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Maxwell's Equations

→ Set of four equations
→ Describes the behavior of Both the electric and Magnetic fields as well as their interaction with.

Maxwell's four eqn's express.
→ How electric charges produce electric field → Gauss's law.
→ The absence of magnetic monopoles. (some books call it Gauss's law in magnet-ism)
→ How currents and changing electric fields produces magnetic fields ⇒ Ampere's law with Maxwell's correction "

→ How changing magnetic fields produce electric fields (Faraday's law of induction)

Electrodynamics before Maxwell's Correction.

(i) $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$ — Gauss's law

(ii) $\vec{\nabla} \cdot \vec{B} = 0$

(iii) $\vec{\nabla} \times \vec{E} = -\frac{\partial B}{\partial t}$ Faraday's Law

(iv) $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ Ampere's law.

Both Fields in terms of scalar potential V and vector potential $\vec{A}$

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$$

&

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

Apply divergence to eqn (iii)

$$\bar{\nabla}\cdot(\bar{\nabla}\times\bar{E}) = \bar{\nabla}\cdot\left(-\frac{\partial \bar{B}}{\partial t}\right) = -\frac{\partial}{\partial t}(\nabla\cdot B) \quad \text{(v)}$$

LHS is zero ∵ divergence of a curl is zero.

RHS is zero ∵ $\vec{\nabla}\cdot\vec{B} = 0$

Now apply divergence to eqn (iv),

$$\underbrace{\bar{\nabla}\cdot(\nabla\times B)}_{\text{zero}} = \mu_0\underbrace{(\bar{\nabla}\cdot\bar{J})}_{\nabla\cdot J = 0} \quad \text{(vi)}$$

for magnetostatic
i.e for steady currents

⇒ Beyond magnetostatic Ampere's law cannot be right In electrodynamics from equation of continuity (conservation of charge)

$$\Rightarrow \quad \vec{\nabla}\cdot\vec{J} = -\frac{\partial \rho}{\partial t} \quad \text{(vii)}$$

⇒ eqn (iv) is not valid for non-steady currents or in electrodynamics.

Maxwell's correction to Ampere's law.

From Gauss's law

$$\vec{\nabla}\cdot\epsilon_0\vec{E} = \rho$$

Take time derivative

$$\frac{\partial}{\partial t}\left(\bar{\nabla}\cdot\epsilon_0\bar{E}\right) = \frac{\partial\rho}{\partial t}$$

$$\Rightarrow \frac{\partial\rho}{\partial t} = \bar{\nabla}\cdot\epsilon_0\frac{\partial\bar{E}}{\partial t}$$

using this in equation we get.

$$\bar{\nabla}\cdot(\bar{\nabla}\times\bar{B}) = \mu_0(\bar{\nabla}\cdot\bar{J}) + \mu_0\nabla\cdot\left(\epsilon_0\frac{\partial E}{\partial t}\right)$$

$$= \mu_0\left(-\frac{\partial\rho}{\partial t}\right) + \mu_0\nabla\cdot\left(\epsilon_0\frac{\partial E}{\partial t}\right)$$

$$= \mu_0\left(-\frac{\partial\rho}{\partial t}\right) + \mu_0\left(\frac{\partial\rho}{\partial t}\right)$$

$$0 = 0$$

$\Rightarrow$ Ampere's law and the conservation of charge eqn. suggest that there are actually two sources of magnetic field.

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⇒ The current density and displacement current

$$\frac{\partial D}{\partial t} = \frac{\partial}{\partial t}(\epsilon_0 E) \rightarrow \text{Displacement Current.}$$

⇒ Ampere's law with Maxwells correction.

$$\vec{\nabla} \times \vec{B} = \mu_0 \bar{J} + \mu_0 \epsilon_0 \frac{\partial \bar{E}}{\partial t}$$

⇒ Maxwells fixed Ampere's law on pure theoretical arguments

⇒ In Maxwells time there was no experiment reason to doubt that Ampere's law was of wider validity.

→ Maxwells called extra term ~~as~~ the displacement current.

$$J_d \equiv \epsilon_0 \frac{\partial E}{\partial t}$$

it has nothing to do with current except that it adds to J in Amp. law

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Paradox of charging Capcitor.

→ If the capacitor plates are very close togethe — the Electric field between them is

$$E = \frac{\sigma}{\epsilon_0} = \frac{1}{\epsilon_0}\frac{Q}{A}$$

Q — charge on the plate

A — is plate's area.

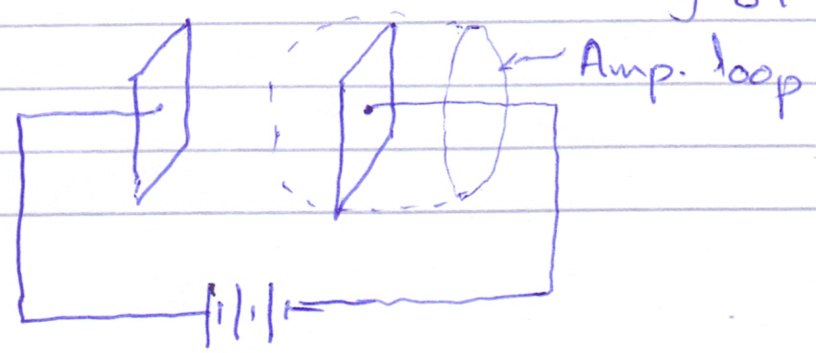
$$\Rightarrow \quad \frac{\partial E}{dt} = \frac{1}{\epsilon_0 A}\frac{dQ}{dt} = \frac{1}{\epsilon_0 A} I$$

From eqn.

$$\bar{\nabla} \times \bar{B} = \mu_0 J_{enc} + \mu_0 \epsilon_0 \frac{\partial E}{\partial t}$$

in integral form.

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc} + \mu_0 \epsilon_0 \int \frac{\partial E}{\partial t} \cdot da$$



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For flat surface $E = 0$ and $I_{enc} = I$

For balloon-shaped surface $\rightarrow I_{enc} = 0$

but $\int \frac{\partial E}{\partial t} \cdot da = \frac{I}{\epsilon_0}$

we get same answer in both cases

or for either surface.

$\rightarrow$ In first case it comes from the genuine current and ~~it~~ in the second from the displacement current.

General Form of Maxwells Equations

Differential Form		Integral Form
i) $\vec{\nabla}\cdot\bar{E} = \frac{\rho}{\epsilon_0}$	$\xrightarrow{\text{Divergence theorem}}$	$\oint \vec{E}\cdot\vec{da} = \frac{1}{\epsilon_0}\int_V \rho\, d\tau$
ii) $\vec{\nabla}\cdot\vec{B} = 0$	"	$\oint \vec{B}\cdot\vec{da} = 0$
iii) $\vec{\nabla}\times\vec{E} = -\frac{\partial \bar{B}}{\partial t}$	$\xrightarrow{\text{Stoke's theorm}}$	$\oint \vec{E}\cdot\vec{dl} = -\frac{d}{dt}\int_S \vec{B}\cdot\vec{da}$
(iv) $\bar{\nabla}\times\bar{B} = \mu_0\bar{J} + \mu_0\epsilon_0\frac{\partial\bar{E}}{\partial t}$		$\oint \vec{B}\cdot\vec{dl} = \mu_0 I_{enc} + \mu_0\epsilon_0\frac{d}{dt}\int \vec{E}\cdot\vec{da}$

Together with the Force law.

$$\vec{F} = q\bar{E} + q(\bar{v}\times\bar{B}) \quad \text{Lorentz force}$$

$\Rightarrow$ It tells how fields affect charges.

and Equation of Continuity.

$$\nabla\cdot J = -\frac{\partial \rho}{\partial t}$$

Summarize the entire theoretical content of classical electrodynamics

Maxwell's Equations:

→ Eqns. i, 2, ii, iii, in this form reinforce the notion that

⇒ Electric fields can be produced either by charges (ρ), or by changing M.F ($\frac{\partial B}{\partial t}$)

And

⇒ Magnetic fields can be produced either by currents (J) or by changing E.F ($\partial E/\partial t$)

⇒ This is misleading because ($\frac{\partial E}{\partial t}$) & ($\frac{\partial B}{\partial t}$) are themselves due to charges and currents

Logical approach can be

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} \rho$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\bar{\nabla} \times \bar{E} + \frac{\partial B}{\partial t} = 0$$

$$\bar{\nabla} \times \vec{B} - \mu_0 \epsilon_0 \frac{\partial E}{\partial t} = \mu_0 \bar{J}$$

⇒ All EM. fields are ultimatelly attributable to charges and currents

Maxwell's Equations in Vacuum.

No free charge & Current i.e ($\rho = J = 0$)

i. $\bar{\nabla} \cdot \vec{\bar{E}} = 0$

ii $\bar{\nabla} \cdot \bar{B} = 0$

iii. $\bar{\nabla} \times \bar{E} = -\dfrac{\partial \bar{B}}{\partial t}$

iv $\bar{\nabla} \times \bar{B} = \mu_0 \epsilon_0 \dfrac{\partial \bar{E}}{\partial t}$

Vacuum is a linear, homogeneous, ~~and~~ isotropic and dispersionless medium.

<u>Linear.</u>

<u>Homogeneous medium:</u> has the same properties at every point $\Rightarrow$ uniform without irregularities.

<u>Isotropic:</u> Having identical values of a property in all directions.

<u>Dispersionless medium</u>

In which all frequencies travels with same velocity. Vacuum is dispersionless for EM waves.

From Gauss's law in dielectric

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho_t}{\epsilon_0} = \frac{1}{\epsilon_0}(\rho_f + \rho_b)$$

where $\rho_f \rightarrow$ Free charges

$\rho_b \rightarrow$ bound volume charges.

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}\left(\rho_f - \bar{\nabla}\cdot\bar{P}\right)$$

where $\rho_b = -\bar{\nabla}\cdot\bar{P}$

and $\sigma_b = \vec{P}\cdot\hat{n} \longrightarrow$ surface bound charges.

$$\vec{\nabla}\cdot\epsilon_0 E + \bar{\nabla}\cdot P = \frac{\rho_f}{\cancel{\epsilon_0}}$$

$$\vec{\nabla}\cdot(\epsilon_0 E + P) = \frac{\rho_f}{\cancel{\epsilon_0}}$$

Define $D \equiv \epsilon_0 E + P$ Electric displacement vector.

$$\Rightarrow \vec{\nabla}\cdot\bar{D} = \rho_f$$

$$\oint D\cdot da = Q_{fenc}$$

Using $\vec{P} = \epsilon_0 \chi_e E$

$$\Rightarrow \vec{D} = \epsilon_0 E + \epsilon_0 \chi_e E$$

$$= \epsilon_0(1 + \chi_e)E$$

Ampere's law in Magnetized Materials.

$$\nabla \times B = \mu_0 J$$

The effect of magnetization is to establish bound currents

$\vec{J}_b = \bar{\nabla} \times \bar{M}$ within the material

and $\vec{K}_b = \vec{M} \times \hat{n}$ on the surface.

$$\Rightarrow \quad J = \vec{J}_f + \vec{J}_b$$

free current is due to wire connected to a battery

The bound current is there because of magnetization.

$$\frac{1}{\mu_0}(\bar{\nabla} \times \bar{B}) = \bar{J} = \bar{J}_f + \bar{J}_b = \bar{J}_f + (\bar{\nabla} \times \bar{M})$$

$$\bar{\nabla} \times \left(\frac{B}{\mu_0} - M\right) = \bar{J}_f \quad ; \quad H = \frac{1}{\mu_0} B - M$$

$$\nabla \times H = \bar{J}_f \quad ; \quad \oint H \cdot d\vec{l} = I_{f\,enc}$$

As $M = \chi_m H$

$$\Rightarrow \vec{B} = \mu_0 H + \mu_0 \chi_m H = \mu_0 (1 + \chi_m) H$$

$$= \mu H.$$

Maxwells Equations inside Matter.

⇒ Eqns. are modified for polarized and magnetized materials.

For linear materials

$$\vec{P} = \epsilon_0 \chi_e \vec{E}$$

$$\vec{M} = \chi_m \vec{H}$$

The fields $\vec{E}$ & $\vec{B}$ are related with $\vec{D}$ & $\vec{H}$ by.

$$\vec{D} = \epsilon_0 E + P \Rightarrow (1+\chi_e)\epsilon_0 E = \epsilon E$$

where $\frac{\epsilon}{\epsilon_0} = \epsilon_r = (1+\chi_e) \rightarrow$ Dielectric Constant.

and

$$\vec{B} = \mu_0(\vec{H} + \vec{M}) = (1+\chi_m)\mu_0 H = \mu H$$

$\chi_e \longrightarrow$ electric susceptibility of material

$\chi_m \longrightarrow$ magnetic " " "

In electrodynamics any change in the electric polarization involves a flow of bound charges resulting in polarization current, J_p.

$$\vec{J}_p = \frac{\partial \vec{P}}{\partial t}$$

Polarization current density is due to linear motion of charge when the Electric polarization changes.

$\Rightarrow$ Total charge density is

$$\rho_t = \rho_f + \rho_b$$

Total current density is

$$J_t = J_f + J_b + J_p$$

and

$$J_D = \epsilon_0 \frac{\partial E}{\partial t}$$

Maxwell's Equations Inside Matt q.

i) $\vec{\nabla}\cdot\vec{E} = \frac{\rho_t}{\epsilon_0} \longrightarrow \vec{\nabla}\cdot\vec{D} = \rho_f$ where $\bar{D} = \epsilon_0\bar{E} + \bar{P}$
& $\vec{D} = \epsilon\bar{E}$

ii) $\bar{\nabla}\cdot\bar{B} = 0$

iii) $\bar{\nabla}\times\bar{E} = -\frac{\partial B}{\partial t}$

(iv)
$$\vec{\nabla}\times\vec{B} = \mu_0 J_f + \mu_0 J_b + \mu_0 J_p + \mu_0\epsilon_0\frac{\partial\bar{E}}{\partial t}$$

$$\bar{\nabla}\times\frac{\vec{B}}{\mu_0} = \bar{J}_f + \bar{\nabla}\times\bar{M} + \frac{\partial\vec{P}}{\partial t} + \epsilon_0\frac{\partial\bar{E}}{\partial t}$$

$$\bar{\nabla}\times\frac{\bar{B}}{\mu_0} = \bar{J}_f + \bar{\nabla}\times\bar{M} + \frac{\partial}{\partial t}(\epsilon_0 E + P)$$

$$\nabla\times\left(\frac{\bar{B}}{\mu_0} - \bar{M}\right) = \bar{J}_f + \frac{\partial D}{\partial t}$$

$$\bar{\nabla}\times\bar{H} = \bar{J}_f + \frac{\partial D}{\partial t}$$

; $H = \frac{B}{\mu_0} - M$
& $B = \mu H$

In non-dispersive, isotropic media ϵ & μ are time independent ~~scalars~~ and space independent.

$\bar{\nabla}\cdot\epsilon\bar{E} = \rho$

$\bar{\nabla}\cdot\mu H = 0$

$\bar{\nabla}\times\bar{E} = -\mu\frac{\partial H}{\partial t}$

$\bar{\nabla}\times\bar{H} = \bar{J} + \epsilon\frac{\partial\bar{E}}{\partial t}$

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Electromagnetic Waves in Vacuum.

⇒ In region of free space (i,e the vacuum)

→ where no electric charges, no electric currents and no matter of any are present

→ Maxwell's equations are.

1, $\vec{\nabla} \cdot \vec{E}(\bar{r},t) = 0$

2, $\bar{\nabla} \cdot \bar{B}(\bar{r},t) = 0$

3, $\bar{\nabla} \times \bar{E}(\bar{r},t) = \frac{-\partial \bar{B}(\bar{r},t)}{\partial t}$

4, $\bar{\nabla} \times \bar{B}(\bar{r},t) = \mu_0 \epsilon_0 \frac{\partial \bar{E}(\bar{r},t)}{\partial t}$

$$= \frac{1}{c^2} \frac{\partial \bar{E}}{\partial t} \qquad \text{where } c^2 = \frac{1}{\mu_0 \epsilon_0}$$

These eqns. are set of coupled first-order partial equations

→ Can be decoupled by applying curl operator to eqns 3 & 4

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$$\nabla\times(\nabla\times E) = \nabla\times\left(-\frac{\partial B}{\partial t}\right) \qquad \nabla\times(\nabla\times B) = \nabla\times\left(\frac{1}{c^2}\frac{\partial \bar{E}}{\partial t}\right)$$

using vector identity

$$\nabla\times(\nabla\times A) = \nabla(\bar{\nabla}\cdot A) - \bar{\nabla}^2 A$$

$$\Rightarrow \bar{\nabla}(\cancel{\bar{\nabla}\cdot E}^{\,0}) - \bar{\nabla}^2 E = -\frac{\partial}{\partial t}(\nabla\times B)$$

$$-\nabla^2 \bar{E} = -\frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{\partial \bar{E}}{\partial t}\right)$$

$$\Rightarrow \quad \nabla^2 E = \frac{1}{c^2}\frac{\partial^2 \bar{E}}{\partial t^2}$$

Similarly $\nabla^2 B = \dfrac{1}{c^2}\dfrac{\partial^2 B}{\partial t^2}$

3D- de-coupled wave equations

→ Have exactly the same structure

→ Both are linear, homogenous, 2nd order differential equations.

Both eqn's have explicit dependent on space and time

$$\nabla^2 \bar{E}(\bar{r},t) - \frac{1}{c^2}\frac{\partial^2 \bar{E}(\bar{r},t)}{\partial t} = 0$$

$$\nabla^2 \bar{B}(\bar{r},t) - \frac{1}{c^2}\frac{\partial^2 \bar{B}(\bar{r},t)}{\partial t^2} = 0$$

→ Maxwell's equations implies that empty space - the vacuum - support the propagation of electromagnetic waves - at the speed of light

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 3 \times 10^8 \, m/sec$$

Monochromatic EM Plane waves

→ A plane wave is a constant freq. $\nu(\lambda)$ wave whose
→ wavefronts are infinite parallel planes.
→ Have constant amplitude normal ($\perp$) to the phase velocity vector.
→ Propagates with speed of light in vacuum.

$$c = \nu\lambda = \frac{\omega}{k}$$

Mathematical form is

$$\vec{F}(\bar{r},t) = \vec{F_0}\, e^{i(\bar{k}\cdot\bar{r} - \omega t)}$$

(19)

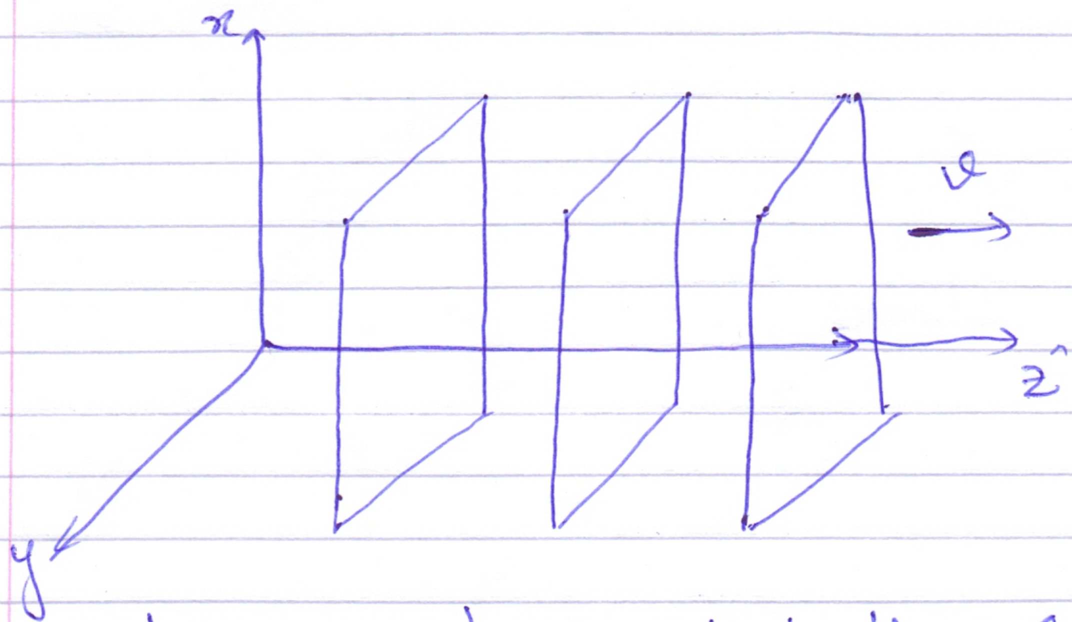
Uniform plane wave

Generally have uniform or constant properties in plane $\perp$ to their direction of propagation.

⇒ The magnitude of the electric and magnetic fields are the same at all points in the direction of propagation

⇒ The Electric & Magnetic fields are orthogonal to the direction of propagation.

⇒ EM wave that propagates in z-direction



Lie in plane $\perp$ to the $\hat{z}$-axis.

$\vec{E}$ & $\vec{B}$ are function of (z,t)

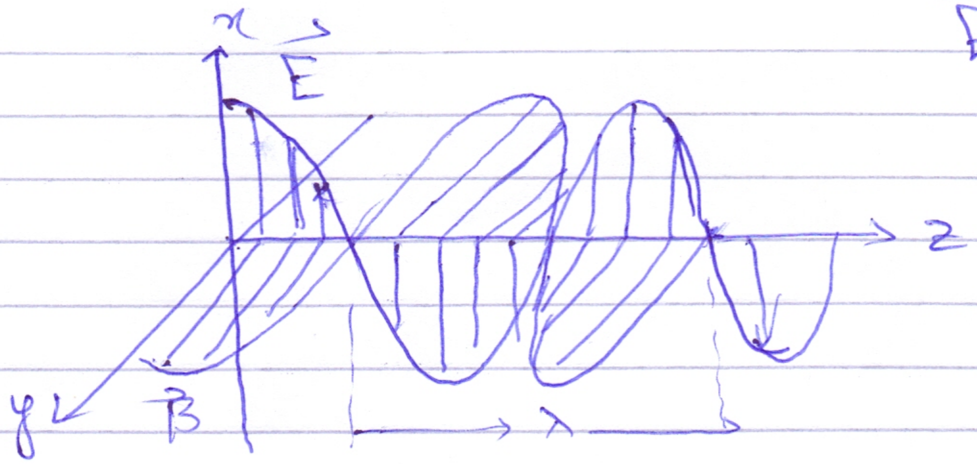
⇒ The direction of propagation is taken to be along z-axis.

⇒ The direction of propagation is normal to the plane formed by the electric & magnetic field vectors.

⇒ The phase of these fields is independent of x & y.

⇒ no phase variation exist over the planer surfaces orthogonal to the direction of the propagation.

$\vec{E} = E_0 \hat{x}$



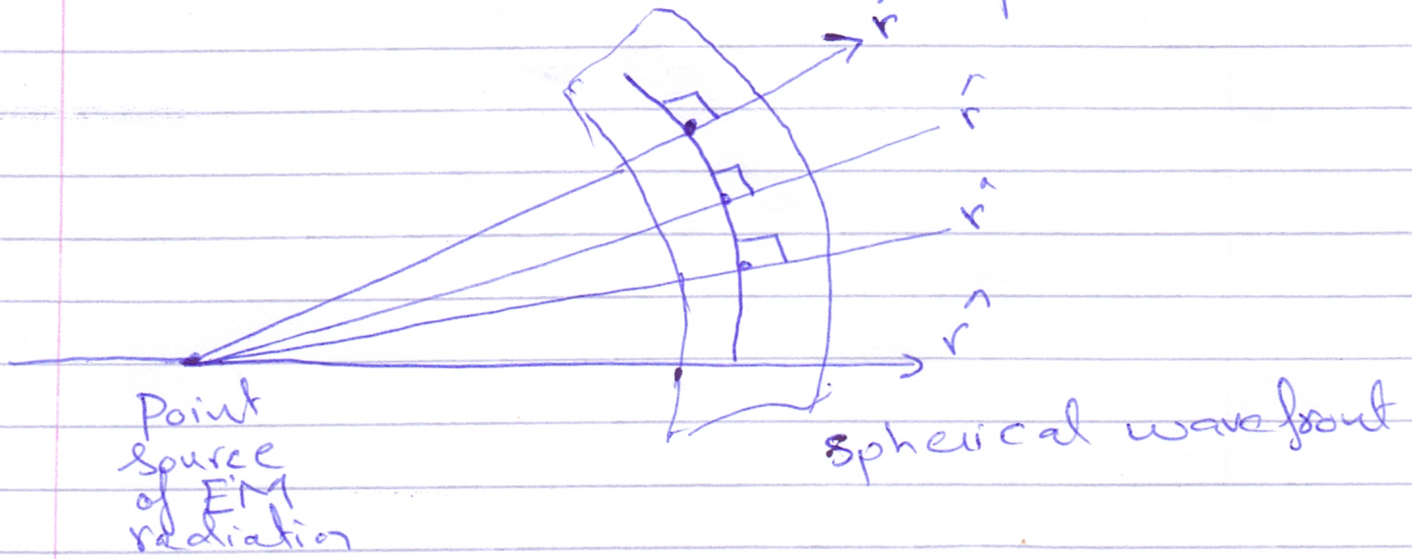
Important properties of waves are Amp, phase or frequency which allows the waves to carry information from source to destination.

$\vec{E}$ is function of $(\bar{x}, t)$ and independent of y & z.

Electomagnetic Spherical waves

→ Another possible solution of wave equation can be spherical EM waves — emitted from a point source

→ Wave-fronts associated with these EM waves are spherical.



Mathematical form

$$\vec{F}(r,t) = \frac{F_o}{r} e^{i(k \cdot r - \omega t)}$$

r – radial distance from the point source to a given pt on wave front.

~~A~~ $\frac{F_o}{r}$ — amplitude

→ If point source is infinitely far

away from field point (observer)

→ A spherical wave → plane wave in this limit ($R_c \to \infty$)

Criterion for a plance wave

$$\lambda \ll R_c$$

Monochromatic Plane waves associated with $\vec{E}$ & $\vec{B}$

Using complex notation i,e

$$e^{i\omega t} = \cos\omega t + i\sin\omega t \quad \text{Euler's eqn.}$$

$$\Rightarrow \quad \vec{\tilde{E}}(\bar{r},t) = \vec{\tilde{E}}_0\left(e^{i(k.r-\omega t)}\right)$$

$$\& \quad \vec{\tilde{B}}(\bar{r},t) = \vec{\tilde{B}}_0\left(e^{i(k.r-\omega t)}\right)$$

For wave propagating in z-direction

w $\vec{K}.\bar{r} = (k_x\hat{x} + k_y\hat{y} + k_z\hat{z})\cdot(x\hat{x} + y\hat{y} + z\hat{z})$

$= kz$

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Monochromatic EM Plane Waves

$\tilde{\vec{E}}(z,t) = \tilde{\vec{E}}_0 \, e^{i(kz-\omega t)}$

propagating in +z direc.

complex vectors

Similarly for M. field

$\tilde{\vec{B}}(z,t) = \tilde{\vec{B}}_0 \, e^{i(kz-\omega t)}$

with $\tilde{\vec{E}}_0 = \vec{E}_0 e^{i\delta_x} = E_0 e^{i\delta_x} \hat{x}$

& $\tilde{\vec{B}}_0 = \bar{B}_0 e^{i\delta_y} \hat{y}$

$\Rightarrow$ The real, physical (instantaneous) fields are:

$\vec{E}(\bar{r},t) \equiv \mathrm{Re}\left(\tilde{\vec{E}}(\bar{r},t)\right)$

& $\vec{B}(\bar{r},t) \equiv \mathrm{Re}\left(\tilde{\vec{B}}(\bar{r},t)\right)$

$\Rightarrow$ Maxwell's equations impose additional constraints on $\tilde{\vec{E}}_0$ & $\tilde{\vec{B}}_0$

As $\bar{\nabla}\cdot\bar{E} = 0$ & $\bar{\nabla}\cdot\bar{B} = 0$

$\mathrm{Re}(\bar{\nabla}\cdot\tilde{\vec{E}}) = 0$ $\quad$ $\mathrm{Re}(\bar{\nabla}\cdot\tilde{\vec{B}}) = 0$

only satisfied if:

if $\bar{\nabla} \cdot \vec{\tilde{E}} = 0$ for all $\bar{r}, t$

and $\bar{\nabla} \cdot \vec{\tilde{B}} = 0 \quad \forall (\bar{r}, t)$

In cartesian Co-ordinates.

$$\nabla \cancel{\ast} = \quad \vec{\nabla} = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}$$

$$\bar{\nabla} \cdot \vec{\tilde{E}} = 0 \quad ; \quad \bar{\nabla} \cdot \vec{\tilde{B}} = 0$$

$$\left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z} \right) \cdot \left(\vec{\tilde{E}}_0 e^{i(kz - \omega t)} \right) = 0$$

If we allow all polarization direction

$$\Rightarrow \vec{\tilde{E}}_0 = \left(E_{0x}\hat{x} + E_{0y}\hat{y} + E_{0z}\hat{z} \right) e^{i\delta} \equiv \vec{E}_0 e^{i\delta}$$

$$\bar{\nabla} \cdot \vec{\tilde{E}} = 0$$

$$\Rightarrow \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z} \right) \cdot \left(E_{0x}\hat{x} + E_{0y}\hat{y} + E_{0z}\hat{z} \right) e^{i\delta} e^{i(kz - \omega t)} = 0$$

E_{0x}, E_{0y} & E_{0z} $\Rightarrow$ Amplitudes of the E.F components in x, y, z directions

$$\Rightarrow \frac{\partial}{\partial x} \hat{x} \cdot E_{0x} \hat{x} e^{i(kz - \omega t)} e^{i\delta} = 0$$

$$\frac{\partial}{\partial y} \hat{y} \cdot E_{0y} \hat{y} e^{i(kz - \omega t)} e^{i\delta} = 0$$

$$\frac{\partial}{\partial z} \hat{z} \cdot E_{0z} \hat{z} e^{i(kz - \omega t)} e^{i\delta} = i k E_{0z} e^{i(kz - \omega t)} e^{i\delta}$$

This will = zero if and only if

$$E_{0z} \equiv 0$$

Similarly $B_{0z} \equiv 0$ iff r

$\Rightarrow$ Maxwell's eqns impose the restriction that an electromagnetic plane wave cannot have any component of $\vec{E}$ or $\vec{B}$ ~~parallel~~ parallel ~~and~~ or anti-parallel to the propagation direction.

$\Rightarrow$ EM wave is a transverse wave (at least for propagation in free space)

→ Maxwell's eqns. impose another restriction on the allowed form of $\vec{E}$ and $\vec{B}$ for an EM wave.

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \text{or} \qquad \vec{\nabla} \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

$$= \mathrm{Re}(\vec{\nabla} \times \vec{\tilde{E}}) = \mathrm{Re}\left(-\frac{\partial \vec{\tilde{B}}}{\partial t}\right)$$

It can only be satisfied if and only if.

$$\vec{\nabla} \times \vec{\tilde{E}} = -\frac{\partial \vec{\tilde{B}}}{\partial t} \qquad \text{or} \qquad \vec{\nabla} \times \vec{\tilde{B}} = \frac{1}{c^2}\frac{\partial \vec{\tilde{E}}}{\partial t}$$

$$\vec{\nabla} \times \vec{\tilde{E}} = \left(\frac{\partial \tilde{E}_z}{\partial y} - \frac{\partial \tilde{E}_y}{\partial z}\right)\hat{x} + \left(\frac{\partial \tilde{E}_x}{\partial z} - \frac{\partial \tilde{E}_z}{\partial x}\right)\hat{y} + \underbrace{\left(\frac{\partial \tilde{E}_y}{\partial x} - \frac{\partial \tilde{E}_x}{\partial y}\right)}_{\text{zero} \because \text{no } x \,\&\, y \text{ dependence in } \tilde{E}_x \,\&\, \tilde{E}_y}$$

$$= -\frac{\partial \tilde{B}_x}{\partial t}\hat{x} - \frac{\partial \tilde{B}_y}{\partial t}\hat{y} - \frac{\partial \tilde{B}_z}{\partial t}\hat{z}$$

As E_z & $B_z = 0$

$$\vec{\nabla} \times \vec{\tilde{E}} = -\frac{\partial \tilde{E}_y}{\partial z} + \frac{\partial \tilde{E}_x}{\partial z}\hat{y} = -\frac{\partial \tilde{B}_x}{\partial t}\hat{x} - \frac{\partial \tilde{B}_y}{\partial t}\hat{y}$$

hold it till next page

Complex Electric field vector $\vec{\tilde{E}}$ is given by

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$$\vec{\tilde{E}} = \tilde{E}_x \hat{x} + \tilde{E}_y \hat{y} + \cancel{\tilde{E}_z \hat{z}}^{\,0}$$

$$= \left(E_{0x} \hat{x} + E_{0y} \hat{y} \right) e^{i(kz-\omega t)} \, e^{i\delta}$$

$\tilde{E}_x = E_{0x} e^{i\delta} \times e^{i(kz-\omega t)}$

Similarly:

$$\vec{\tilde{B}} = \tilde{B}_x \hat{x} + \tilde{B}_y \hat{y} = \left(B_{0x} \hat{x} + B_{0y} \hat{y} \right) e^{i(kz-\omega t)} \, e^{i\delta}$$

$$\vec{\nabla} \times \vec{\tilde{E}} = -\frac{\partial \tilde{E}_y}{\partial z} \hat{x} + \frac{\partial \tilde{E}_x}{\partial z} \hat{y} = -\frac{\partial \tilde{B}_x}{\partial t} \hat{x} - \frac{\partial \tilde{B}_y}{\partial t} \hat{y} \quad (1)$$

$$\vec{\nabla} \times \vec{\tilde{B}} = -\frac{\partial \tilde{B}_y}{\partial t} \hat{x} + \frac{\partial \tilde{B}_x}{\partial t} \hat{y} = \frac{1}{c^2} \frac{\partial \tilde{E}_x}{\partial t} \hat{x} + \frac{1}{c^2} \frac{\partial \tilde{E}_y}{\partial t} \hat{y} \quad (2)$$

Comparing same components

x-component

$$\vec{\nabla} \times \vec{\tilde{E}} \boxed{-\frac{\partial \tilde{E}_y}{\partial z} \hat{x} = -\frac{\partial \vec{B}_x}{\partial t} \hat{x}}$$

$$\Rightarrow \quad \frac{\partial \tilde{E}_y}{\partial z} = \frac{\partial \tilde{B}_x}{\partial t} \Rightarrow ik E_{0y} e^{i(kz-\omega t)} e^{i\delta} = -i\omega B_{0x} e^{i(kz-\omega t)} e^{i\delta}$$

$$\Rightarrow \quad ik E_{0y} = -i\omega B_{0x} \quad (3)$$

For y-compt.

$$\frac{\partial E_x}{\partial z} \hat{y} = -\frac{\partial \tilde{B}_y}{\partial t} \hat{y} \Rightarrow ik E_{0x} = i\omega B_{0y} \quad (4)$$

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For $\bar{\nabla} \times \bar{B} = \mu_0 \epsilon_0 \frac{\partial \bar{E}}{\partial t}$

$\nabla \times B$: $\boxed{-\frac{\partial \tilde{B}_y}{\partial z} \hat{x} = \frac{1}{c^2} \frac{\partial E_x}{\partial t} \hat{x}} \Rightarrow -ikB_{oy} = \frac{-1}{c^2} i\omega E_{ox}$ (5)

&

$+\frac{\partial \tilde{B}_x}{\partial z} \hat{y} = \frac{1}{c^2} \frac{\partial \tilde{E}_y}{\partial t} \hat{y} \Rightarrow ikB_{ox} = -\frac{1}{c^2} i\omega E_{oy}$ (6)

From eqn 3, 4, 5 & 6 we have

$ikE_{oy} = -i\omega B_{ox} \Rightarrow E_{oy} = -\left(\frac{\omega}{k}\right) B_{ox}$ (7)

$ikE_{ox} = i\omega B_{oy} \Rightarrow E_{ox} = \left(\frac{\omega}{k}\right) B_{oy}$ - (8)

$-ikB_{oy} = -\frac{1}{c^2} i\omega E_{ox} \Rightarrow B_{oy} = \frac{1}{c^2}\left(\frac{\omega}{k}\right) E_{ox}$ - 9

$ikB_{ox} = -\frac{1}{c^2} i\omega E_{oy} \Rightarrow B_{ox} = -\frac{1}{c^2}\left(\frac{\omega}{k}\right) E_{oy}$ - (10)

As $c = f\lambda$ or $c = \nu\lambda = (2\pi f)\left(\frac{\lambda}{2\pi}\right) = \frac{\omega}{k}$

$\frac{1}{c} = \frac{k}{\omega}$ and $k = \frac{2\pi}{\lambda}$

eqn (7) in terms of $B_{o\cancel{y}x} = -\left(\frac{k}{\omega}\right) E_{oy}$

" (8) " " of $B_{o\cancel{y}} = \cancel{\tfrac{1}{c^2}}\left(\frac{\omega}{k}\right) E_{ox}$

$\vec{\nabla} \times \vec{\tilde{E}}$: $B_{ox} = -\frac{1}{c} E_{oy}$

$B_{oy} = \frac{1}{c} E_{ox}$

$\vec{\nabla} \times \vec{B}$: $B_{oy} = \frac{1}{c} E_{ox}$

$B_{ox} = -\frac{1}{c} E_{oy}$

Redundancy of relations.

Two independent relations are

$B_{ox} = -\frac{1}{c} E_{oy} \Rightarrow -\hat{x} = \hat{z} \times \hat{y}$

$B_{oy} = \frac{1}{c} E_{ox} \Rightarrow \hat{y} = \hat{z} \times \hat{x}$

In compact form.

$$\vec{\tilde{B}}_o = \frac{1}{c}\left(\hat{z} \times \vec{\tilde{E}}_o\right)$$

Physically this relation states that $\vec{E}$ & $\vec{B}$ are

$\Rightarrow$ In phase with each other.

$\Rightarrow$ Mutually $\perp$ to each other $-$ $(E \perp B) \perp \hat{z}$

The real amplitudes of $\vec{E}$ & $\vec{B}$ are related to each other.

$$B_0 = \frac{1}{c} E_0$$

where $B_0 = \sqrt{B_{0x}^2 + B_{0y}^2}$
& $E_0 = \sqrt{E_{0x}^2 + E_{0y}^2}$

Instantaneous Poynting's vector associated with an EM wave is

$$\vec{S}(\bar{r},t) = \frac{1}{\mu_0} \vec{E}(\bar{r},t) \times \vec{B}(\bar{r},t) \qquad = \frac{\text{Watt}}{m^2}$$

$$\Rightarrow \vec{S}(z,t) = \frac{1}{\mu_0} \vec{E}(z,t) \times \vec{B}(z,t)$$

$$= \frac{1}{\mu_0} Re\{\tilde{\vec{E}}(z,t)\} \times Re\{\vec{B}(z,t)\}$$

It represents the energy that is instant-taneously flowing out of boundry surface.

$\Rightarrow$ The energy transported by $\vec{E}$ & $\vec{B}$

Also called energy flux density.

For linearly polarized plane wave

propagating in z-direction

$$\vec{S}(z,t) = c\,\epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta)$$

Energy density stored in E & B

$$= \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)$$

for free space propagation

$U_E = U_M$ i.e. | energy stored in Electric field is equal to that stored in M.F

Instantaneous EM field energy contained in volume V is

$$U_{EM}(t) = \int_V u_{EM}(\bar{r},t)\, d\tau \quad (\text{Joule})$$

EM linear momentum density

$$\wp_{EM}(\bar{r},t) \equiv \epsilon_0 \mu_0 \bar{S} = \frac{1}{c^2} \bar{S}(\bar{r},t)$$

$$= \epsilon_0 \left(\bar{E}(r,t) \times \bar{B}(\bar{r},t) \right) \quad kg/m^2\text{-}sec$$

$$P_{EM}(t) \equiv \int \wp_{EM}(\bar{r},t)\, d\tau$$

EM angular momentum Density

$$\ell_{EM}(\bar{r},t) \equiv \bar{r} \times \vec{P}_{EM}$$

$$= \epsilon_0 \mu_0 \bar{r} \times S(\bar{r},t) = \frac{1}{c^2} \bar{r} \times \vec{S}(\bar{r},t)$$

$$\mathcal{L}_{EM}(t) \equiv \int_V \ell_{EM}(\bar{r},t)\, d\tau$$

$$= \epsilon_0 \int_V \left[\bar{r} \times \left[\vec{E}(\bar{r},t) \times \vec{B}(\bar{r},t) \right] \right] d\tau$$

Intensity of EM field is

$$I(\bar{r}) \equiv \langle | \vec{S}(\bar{r},t) | \rangle = c \langle u_{EM}(\bar{r},t) \rangle$$

$$= \frac{1}{2} c\, \epsilon_0 E_0^2(\bar{r}) \qquad \frac{\text{Watt}}{m^2}$$

time averaged Poynting vector