Linear Algebra

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- Vectors & Vector spaces
- Matrix transformations
- Bases

Salar $\longrightarrow$ Vec ?

What's a vector?

$\mathcal{H} \longrightarrow$ Vec space

$\mathcal{L}(\mathcal{H}) : \mathcal{H} \longrightarrow \mathcal{H}$

Types of operations

* Hermitian
* Normal
* Unitary

Eigen value Decomposition

$\longrightarrow$ Need exact definitions

Why linear algebra

Group: Finite or infinite set of elements

and an operation $*$ that satisfy:

- Closure $\longrightarrow$ $\forall a,b,c \in G : a*b \in G$
- Associativity $\longrightarrow$ $(a*b)*c = a*(b*c)$
- Identity $\longrightarrow$ $\exists\, 1 : a*1 = 1*a = a$
- Inverse $\xrightarrow{\forall a}$ $\exists\, a^{-1} : a*a^{-1} = a^{-1}*a = 1$

Example: $\mathbb{Z}$, $2\mathbb{Z}$, $\mathbb{R}$ ....

Ring: $\longrightarrow$

Field: $\longrightarrow$

Vectorspace

A set V over a Field with two operations

$\cdot : F \times V \longrightarrow V$

$+ : V \times V \longrightarrow V$

such that $a, b, \ldots \in V$ & $\alpha, \beta, \ldots \in F$

$$a+b = b+a$$

$$(a+b)+c = a+(b+c)$$

$$\exists\, 0 \in V: \quad a+0 = 0+a = a$$

$$\forall a\ \exists (-a) \in V: \quad a+(-a) = (-a)+a = 0$$

— Group with respect to +

$$\alpha(a+b) = \alpha a + \alpha b$$

$$(\alpha+\beta)a = \alpha a + \beta a$$

$$\alpha(\beta a) = (\alpha\beta)a$$

$$\exists\, 1 \in F: \quad 1.a = a$$

Examples: $\mathbb{R}^2$. $\vec{v} = \begin{pmatrix}\alpha \\ \beta\end{pmatrix}$, $\vec{w} = \begin{pmatrix}\gamma \\ \delta\end{pmatrix}$

$\mathbb{R}^3$: $\vec{v} = \begin{pmatrix}\alpha \\ \beta \\ \gamma\end{pmatrix}$

$M_{m\times n}(\mathbb{R})$, $M_{m\times n}(\mathbb{C})$

$C^n([a,b])$

Inner product $\langle\ ,\ \rangle$

$\langle\ ,\ \rangle : V \times V \longrightarrow C$ such that

$*\quad \langle a, b+\alpha c\rangle = \langle a,b\rangle + \alpha\langle a,c\rangle$

$*\quad \langle a, b\rangle = \langle b, a\rangle^*$

* $\langle a, b\rangle = \langle b, a\rangle^*$

* $\langle a, a\rangle \geq 0$

* $\langle a, a\rangle = 0 \iff a = 0$

If V has an inner product, it is called an inner product vector space.

Examples:

$\vec{a}, \vec{b} \in \mathbb{R}^n$: $\langle a, b\rangle = \sum_i^n a_i^* b_i$

$\vec{a}, \vec{b} \in M_{m,n}(\mathbb{C})$: $\langle a, b\rangle = \mathrm{Tr}\left[a^\dagger b\right]$

$\vec{f(x)}, \vec{g(x)} \in C^k[a, b]$

$$\langle f, g\rangle = \int_a^b f^*(x)\, g(x)\, dx$$

Norm: $\| \quad \| : V \to \mathbb{R}$

* $\|\vec{a}\| \geq 0$

* $\|\vec{a}\| = 0 \to \vec{a} = 0$

* $\|\alpha \vec{a}\| = |\alpha| \|\vec{a}\|$

* $\|\vec{a} + \vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|$

→ A space with a norm, is called "Normed space".

Distance. $d(\vec{a},\vec{b}) : V \times V \to \mathbb{R}$

$$d(\vec{a},\vec{b}) \geq 0$$
$$d(a,b) = 0 \iff \vec{a} = \vec{b}$$
$$d(a,b) \leq d(a,c) + d(c,b)$$

Show that if V has

* an inner product, it has a norm.

* a norm, it has a distance.

Complete Vector spaces

V is complete if any Caushy Sequence in V converges to an element in V

Cauchy Sequence : $\{\vec{a}_i\} \subset V$ such that:

$\forall \varepsilon \;\; \exists N . \;\; \forall m,n \geq N$

$$\| a_m - a_n \| \leq \varepsilon$$

Banach space:

A complete vector space with a norm.

Hilbert space

A complete vector space with an inner product

Bases

Linear independence: A set of vectors $\{v_i\} \subset V$ are linearly indep. if

$$\sum_i \alpha_i v_i = 0 \Rightarrow \alpha_i = 0 \quad \forall i$$

Basis: A set of linearly indep. vectors $\{e_i\} \subset V$ such that, $\forall \vec{w} \in V \quad \exists \omega_i$

$$\vec{w} = \sum_i \omega_i \vec{e}_i$$

Orthonormal Basis:

$\forall i,j$: $\quad \langle e_i, e_j \rangle = \delta_{ij}$

For an orthonormal basis it's easy to find ω_i:

$$\langle \vec{e}_i, \vec{\omega} \rangle = \langle e_i, \sum_j \omega_j e_j \rangle = \sum_j \omega_j \langle e_i, e_j \rangle$$

$$= \omega_i$$

(A) Propose a method to construct an orthonormal Basis.

Examples:

$\mathbb{R}^3$: $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

or

$\left\{ \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

(A) Find a basis for $M_{2,3}(\mathbb{R})$

Change of basis. Take two bases: $\{e_i\}$ & $\{e'_i\}$

$\vec{w} = \sum_i \omega_i \vec{e}_i \Rightarrow \omega_i$ give a representation for $\vec{\omega}$ in $\{e_i\}$ basis.

But we can also experss it in some other basis:

$$\vec{w} = \sum_i w_i' \vec{e}_i'$$

Example:

$$\vec{w} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 1 \end{pmatrix} : \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

$(2 , 3)$

$$\vec{w} = \frac{5}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}/\sqrt{2}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}/\sqrt{2} \right\}$$

$\left(\frac{5}{\sqrt{2}} , \frac{1}{\sqrt{2}} \right)$

How do we change the basis?

Assume $\vec{w}$ in $\{e_i\}$ is given $\longrightarrow w_i$

What is the representation in $\{e_i'\}$?

Remember that since $\{e_i'\}$ is also a basis

$$e_i = \sum_j S_{ij} e_j'$$

(A) Show that $\sum_i |S_{ij}|^2 = 1$.

$$\sum_j |S_{ij}|^2 = 1$$

$$\Rightarrow \vec{w} = \sum_i w_i e_i = \sum_i \sum_j w_i S_{ij} e_j'$$

$$= \sum_j \left(\sum_i \omega_i S_{ij} \right) e_j'$$

$$\Rightarrow \omega_i' = \sum_i \omega_i S_{ij}$$

Linear Operations

$$\mathcal{L}(\mathcal{H}) : \mathcal{H} \longrightarrow \mathcal{H}$$

Linear transformation of $\vec{V} \in V$:

Assume that $\{e_i\}$ forms a basis for V.

$A \in \mathcal{L}(\mathcal{H})$:

$$A \vec{\omega} = \vec{V} \quad : \quad A \sum_i \omega_i e_i = \sum_i V_i e_i$$

$$A \cdot \omega_i \longrightarrow V_i \quad :$$

To understand the effect of A, we apply it to the basis elements:

$$A e_i = \sum_j a_{ij} e_j \Rightarrow a_{ij} = \langle e_j, A e_i \rangle$$

Example : $\mathcal{H} = \mathbb{C}^3 \Rightarrow \mathcal{L}(\mathcal{H}) = M_{3\times 3}(\mathbb{C})$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad : \quad \{e_i\} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$A e_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = e_2$$

$$A e_2 = e_1 + e_3$$

Some important subspaces of $\mathcal{L}(\mathcal{H})$

For any operator $A \in \mathcal{L}$, we can define $A^\dagger$ such that $\forall \vec{v}, \vec{w} \in \mathcal{H}$. $\langle \vec{w}, A\vec{v} \rangle = \langle A^\dagger \vec{w}, \vec{v} \rangle$.

Ⓐ What is $(A^\dagger)^\dagger$?

1. Hermitian Operators

$$A = A^\dagger \Rightarrow \langle A\vec{w}, \vec{v} \rangle = \langle \vec{w}, A\vec{v} \rangle$$

2. Unitary $\quad AA^\dagger = A^\dagger A = \mathbb{1}$

Identity

$$\Rightarrow \langle U\vec{w}, U\vec{v} \rangle = \langle \vec{w}, \vec{v} \rangle \longrightarrow \text{Preserves the inner-product.}$$

Ⓐ Show that S defined above is unitary.

3 Normal operators

$$AA^\dagger = A^\dagger A$$

Ⓐ Show that Hermitian operators are normal.

Ⓐ " " Unitary " " " "

Eigen values & Eigen vectors

Imagine that there exists a vector $\vec{v}$ such that

$$A\vec{v} = \lambda\vec{v}$$

then $\vec{v}$ is an eigenvector of A & λ is the corresponding eigenvalue

Ⓐ For an operator $A \in \mathcal{L}$, what's the possible max & min number of eigen vectors

Normal operators & Spectral decomposition

It is possible that if and only if A is normal, its eigenvectors would form a complete basis.

$$\Rightarrow A\vec{v}_i = \lambda_i \vec{v}_i \Rightarrow \{\vec{v}_i\} \text{ form a basis.}$$

$$A\vec{w} = A\sum w_i \vec{v}_i = \sum_i w_i (A\vec{v}_i) = \sum_i (w_i \lambda_i)\vec{v}_i .$$

The eigenvectors provide a convenient basis to apply the operator.

Example: $M_{3\times 3}(\mathbb{R}) \to A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

To find the eigenvectors & eigen values:

$$A\vec{v} = \lambda \vec{v} \Rightarrow (A - \lambda \mathbb{1})\vec{v} = 0$$

$$\vec{v} \neq 0 \Rightarrow A - \lambda \mathbb{1} = 0 \Rightarrow \det(A - \lambda \mathbb{1}) = 0$$

Solving this would give λ's

$A\vec{v} = \lambda \vec{v}$ gives $\vec{v}$s.

Ⓐ Do it for $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ & $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Bra-Ket notation

$\vec{v} \longrightarrow |v\rangle \in \mathcal{H}$

$\vec{v}^{\dagger} \longrightarrow \langle v| \in$ Dual space

Inner-product $\langle v, w\rangle \longrightarrow \langle v | w\rangle$

$$A = \sum_{ij} a_{ij} |i\rangle\langle j|$$

Eigenvectors $A|v\rangle = \lambda |v\rangle$

Eigenvalue decomposition $A = \sum_i \lambda_i |v_i\rangle\langle v_i|$

$$\langle \vec{v}, A\vec{w} \rangle = \langle v | Aw \rangle = \langle v | A | w \rangle$$

Hermitian $\langle \vec{v}, A\vec{w} \rangle = \langle A\vec{v}, \vec{w} \rangle \longrightarrow \langle v|A|w \rangle = \langle v|A^{\dagger}|w \rangle$