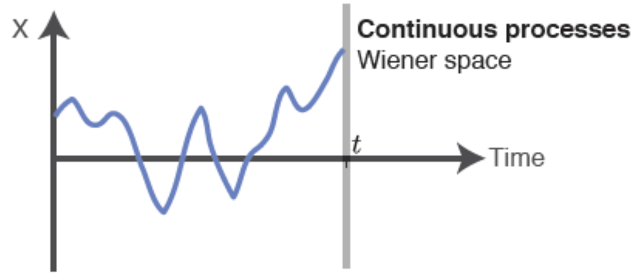


$$p_T(t; -L) = \frac{|L|}{\sqrt{4\pi D t^3}} e^{-(-L-vt)^2/(4Dt)}$$

$$p_T(t; -L) = \frac{|L|}{\sqrt{4\pi D t^3}} e^{-(L-vt)^2/(4Dt)}$$

$$\frac{p_T(t; L)}{p_T(t; -L)} = e^{vL/D}$$

Continuous processes



$$p_+ = \frac{e^{s_-} - 1}{e^{s_-} - e^{-s_+}}$$

$$p_- = \frac{1 - e^{-s_+}}{e^{s_-} - e^{-s_+}}$$

$$P\left(S_{\min} > -s_-\right) = 1 - e^{-s_-}$$

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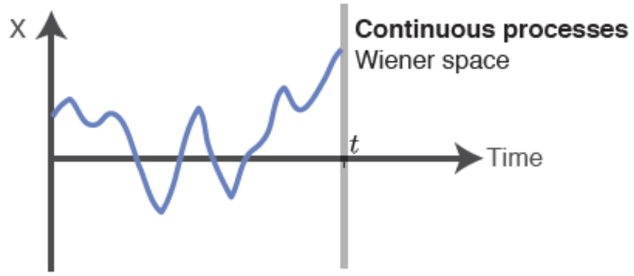
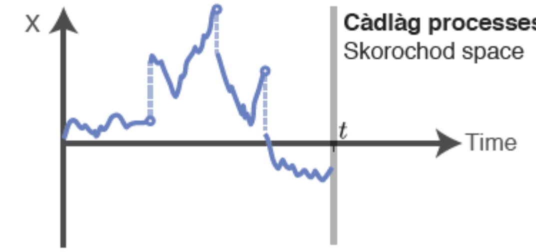
Continuous processes



Processes with jumps



$$p_+ = \frac{e^{s_-} - 1}{e^{s_-} - e^{-s_+}}$$

$$p_- = \frac{1 - e^{-s_+}}{e^{s_-} - e^{-s_+}}$$

$$p_+ \geq 1 - \frac{1}{e^{s_-} - e^{-s_+}}$$

$$p_- \leq \frac{1}{e^{s_-} - e^{-s_+}}$$

$$P\left(S_{\min} > -s_-\right) = 1 - e^{-s_-}$$

$$\langle S_{\min} \rangle = -1$$

$$P\left(S_{\inf} \geq -s_-\right) \geq 1 - e^{-s_-}$$

$$\langle S_{\inf} \rangle \geq -1$$

$$p_T(t|S_{\text{tot}} = s) = p_T(t|S_{\text{tot}} = -s)$$

???

Example: overdamped Langevin processes

First law of thermodynamics for overdamped Langevin process

System set-up

$$\frac{dX}{dt} = \mu F + \nabla D + \sqrt{2}\sigma \cdot \xi \ , \qquad \langle \xi \rangle = 0, \quad \langle \xi_i(t)\xi_j(t') \rangle = \delta_{i,j}\delta(t-t')$$

$$F = -\nabla u + f \ , \quad \sigma\sigma^T = D \ , \quad D = \mu \, \mathsf{T}_{\text{env}}$$

First law of thermodynamics for overdamped Langevin process

System set-up

Ito product

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First law of thermodynamics

$$dW = f \circ dX \;, \qquad dQ = du - dW$$

K Sekimoto, Prog. Theory. Phys. Suppl. ***130****, 17 (1998)*

First law of thermodynamics for overdamped Langevin process

System set-up

Ito product

$$\frac{dX}{dt} = \mu F + \nabla D + \sqrt{2}\sigma \cdot \xi \ , \qquad \langle \xi \rangle = 0, \quad \langle \xi_i(t)\xi_j(t') \rangle = \delta_{i,j}\delta(t-t')$$

$$F = -\nabla u + f \ , \quad \sigma\sigma^T = D \ , \quad D = \mu \, \mathsf{T}_{\text{env}}$$

First law of thermodynamics

Stratanovich product

$$dW = f \circ dX \ , \qquad dQ = du - dW$$

K Sekimoto, Prog. Theory. Phys. Suppl. ***130****, 17 (1998)*

Second law of thermodynamics for overdamped Langevin process

Definition of entropy production:

$$dS_{\rm tot}(t) = -\frac{dQ}{\mathsf{T}_{\rm env}} - d\log p_{\rm ss}(X(t))$$

where

$$\nabla \cdot ((\mu F + \nabla D)p_{\rm ss} - \nabla(D p_{\rm ss})) = 0$$

Udo Seifert, Physical review letters (2005)

Second law of thermodynamics for overdamped Langevin process

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Rules of stochastic calculus imply:

$$\frac{dS_{\rm tot}(t)}{dt} = v_S + \sqrt{2v_S}\cdot\xi_S \ ,$$

$$\xi_S = \frac{\xi\sigma^{-1}J}{\sqrt{JD^{-1}J}}\ , \quad v_S = \frac{J_{\rm ss}D^{-1}J_{\rm ss}}{p_{\rm ss}}$$

Second law of thermodynamics for overdamped Langevin process

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$$\xi_S = \frac{\xi \sigma^{-1} J}{\sqrt{J D^{-1} J}} \ , \quad v_S = \frac{J_{\mathrm{ss}} D^{-1} J_{\mathrm{ss}}}{p_{\mathrm{ss}}}$$

Second law of thermodynamics for overdamped Langevin process

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Rules of stochastic calculus imply:

$$\frac{dS_{\mathrm{tot}}(t)}{dt} = v_S + \sqrt{2v_S} \cdot \xi_S \ ,$$

$$\xi_S = \frac{\xi \sigma^{-1} J}{\sqrt{J D^{-1} J}} \ , \quad v_S = \frac{J_{\mathrm{ss}} D^{-1} J_{\mathrm{ss}}}{p_{\mathrm{ss}}}$$

→

$$\frac{d\langle S_{\mathrm{tot}}(t)\rangle}{dt} = \langle v_S \rangle \geq 0$$

Exponential martingale structure of entropy production

$$\frac{dS_{\mathrm{tot}}(t)}{dt} = v_S + \sqrt{2v_S} \cdot \xi_S \ , \qquad \frac{dX}{dt} = \mu F + \nabla D + \sqrt{2}\sigma \cdot \xi$$

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No drift term —-> martingale

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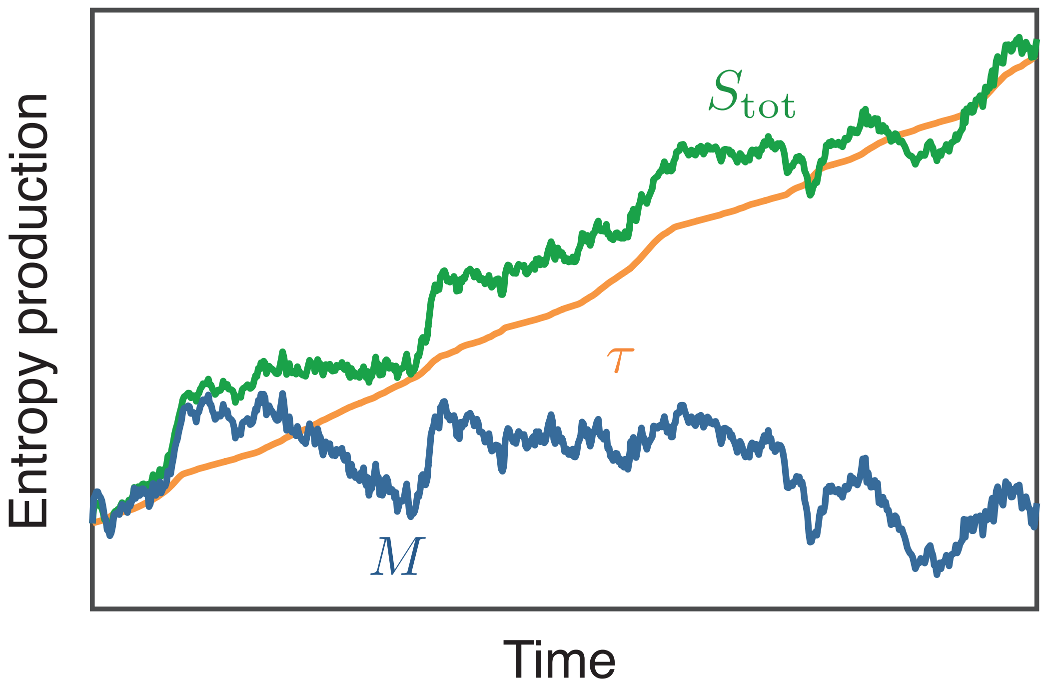
Random-time transformation

$$\frac{dS_{\rm tot}(t)}{dt} = v_S + \sqrt{2v_S} \cdot \xi_S$$



E*ntropic time:*

$$\tau = \int_0^t v_S(X(t'))dt'$$

$$S_{\rm tot}(t) = \tau(t) + M(t) \ ,$$

Nondecreasing Martingale

$$M(t) = \int_0^{t'} {\rm d}t \ \sqrt{2v_S(t')}\xi_S(t')$$

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Random-time transformation renders certain properties of entropy production universal

$$\frac{dS_{\rm tot}(\tau)}{d\tau} = 1 + \sqrt{2}\eta(\tau) \ , \quad \eta(\tau) = \frac{\xi_S(\tau)}{\sqrt{v_S(\tau)}}$$

Fluctuation properties independent of the time-scale are universal

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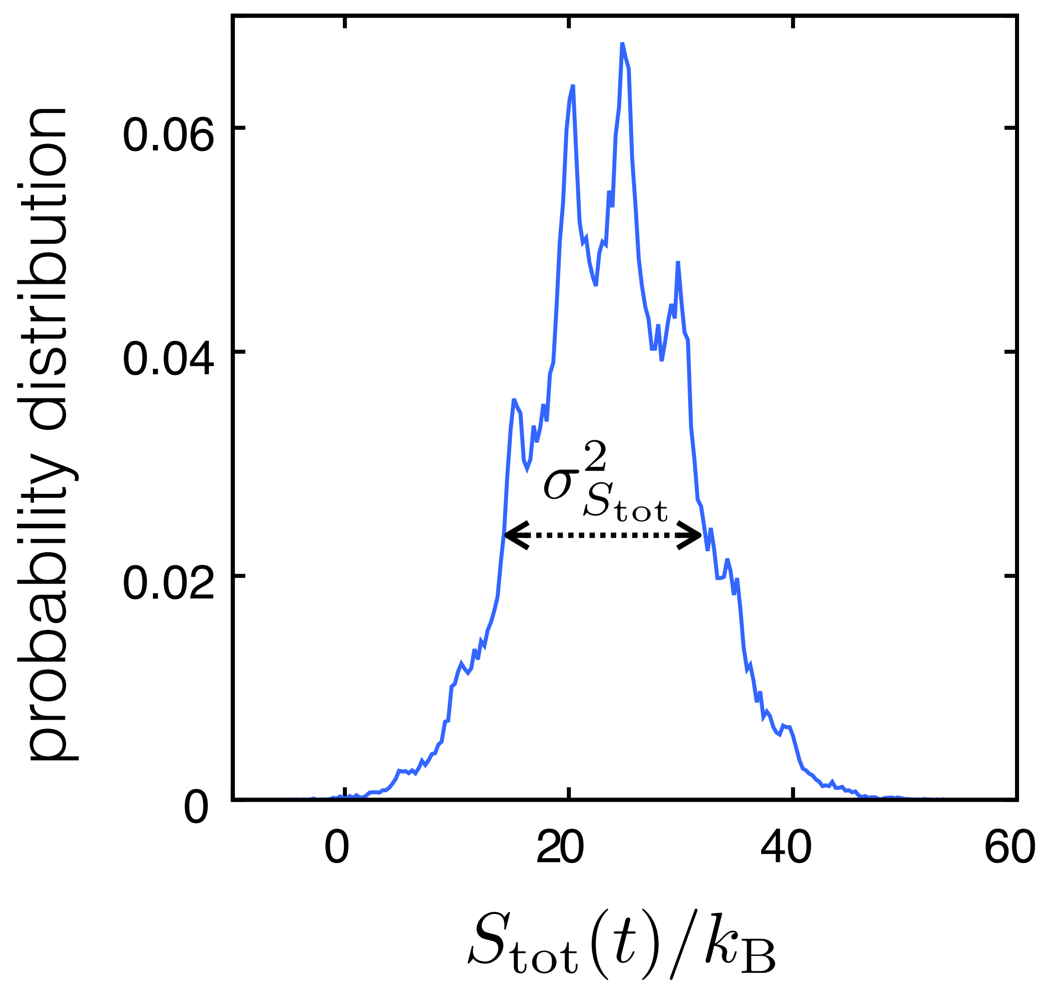
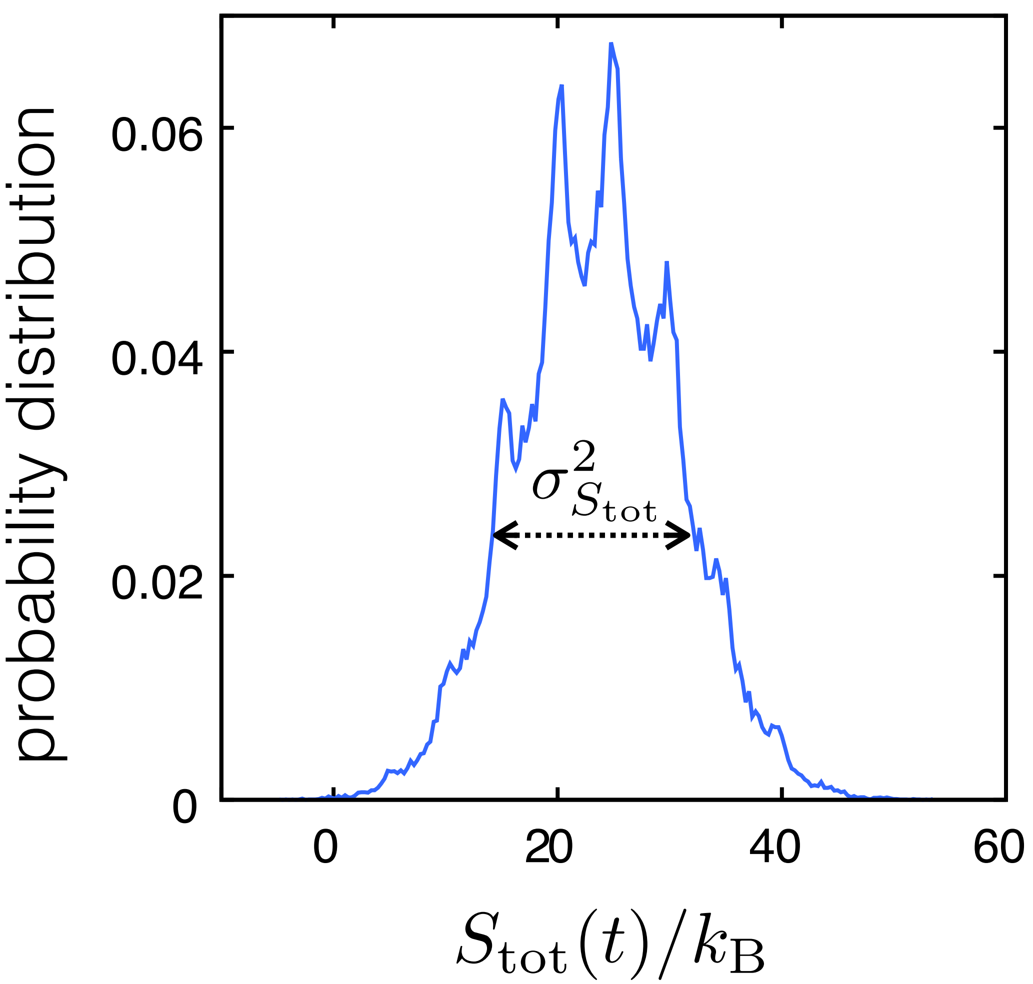
Inequality for the Fano-factor of entropy production



Inequality for the Fano-factor of entropy production

$$\frac{\sigma^2_{S_{\rm tot}}}{\langle S_{\rm tot}\rangle} = 2 + \frac{\sigma^2_{\tau}}{\langle \tau \rangle}$$

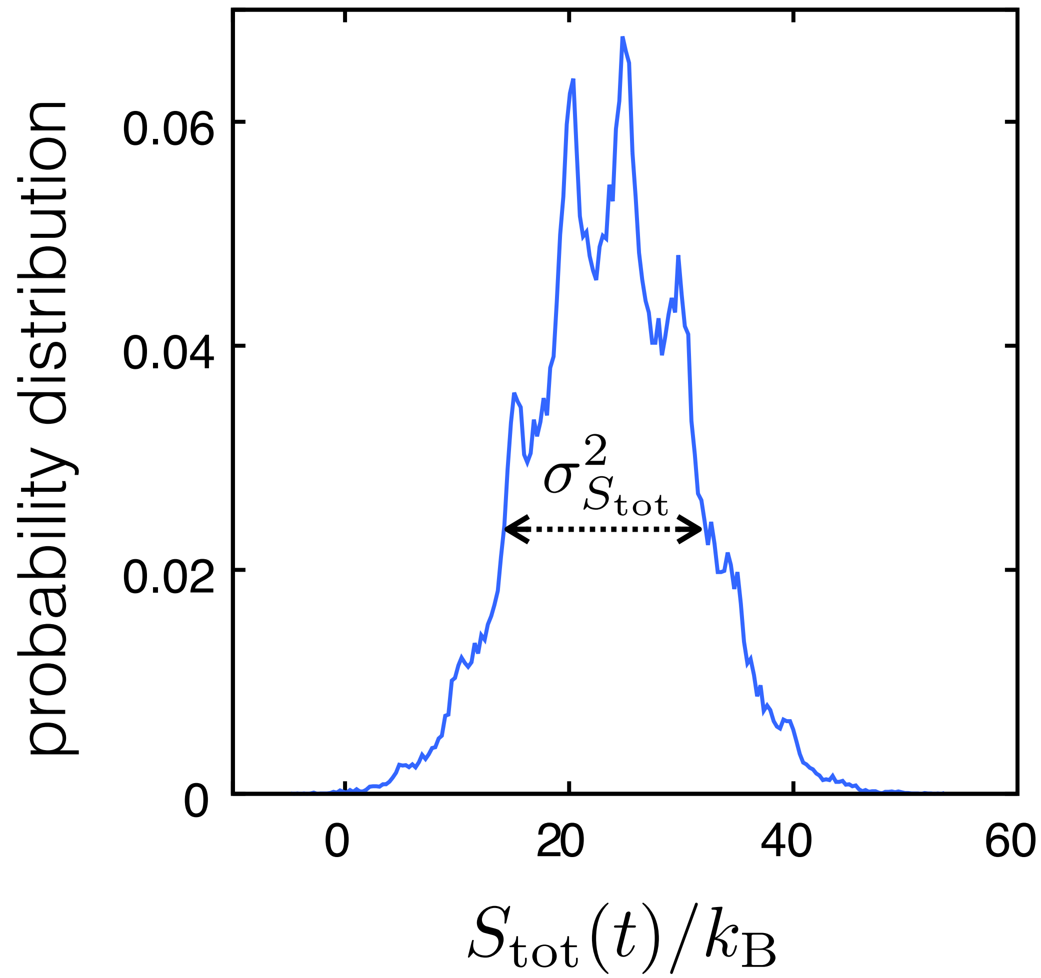


Inequality for the Fano-factor of entropy production

$$\frac{\sigma^2_{S_{\rm tot}}}{\langle S_{\rm tot}\rangle} = 2 + \frac{\sigma^2_{\tau}}{\langle \tau \rangle}$$

$$\downarrow$$

$$\frac{\sigma^2_{S_{\rm tot}}}{\langle S_{\rm tot}\rangle} \geq 2$$



Fano factor inequality for entropy production

$$S_{\rm tot}(t) = \tau(t) + M(t)\,, \qquad \frac{dX}{dt} = \mu F + \nabla D + \sqrt{2}\sigma \cdot \xi$$

$$\tau = \int_0^t v_S(X(t'))dt' \quad , \quad M(t) = \int_0^{t'} \mathrm{d}t\,\sqrt{2 v_S(X(t'))}\xi_S(t')$$

$$\frac{\sigma^2_{S_{\rm tot}(t)}}{\langle S_{\rm tot}(t)\rangle} = \frac{\sigma^2_{\tau(t)}}{\langle \tau(t)\rangle} + \frac{\sigma^2_{M(t)}}{\langle S_{\rm tot}(t)\rangle} + 2\,\frac{\langle \tau(t) M(t)\rangle}{\langle S_{\rm tot}(t)\rangle}$$

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= 2

Ito isometry

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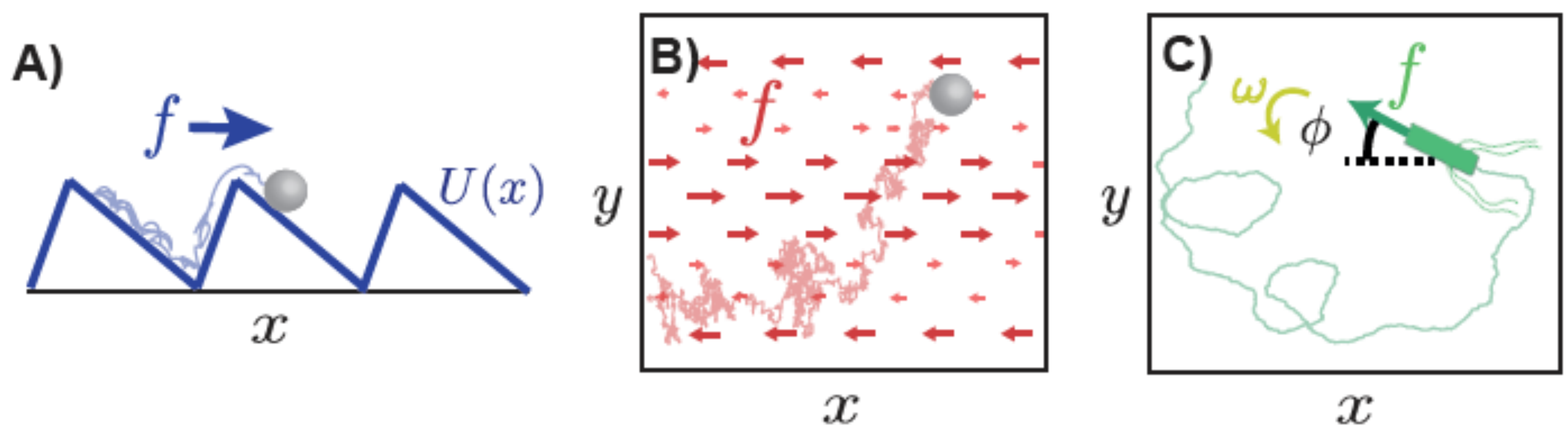
= 2

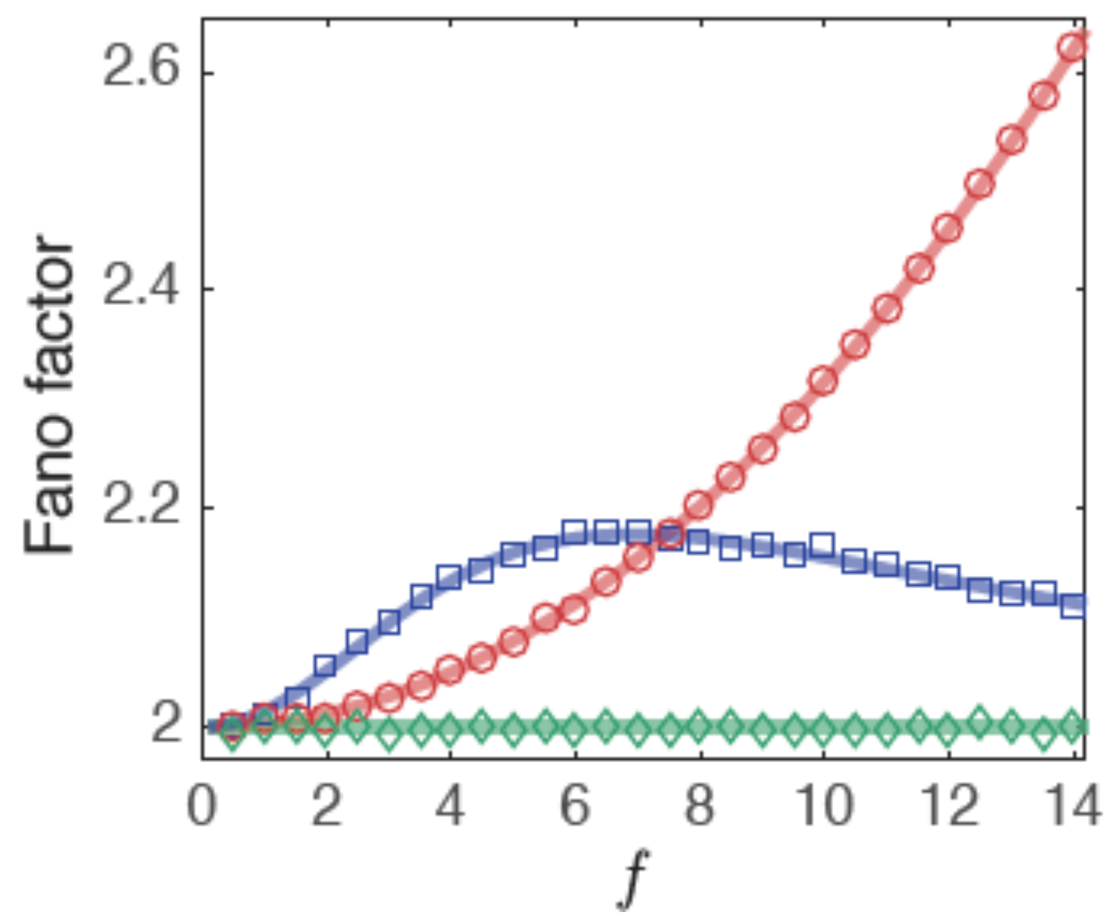
= 0

Ito isometry

Doob h-transform

Fano factor inequality for entropy production





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Thank you for your attention!