Weak time-derivatives and pricing equations

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Plan

- I illustrate a novel mathematical tool for the characterization of martingales in continuous time: the *weak time-derivative* of Marinacci, Severino (*Finance & Stochastics*, 2018).
- I compare weak time-differentiability with other existing notions (infinitesimal generator).
- I discuss some fundamental asset pricing equations related to martingale identification.
- I present some measure changes that originate useful martingale processes for pricing.

General set-up

- Time interval $[0, T]$.
- Filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$.
- $\mathcal{U}$ is the space of adapted processes $u : [0, T] \to L^1(\mathcal{F}_T)$ that
  - are L^1-right-continuous in $[0, T)$,
  - are L^1-left-continuous at T,
  - have finite $\int_0^T \mathbb{E}[|u_\tau|]d\tau$.
- Martingales belong to $\mathcal{U}$.

Weak time-differentiability

Definition

A process $u \in \mathcal{U}$ is *weakly time-differentiable* when there exists a process $\mathcal{D}u \in \mathcal{U}$ such that, for every $t \in [0, T]$,

$$\int_t^T \mathbb{E}\left[(\mathcal{D}u)_\tau \mathbf{1}_{A_t}\right] \varphi(\tau) d\tau = - \int_t^T \mathbb{E}\left[u_\tau \mathbf{1}_{A_t}\right] \varphi'(\tau) d\tau$$

for all $A_t \in \mathcal{F}_t$ and $\varphi \in C_c^1([t, T])$.

$\mathcal{D}u$ is the *weak time-derivative* of u.

- A bridge between variational and stochastic calculus.
- Purpose: capture the behaviour of the conditional expectation over time.

Martingales via weak time-derivatives

- $\mathcal{U}^1$ denotes the space of weakly time-differentiable processes $u \in \mathcal{U}$.

Proposition

u belongs to $\mathcal{U}^1$ and has $\mathcal{D}u = 0$ if and only if u is a martingale.

Proposition

Let $u \in \mathcal{U}^1$. Then,

- $\mathcal{D}u \geqslant 0$ if and only if u is a submartingale.
- $\mathcal{D}u \leqslant 0$ if and only if u is a supermartingale.

Properties of weak time-derivatives

Proposition

Consider $g \in \mathcal{U}$, m a martingale and

$$u_t = \int_0^t g_s ds + m_t,$$

Then, $\mathcal{D}u = g$.

Examples: deterministic drift + martingale

- Consider $\alpha \in \mathbb{R}$ and m a martingale. Then, $u_t = \alpha t + m_t$ has $\mathcal{D}u = \alpha$.
- E.g. in Black-Scholes (1973) log prices satisfy

$$\log(X_t) = (r - \sigma^2/2)t + \sigma W_t^Q,$$

where W^Q is a Wiener process under the risk-neutral measure Q. Then, $\mathcal{D}(\log X) = r - \sigma^2/2$.

Examples: continuous Itô semimartingales

- Consider $g \in \mathcal{U}$, h adapted and $\int_0^T \mathbb{E}[h_s^2] ds$ finite. Then, the process $X \in \mathcal{U}$ defined by

$$dX_t = g_t dt + h_t dW_t$$

has $\mathcal{D}X = g$.

The weak time-derivative is the drift.

- If $u_t = f(t, X_t)$ with f regular, then by Itô's formula

$$\mathcal{D}u = g \frac{\partial f}{\partial x} + \frac{\partial f}{\partial t} + \frac{1}{2} h^2 \frac{\partial^2 f}{\partial x^2}.$$

Characterization of weak time-differentiable processes

Theorem

$u \in \mathcal{U}$ is weakly time-differentiable if and only if it is a special martingale

$$u = a + m,$$

with $a_t = \int_0^t (\mathcal{D}u)_s ds$ and m a martingale.

- $\mathcal{U}^1$ is the space of special semimartingales that feature a (unique) absolutely continuous finite variation term and a (unique) local martingale term which is actually a martingale.

Example: jump-diffusion processes

- Consider

$$\frac{dX_t}{X_{t^-}} = \mu dt + \sigma dW_t + dH_t,$$

where H is a compound Poisson process: $H_t = \sum_{k=1}^{N_t} z_k$, where
  - N is a Poisson process independent of W with intensity λ,
  - z_k are i.i.d., independent of W and N,
  - $\mathbb{E}[z_k] = z$,
  - $z_k \geqslant -1$.

- The *compensated* Poisson process $\hat{H}_t = H_t - \lambda z t$ is a martingale. Hence,

$$\frac{dX_t}{X_{t^-}} = (\mu + \lambda z)dt + \sigma dW_t + d\hat{H}_t$$

has $\mathcal{D}X_t = (\mu + \lambda z)X_{t^-}$

Infinitesimal generator

- Let X be a *Feller* process.
- The infinitesimal generator $\mathcal{A}$ maps any continuous bounded function f belonging to dom($\mathcal{A}$) into the function $\mathcal{A}f$ such that

$$\mathcal{A}f\left(X_t\right) = \lim_{h\to 0^+} \frac{\mathbb{E}_t\left[f\left(X_{t+h}\right)\right] - f\left(X_t\right)}{h} \qquad \forall t \in [0, T].$$

The limit is in the uniform topology over all states $\omega \in \Omega$ and $\mathcal{A}f$ is continuous and bounded.

The weak time-derivative coincides with the infinitesimal generator.

Extended infinitesimal generator

- Let X be a *Markov* process.
- The extended infinitesimal generator of a measurable function f of X_t is a measurable function g such that $g(X_t)$ is integrable and the process

$$z_t = f(X_t) - f(X_0) - \int_0^t g(X_\tau) d\tau$$

is a martingale.

The weak time-derivative coincides with
the extended infinitesimal generator.

No arbitrage pricing

- Consider an arbitrage-free market with constant interest rate r, several risky securities and a bond.
- The value $B_t = e^{rt}$ of the bond satisfies

$$dB_t = rB_t dt \qquad t \in [0, T).$$

- P is the given (physical) measure.
- Q is a risk-neutral measure that makes discounted prices Q-martingales.

Weak time-derivatives and no arbitrage pricing

- Consider the price π of a marketed payoff $h_T \in L^1(\mathcal{F}_T, Q)$.

Proposition

Under Q the following conditions are equivalent:

(i) π is a no arbitrage price process;

(ii) $\mathcal{D}(\pi/B) = 0$;

(iii) $\mathcal{D}\pi = r\pi$.

- $\mathcal{D}\pi = r\pi$ generalizes the bond equation to random payoffs.

The no arbitrage pricing equation

Theorem

Under Q there exists a unique solution π in $\mathcal{U}^1$ of

$$\begin{cases} (\mathcal{D}\pi)_t = r\pi_t & t \in [0, T) \\ \pi_T = h_T & \end{cases}$$

given by

$$\pi_t = e^{-r(T-t)} \mathbb{E}_t^Q [h_T] .$$

- The proof exploits the martingale property of π / B under Q.

Example: Black-Scholes model

- Under P the bond and the risky asset follow:

$$dB_t = rB_t dt, \qquad dX_t = \mu X_t dt + \sigma X_t dW_t^P.$$

- Under Q the two securities share the same drift coefficient r:

$$dB_t = rB_t dt, \qquad dX_t = rX_t dt + \sigma X_t dW_t^Q.$$

The no arbitrage pricing equation captures
the drift change due to risk-neutrality.

Risk neutrality and discounting

- The usefulness of martingales goes beyond discounted prices under Q.
- Indeed, different ways of discounting originate different martingales.
- E.g., if interest rates are stochastics (and denoted by r_t), the previous *bond* can be replaced
  - by the *money market account* with
    - value 1 at time 0
    - value $e^{\int_0^T r_\tau d\tau}$ at time T
  - or by the *zero-coupon bond* with
    - value 1 at time T
    - value $\mathbb{E}^Q[e^{-\int_0^T r_\tau d\tau}]$ at time 0.

The forward measure

- The measure Q corresponds to discounting by the money market account.
- Discounting by zero-coupon bonds generates the *forward measure* F, which is still an equivalent martingale measure.
- Drifts of prices under different measures may be very different, although drifts of *discounted prices* are null.
- Suppose that $dr_t = \mu(t, r_t)dt + \sigma(t, r_t)dW_t^P$.
  E.g. r_t follows a Vasicek (1977), or Ornstein-Uhlenbeck, process.

Example: dynamics of zero-coupon bond prices $\pi_t(1_T)$

- By Itô's formula, the zero-coupon bond price satisfies under P

$$\frac{d\pi_t(1_T)}{\pi_t(1_T)} = \widetilde{\mu}(t, r_t)\, dt + \widetilde{\sigma}(t, r_t)\, dW_t^P.$$

- Under Q the same price follows

$$\frac{d\pi_t(1_T)}{\pi_t(1_T)} = r_t\ dt + \widetilde{\sigma}(t, r_t)\, dW_t^Q.$$

- Under F the dynamics is

$$\frac{d\pi_t(1_T)}{\pi_t(1_T)} = \left(r_t + \widetilde{\sigma}^2(t, r_t)\right) dt + \widetilde{\sigma}(t, r_t)\, dW_t^F.$$

See further details and examples in Severino (2019).

Changes of numéraires and martingales

- Martingales under the forward measure are very important: they identify *forward prices*.
- Forward prices are related to contracts that fix a price at time 0 for delivering a commodity/payoff at time T.
- Differential tools that are able to characterize martingales may be useful for studying these objects.
- Moreover, many changes of numéraires (and the related martingales) are illustrated in the option pricing literature, in very diverse contexts.

Conclusions

- The weak time-derivative captures the drift of semimartingale processes and provides a characterization of martingales.
- The no arbitrage pricing equation for random payoffs exploits the martingale property of discounted prices.
- Alternative discounting ways (together with suitable measure changes) deliver different martingales associated to asset prices.

Thank you for your attention!