Electromagnetic Wave Propagation in Linear Media

→ EM wave is propagating inside matter.

→ There are no free charges and no free currents

→ The medium is an insulator / non conductor

Maxwell's equations become

1, $\vec{\nabla} \cdot \vec{D}(\bar{r},t) = 0$ $\qquad \vec{D} = \epsilon_0 \bar{E} + \bar{P} = \epsilon \bar{E}$

2, $\bar{\nabla} \cdot \bar{B}(\bar{r},t) = 0$ $\qquad \vec{B} = \mu_0(\bar{H} + \bar{M}) = \mu \vec{H}$

3, $\vec{\nabla} \times \vec{E}(\bar{r},t) = -\dfrac{\partial B(\bar{r},t)}{\partial t}$

4, $\bar{\nabla} \times \bar{H}(\bar{r},t) = \dfrac{\partial \vec{D}(\bar{r},t)}{\partial t}$

$P = \epsilon_0 \chi_e E$

$M = \chi_e H$

$\frac{\epsilon}{\epsilon_0} = 1 + \chi_e$

$\frac{\mu}{\mu_0} = (1 + \chi_m)$

Medium is assumed to be linear, homogeneous and isotropic

$\Rightarrow \vec{D} = \epsilon \vec{E}(\bar{r},t)$ and $\vec{H}(\bar{r},t) = \frac{1}{\mu} \vec{B}(\bar{r},t)$

and $P = \epsilon_0 \chi_e E$ and $M = \chi_e H$

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Maxwell's equation interms of $\vec{E}$ & $\vec{B}$

1, $\vec{\nabla} \cdot \vec{E}(\bar{r},t) = 0$

2, $\vec{\nabla} \cdot \vec{B}(\bar{r},t) = 0$

3, $\vec{\nabla} \times \vec{E}(\bar{r},t) = \frac{-\partial B(\bar{r},t)}{\partial t}$

4, $\vec{\nabla} \times \bar{B}(\bar{r},t) = \mu\epsilon \frac{\partial \vec{E}(\bar{r},t)}{\partial t}$

The $\vec{E}$ and $\vec{B}$ fields in medium obey the following wave equation.

$$\nabla^2 \bar{E}(\bar{r},t) = \epsilon\mu \frac{\partial^2 \bar{E}(\bar{r},t)}{\partial t^2} = \frac{1}{V_{prop}^2} \frac{\partial^2 E(\bar{r},t)}{\partial t^2}$$

$$\nabla^2 \bar{B}(\bar{r},t) = \frac{\epsilon\mu \, \partial^2 \bar{B}(\bar{r},t)}{\partial t^2} = \frac{1}{V_{prop}^2} \frac{\partial^2 B(\bar{r},t)}{\partial t^2}$$

where $V_{prop} = \frac{1}{\sqrt{\epsilon\mu}}$

is ~~V_{prop} is~~ Speed of propagation of EM wave in linear, homogeneous isotropic medium.

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For linear, homogeneous and isotropic media.

$$\epsilon = k_e \epsilon_0 = (1+\chi_e)\epsilon_0 \Rightarrow k_e = \frac{\epsilon}{\epsilon_0} = (1+\chi_e)$$

relative permittivity or dielectric constant

$$\mu = k_m \mu_0 = (1+\chi_m)\mu_0$$

$$\Rightarrow k_m = \frac{\mu}{\mu_0} = (1+\chi_m)$$ relative magnetic permeability.

as $$V_{prop} = \frac{1}{\sqrt{\epsilon\mu}} = \frac{1}{\sqrt{k_e \epsilon_0 k_m \mu_0}} = \frac{1}{\sqrt{k_e k_m}} \frac{1}{\sqrt{\epsilon_0 \mu_0}}$$

$$= \frac{1}{\sqrt{k_e k_m}} c$$ where $c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$

If $k_e k_m \geq 1$

$$\Rightarrow \frac{1}{\sqrt{k_e k_m}} \leq 1$$

$$\Rightarrow V_{prop} = \frac{1}{\sqrt{k_e k_m}} c \leq c$$

as $k_e = \frac{\epsilon}{\epsilon_0}$ and $k_m = \frac{\mu}{\mu_0}$ are dimensionless.

$\Rightarrow \frac{1}{\sqrt{k_e k_m}}$ is also dimensionless

Define the index of refraction of Linear, Homogeneous and isotropic medium as:

$$n \equiv \sqrt{k_e k_m} = \sqrt{\frac{\epsilon \mu}{\epsilon_0 \mu_0}}$$

$$\Rightarrow \qquad v_p = \frac{c}{n} \leq c$$

OR

$$c = n\, v'_{prop}$$

For many paramagnetic and dia-magnetic-type materials

$$\mu = \mu_0 (1 + \chi_m) \simeq \mu_0$$

$$\text{As } |\chi_m| \sim \theta(10^{-8}) \sim 0$$

$$\Rightarrow k_m = \frac{\mu}{\mu_0} = (1 + \chi_m) = 1$$

$$\Rightarrow n = \sqrt{k_e} \quad \& \quad v' = \frac{c}{n} = \frac{c}{\sqrt{k_e}}$$

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The plane EM wave propagating in a linear/Homogeneous/isotropic medium

E.F & M.F obeys the wave eqns.

$$\nabla^2 \bar{E}(\bar{r},t) = \epsilon\mu \frac{\partial^2 \bar{E}}{\partial t^2} = \frac{1}{v^2}\frac{\partial^2 \bar{E}}{\partial t^2}$$

&

$$\nabla^2 \bar{B}(\bar{r},t) = \epsilon\mu \frac{\partial^2 B}{\partial t^2} = \frac{1}{v^2}\frac{\partial^2 B}{\partial t^2}$$

Solutions are of the form:

$$\bar{E}(\bar{r},t) = E_0 \cos(\bar{k}\cdot\bar{r} - \omega t + \delta)\,\hat{n}$$

&

$$\bar{B}(\bar{r},t) = B_0 \cos(\bar{k}\cdot\bar{r} - \omega t + \delta)\,(\hat{k}\times\hat{n})$$

where $\hat{n}$ — polarization direction

& $\hat{k}$ — direction of propagation direction

with $\hat{n} \perp \hat{k}$

& $$\bar{B}(\bar{r},t) = \frac{1}{v}\,\hat{k}\times\bar{E}(\bar{r},t)$$

$$|\bar{B}(\bar{r},t)| = \frac{1}{v}\,|E(\bar{r},t)|$$

$$\Rightarrow B_0 = \frac{1}{v} E_0$$

The intensity of an EM plane wave in L/H/I medium is

$$I(\bar{r}) \equiv \langle |S(\bar{r},t)| \rangle = \frac{1}{2} v \epsilon E_0^2(\bar{r})$$

$$= \frac{1}{2}\left(\frac{c}{n}\right) \epsilon E_0^2(\bar{r})$$

$$U_{EM} = \frac{1}{2}\left(\epsilon E^2(\bar{r},t) + \frac{1}{\mu} B^2(\bar{r},t)\right]$$

$$= \frac{1}{2}\left(\vec{E}(\bar{r},t) \cdot \bar{D}(\bar{r},t) + B(\bar{r},t) \cdot H(\bar{r},t)\right]$$

The instantaneous linear momentum density

$$p_{EM}(\bar{r},t) = \epsilon\mu \vec{S}(\bar{r},t) = \epsilon\left[\bar{E}(\bar{r},t) \times \bar{B}(\bar{r},t)\right]$$

Angular momentum density

$$l_{EM}(\bar{r},t) = \bar{r} \times p_{EM}(\bar{r},t)$$

$$= \epsilon\, \bar{r} \times \left[\vec{E}(\bar{r},t) \times \vec{B}(\bar{r},t)\right]$$

Boundary Conditions b/w two <u>linear</u> / <u>homogeneous</u> / <u>Isotropic</u> <u>media</u>

When a wave passes from one transparent medium to another.

⇒ air ⟶ water, water ⟶ oil, glass ⟶ plastic

→ Using integral form of Maxwell's equation

1, $\oint \bar{D}\cdot\bar{da} = 0 \Rightarrow \bar{\nabla}\cdot\bar{D} = 0$

2, $\oint \bar{E}\cdot\bar{dl} = -\frac{d}{dt}\int \bar{B}\cdot\vec{da} \Rightarrow \bar{\nabla}\times\bar{E} = -\frac{\partial B}{\partial t}$

3, $\oint \bar{B}\cdot da = 0 \Rightarrow \bar{\nabla}\cdot\bar{B} = 0$

4, $\oint \bar{H}\cdot\bar{dl} = \frac{d}{dt}\int \bar{D}\cdot\bar{da} \Rightarrow \bar{\nabla}\times\bar{H} = \frac{\partial D}{\partial t}$

BC:1

Normal component of $\vec{D}$ is continuous across the interface when there is no free surface charge.

Mathematically when $\sigma_f = 0$

$$D_1^{\perp}(\bar{r},t) = D_2^{\perp}(\bar{r},t)$$

$$\epsilon_1 E_1^{\perp}(\bar{r},t) = \epsilon_2 E_2^{\perp}(r,t)$$

B.C: 2

The tangential compt of $\vec{E}$ is continous across the interface.

$$E_1^{\parallel}(\bar{r},t)\Big|_{\text{interfac}} = E_2^{\parallel}(\bar{r},t)$$

BC: 3

The normal compt of $\vec{B}$ is (always) continous across the interface.

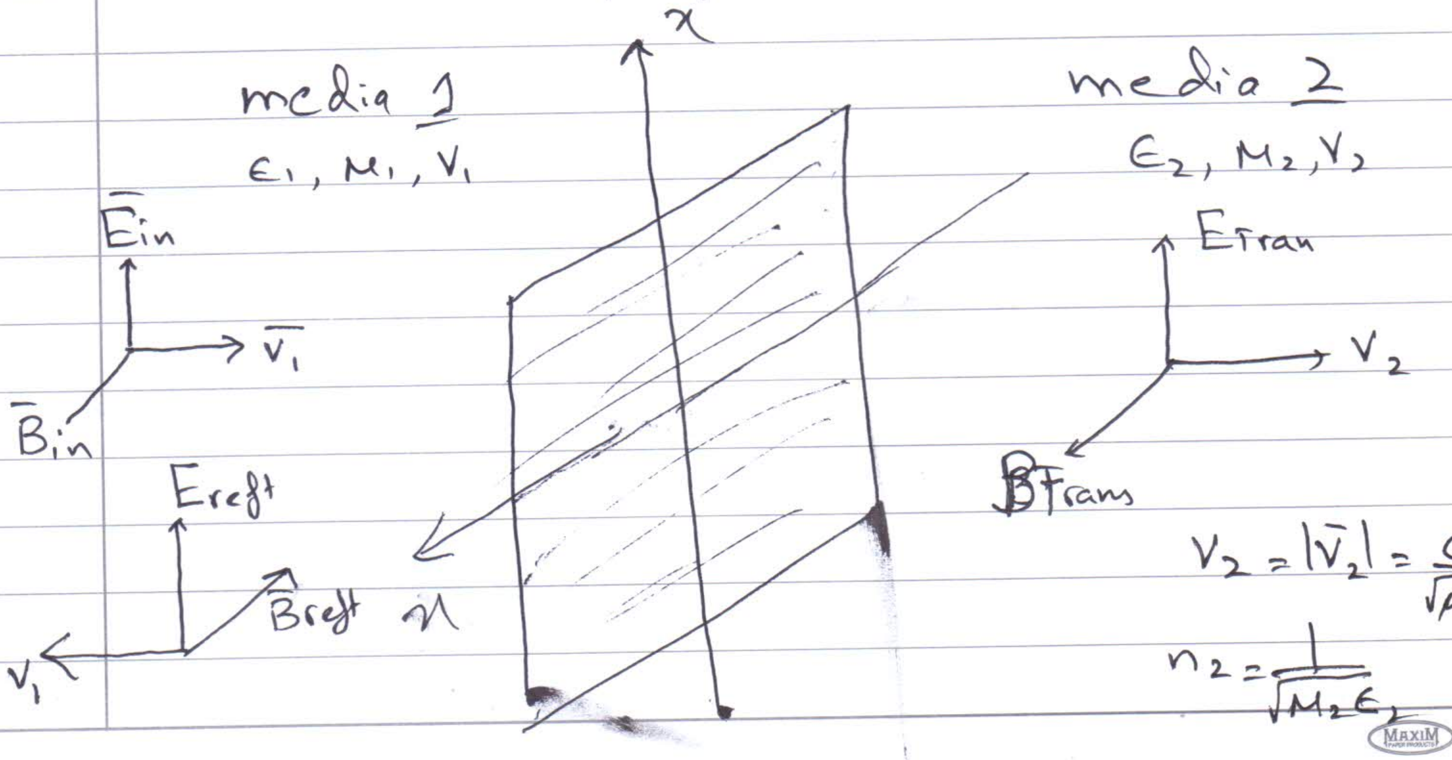
$$B_1^{\perp}\Big|_{\text{interfac}} = B_2^{\perp}\Big|$$

BC: 4 The tangential compt H is continous across the interface when there is no free surface current flowing @ the interface.

$$\Rightarrow \quad H_1^{\parallel}(\hat{r},t) = H_2^{\parallel}(\hat{r},t)$$

$$\frac{1}{\mu_1} B_1^{\parallel}(\hat{r},t) = \frac{1}{\mu_2} B_2^{\parallel}(\hat{r},t)$$

Reflection & Transmission of Linear Polarized Plane EM waves at Normal incidence at boundary b/w Two - linear/H/I media.



$$v_2 = |\bar{v}_2| = \frac{c}{\sqrt{\mu_2 \epsilon_2}}$$

$$n_2 = \frac{1}{\sqrt{\mu_2 \epsilon_2}}$$

$\Rightarrow$ Inteface lies in x-y plane.

$\Rightarrow$ EM wave of freq ω is polarized in $\hat{x}$-direction

$\Rightarrow$ Propagating in z-direction.

$\Rightarrow$ Incident wave in medium 1:

$\rightarrow$ Propagation direction $\hat{z}$-direction $\Rightarrow \hat{k} = \hat{k}_1 = \hat{z}$
Polarization $\hat{n}_{in} = \hat{x}$

$$\vec{\tilde{E}}_{in}(z,t) = \tilde{E}_{0in}\, e^{i(k_1 z - \omega t)}\, \hat{x}$$

$$|\vec{k}_1| = \frac{2\pi}{\lambda_1} = \frac{\omega}{v_1}$$

$$\Rightarrow \vec{\tilde{B}}_{in}(z,t) = \frac{1}{v_1}\, \hat{k}_1 \times \vec{\tilde{E}}_{in}(z,t)$$

$$= \frac{1}{v_1} \underbrace{(\hat{z} \times \hat{x})}_{\hat{y}}\, \tilde{E}_{0in}\, e^{i(k_1 z - \omega t)}$$

$$= \frac{1}{v_1}\, \hat{y}\, \tilde{E}_{0in}\, e^{i(k_1 z - \omega t)}$$

Reflected wave in medium 1.

Propagation direction $(-\hat{z}) \Rightarrow \hat{k}_{ref} = -\hat{k}_1 = -\hat{z}$
Polarization " $\hat{n} = \hat{x}$

$$\tilde{\vec{E}}_{reft} = \tilde{E}_{orefr}\, e^{i(-k_1 z - \omega t)}\, \hat{x}$$

$$\Rightarrow \tilde{\vec{B}}_{ref} = \frac{1}{v_1}\left(-\hat{z}_z \times \hat{x}\right) e^{i(-k_1 z - \omega t)} \qquad \hat{y}$$

$$= \frac{1}{v_1}\left(-\hat{y}\right) e^{i(-k_1 z - \omega t)}$$

Transmitted EM plane wave (medium 2)

$\Rightarrow$ Propagation in $+\hat{z}$ $\Rightarrow$ $\hat{k}_{trans} = \hat{k}_2 = +\hat{z}$

Polarization $\hat{n}_{trans} = \hat{x}$ $\quad\Big|\; k_2 = \frac{2\pi}{\lambda_2}$

$$\tilde{\vec{E}}_{trans}(z,t) = \tilde{E}_{o_{trans}}\, e^{i(k_2 z - \omega t)}\, \hat{x}$$

$$\tilde{\vec{B}}_{trans}(z,t) = \frac{1}{v_2}\left[\hat{k}_{trans} \times \tilde{\vec{E}}_{trans}(z,t)\right]$$

$$= \frac{1}{v_2}\left(\hat{z} \times \hat{x}\right)\tilde{E}_0 e^{i(k_2 z - \omega t)}$$

$$= \frac{1}{v_2}\, \tilde{E}_{o_{trans}}\, e^{i(k_2 z - \omega t)}\, \hat{y}$$

$\Rightarrow$ $\hat{n}_{inc} = \hat{n}_{ref} = \hat{n}_{trans} = \hat{x}$

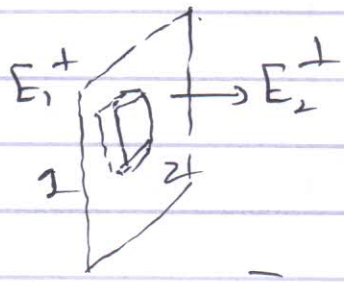
$E_{in}(z,t) \parallel E_{ref}(z,t) \parallel E_{tran.}(z,t)$

At the interface b/w the two media at $z=0$ (in the x-y plane) the B.C's (1-4) must be satisfied for total $\bar{E}$ & $\bar{B}$ immediately present on either side of the Interface.

BC:1 Normal compt. of D is continous.

$$D_1^{\perp} = D_2^{\perp} \Rightarrow \epsilon_1 E_{1_{tot}}^{\perp} = \epsilon_2 E_{2_{tot}}^{\perp}$$

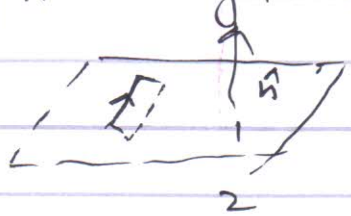
in z-direction



BC:2 Tangential componets of $\bar{E}$ is continous

$$E_{1_{tot}}^{\parallel} = E_{2_{tot}}^{\parallel}$$

$\parallel$ — x-y plane



BC: 3 Normal component of $\bar{B}$ is continous

$$\bar{B}_{1_{Tot}}^{\perp} = B_{2_{tot}}^{\perp}.$$

(~~in~~ z-direct-

BC:4 Tangential $\bar{H}$ is continous.

$$\frac{1}{\mu_1} B^{\parallel}_{1_{tot}} = \frac{1}{\mu_2} B^{\parallel}_{2_{tot}}$$

For plane E.M wave at normal incidence on interface at $z=0$ lying in x-y plane, no compt. of $\bar{E}$ & $\bar{B}$ are allowed to be along $\pm z$ - propagation direction.

$\Rightarrow$ The BC:1 & BC:3 impose no restriction on such EM wave.

As $E^{\perp}_{1tot} = E^{z}_{1tot} = 0 \quad E^{\perp}_{2tot} = E^{z}_{2tot} = 0$

and $B^{\perp}_{1tot} = B^{z}_{1tot} = 0 \quad B^{\perp}_{2tot} = B^{z}_{2tot} = 0$

$\Rightarrow$ Restrictions are imposed by BC: 2 & BC:4.

At $z=0$ at interface:

BC: 2 requires that

$$\vec{\tilde{E}}_I(z=0,t) + \vec{\tilde{E}}_R(z=0,t) = \vec{\tilde{E}}_T(z=0,t) \quad (A)$$

BC: 4

$$\frac{1}{\mu_1}\vec{\tilde{B}}_I(z=0,t) + \frac{1}{\mu_1}\vec{\tilde{B}}_R(z=0,t) = \frac{1}{\mu_2}\vec{\tilde{B}}_T(z=0,t) \quad (B)$$

Putting explicit expressions for the complex fields

$$\tilde{E}_{0I}\, e^{i(k_1 z-\omega t)}\hat{x} + \tilde{E}_{0R}\, e^{i(-k_1 z-\omega t)}\hat{x}\Big|_{z=0} = \tilde{E}_{0T}\, e^{i(k_2 z-\omega t)}\Big|_{z=0}$$

& eq (B) in terms of $\vec{E}$ fields

$$\frac{1}{\mu_1 v_1}\tilde{E}_{0I}\, e^{i(k_1 z-\omega t)}\hat{y} + \frac{1}{\mu_1 v_1}\tilde{E}_{0R}\, e^{i(-k_1 z-\omega t)}(-\hat{y})\Big|_{z=0} =$$

$$\frac{1}{\mu_2 v_2}\tilde{E}_{0T}\, e^{i(k_2 z-\omega t}\hat{y}$$

BC: 2 at $z=0$

$$\tilde{E}_{0I}\,\cancel{e^{-i\omega t}} + \tilde{E}_{0R}\,\cancel{e^{-i\omega t}} = \tilde{E}_{0T}\,\cancel{e^{-i\omega t}}$$

$$\Rightarrow \quad \tilde{E}_{0I} + \tilde{E}_{0R} = \tilde{E}_{0T}$$

BC: 4 at $z=0$

$$\frac{1}{\mu_1 v_1}\tilde{E}_{0I}\cancel{e^{-i\omega t}} - \frac{1}{\mu_1 v_1}\tilde{E}_{0R}\cancel{e^{-i\omega t}} = \frac{1}{\mu_2 v_2}\tilde{E}_{0T}\cancel{e^{-i\omega t}}$$

$$\Rightarrow \frac{1}{\mu_1 v_1}\tilde{E}_{0I} - \frac{1}{\mu_1 v_1}\tilde{E}_{0R} = \frac{1}{\mu_2 v_2}\tilde{E}_{0T}$$

Assuming μ_1, μ_2, v_1 & v_2 are known.

$$\Rightarrow \tilde{E}_{0I} + \tilde{E}_{0\,ref} = \tilde{E}_{0T} \qquad (BC:2) \quad a)$$

and $\tilde{E}_{0I} - \tilde{E}_{0R} = \frac{\mu_1 v_1}{\mu_2 v_2}\tilde{E}_{0T} \quad (BC: 4) \quad b)$

where $\beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 \frac{c}{n_1}}{\mu_2 c/n_2} = \frac{\mu_1 n_2}{\mu_2 n_1}$

Solving eqn's a and b for $\tilde{E}_{0R}$ & $\tilde{E}_{0T}$ in terms of $\tilde{E}_{0I}$

$$\Rightarrow \tilde{E}_{0R} = \frac{(1-\beta)}{(1+\beta)}\tilde{E}_{0I}$$

$$\& \quad \tilde{E}_{0T} = \frac{2}{(1+\beta)}$$

as $n_1 = \frac{c}{v_1} = \sqrt{\frac{\epsilon_1 \mu_1}{\epsilon_0 \mu_0}}$

$n_2 = \sqrt{\frac{\epsilon_2 \mu_2}{\epsilon_0 \mu_0}}$

For paramagnetic or diamagnetic $\chi_m \ll 1$

$\Rightarrow \mu_1 = \mu_2 \cong \mu_0$

$$\Rightarrow \beta = \frac{\mu_1 v_1}{\mu_2 v_2} \simeq \frac{v_1}{v_2} = \frac{n_2}{n_1}$$

$$\Rightarrow \tilde{E}_{0R} = \frac{(1-\beta)}{(1+\beta)} \tilde{E}_{0I} \simeq \frac{(v_2 - v_1)}{(v_2 + v_1)} \tilde{E}_{0I}$$

$$\& \quad \tilde{E}_{0T} \cong \frac{2v_2}{(v_2+v_1)} \tilde{E}_{0I}$$

As $\tilde{E}_{0I} = E_{0I} e^{i\delta}$

$\tilde{E}_{0R} = E_{0R} e^{i\delta}$

& $\tilde{E}_{0T} = E_{0T} e^{i\delta}$

For real amplitudes:

$$E_{0R} = \left(\frac{v_2 - v_1}{v_2 - v_1}\right) E_{0I} = \left(\frac{n_1 - n_2}{n_1 + n_2}\right) E_{0I}$$

&

$$E_{0T} = \left(\frac{2v_2}{v_2+v_1}\right) E_{0I} = \left(\frac{2n_1}{n_1+n_2}\right) E_{0I}$$

under $\mu_1 = \mu_2 = \mu_0$

The reflected wave is in phase if $v_2 > v_1$ and out of phase if $v_2 < v_1$. The transmitted wave is always in phase with Incident wave.

The real amplitudes are

$$E_{0R} = \left| \frac{v_2 - v_1}{v_2 + v_1} \right| E_{0I}$$

$$E_{0T} = \left(\frac{2 v_2}{v_1 + v_2} \right) E_{0I}$$

In terms of indices of refraction:

$$E_{0R} = \left| \frac{n_1 - n_2}{n_1 + n_2} \right| E_{0I} \; ; \; E_{0T} = \left(\frac{2 n_1}{n_1 + n_2} \right) E_{0I}$$

Fresnel equations for normal incident

$$\frac{E_{0R}}{E_{0I}} = \left| \frac{n_1 - n_2}{n_1 + n_2} \right| \; ; \; \frac{E_{0T}}{E_{0I}} = \left(\frac{2 n_1}{n_1 + n_2} \right)$$

Intensity of EM wave is

$$I(\bar{r}) = \langle |S(\bar{r},t)| \rangle = v \langle u_{EM}(\hat{r},t) \rangle$$

$$= \frac{1}{2} \epsilon v E_0^2$$

Average power per unit area.

Reflection co-eff. is defined as

$$R(\bar{r}) \equiv \left(\frac{I_R(\bar{r})}{I_I(r)} \right) = \frac{\frac{1}{2} \epsilon_1 v_1 E_{0R}^2}{\frac{1}{2} \epsilon_1 v_1 E_{0I}^2} = \frac{E_{0R}^2}{E_{0I}^2}$$

& Transmission co-eff as

$$T(\bar{r}) = \left(\frac{I_T(\bar{r})}{I_I(\bar{r})} \right) = \frac{\frac{1}{2} \epsilon_2 v_2 E_{0T}^2}{\frac{1}{2} \epsilon_1 v_1 E_{0I}^2} = \frac{\epsilon_2 v_2}{\epsilon_1 v_1} \frac{E_{0T}^2}{E_{0I}^2}$$

Reflection and Transmission in terms of refractive index

$$R(\bar{r}) \equiv \left(\frac{1-\beta}{1+\beta} \right)^2 \simeq \left(\frac{v_2 - v_1}{v_2 + v_1} \right)^2 = \left(\frac{n_1 - n_2}{n_1 + n_2} \right)^2$$

$$T(\bar{r}) = \left(\frac{\epsilon_2 v_2}{\epsilon_1 v_1} \right) \left(\frac{2}{1+\beta} \right)^2 = \left(\frac{\epsilon_2 v_2}{\epsilon_1 v_1} \right) \left(\frac{2 v_2}{v_2 + v_1} \right) = \left(\frac{\epsilon_2 v_2}{\epsilon_1 v_1} \right) \left(\frac{2 n_1}{n_1 + n_2} \right)$$

$$R(r) + T(r) = 1$$

$\Rightarrow$ EM energy is conserved at the boundary b/w two L/H/I media

Ex: A monochromatic plane EM wave is incident on an air-glass interface at normal incidence, index of refraction are

$$n_1 = n_{air} \approx 1.0$$
$$n_2 = n_{glass} \simeq 1.5$$

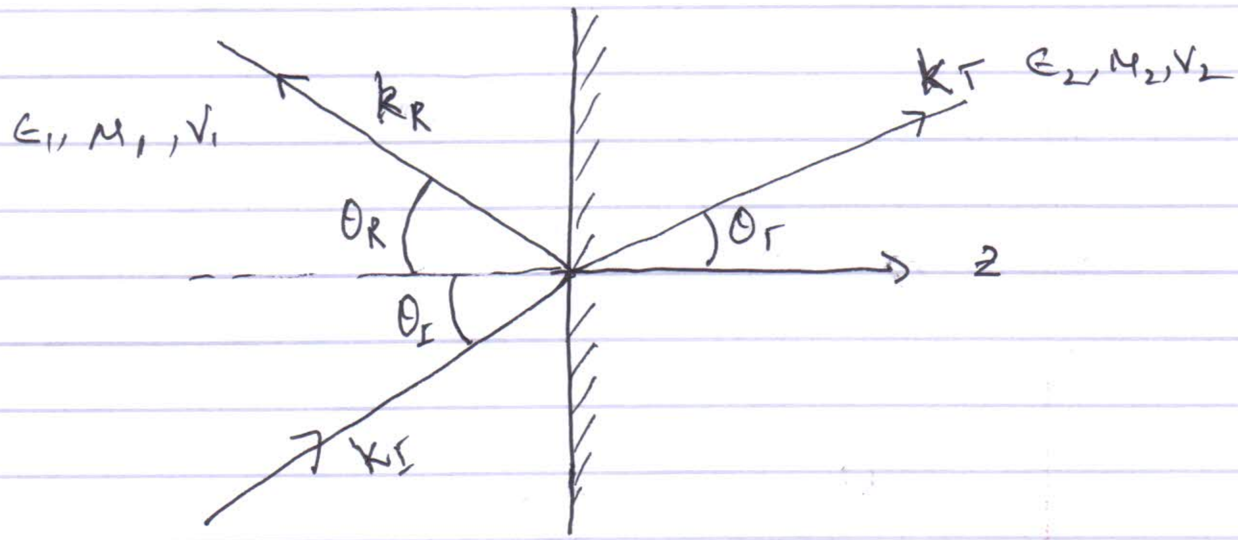
$$R = \left[\frac{n_1 - n_2}{n_1 + n_2}\right] = \frac{1}{25} = 0.04 = 4\%$$

$$T = \frac{4 n_1 n_2}{(n_1 + n_2)^2} = \frac{6.0}{6.25} = 0.96 = 96\%$$

$$\Rightarrow \quad R + T = 0.04 + 0.96 = 1.00$$

Reflection and Transmission at Oblique Incident.

⇒ A monochromatic plane EM wave incident at an oblique angle θ_I on a boundary b/w two linear/isotopic/homogenous media. (x-y plane)



The incident EM is

$$\tilde{\vec{E}}_I(\bar{r},t) = \tilde{\vec{E}}_{0I}\, e^{i(\bar{k}_I \cdot \bar{r} - \omega t)}$$

$$\tilde{\vec{B}}_I(\bar{r},t) = \frac{1}{v_1}\left(\hat{k}_I \times \tilde{\vec{E}}_I(r,t)\right)$$

$\tilde{\vec{E}}_{0I} = \hat{n}\tilde{E}_{0I}$
$= \hat{n} E_{0I} e^{i\ldots}$

The reflected EM wave is

$$\tilde{\vec{E}}_R(\bar{r},t) = \tilde{\vec{E}}_{oR}\, e^{i(\bar{k}_R \cdot \bar{r} - \omega t)}$$

&

$$\tilde{\vec{B}}_R(\bar{r},t) = \frac{1}{v_1}\left(\hat{k}_R \times \tilde{\vec{E}}_R\right)$$

The Transmitted EM wave is

$$\tilde{\vec{E}}_T(\bar{r},t) = \tilde{\vec{E}}_{oT}\, e^{i(\vec{k}_T \cdot \bar{r} - \omega t)}$$

&

$$\tilde{\vec{B}}_T(\bar{r},t) = \frac{1}{v_2}\left(\hat{k}_T \times \tilde{\vec{E}}_T(\bar{r},t)\right)$$

$\Rightarrow$ All the three waves have the same frequenies $f = \frac{\omega}{2\pi} \Rightarrow f_1 = f_2 = f$

$\Rightarrow \omega_1 = \omega_2 = \omega \Rightarrow k_1 v_1 = k_2 v_2$

$\Rightarrow \omega = k_I v_1 = k_R v_1 = k_T v_2$

with $k_I = |\bar{k}_I| = \frac{2\pi}{\lambda_1}$; $k_R = |\bar{k}_R| = \frac{2\pi}{\lambda_1}$

& $k_T = |\vec{k}_T| = \frac{2\pi}{\lambda_2}$

$$\omega = \omega_1 = \omega_2 = 2\pi\left(\frac{v_1}{\lambda_1}\right) = 2\pi\left(\frac{v_2}{\lambda_2}\right)$$

$\because \omega = vk$

$\& \; k = \frac{2\pi}{\lambda}$

$k_0 = \frac{\omega}{c}$

$$\Rightarrow \omega = 2\pi f_I = 2\pi f_R = 2\pi f_T$$

$$\Rightarrow f_I = f_R = f_T$$

Then $\lambda_1 = \frac{\lambda_0}{n_1}$ & $\lambda_2 = \frac{\lambda_0}{n_2}$

λ_0 — vacuum wave length $= c/f$

& $v_1 = \frac{c}{n_1}$ & $v_2 = \frac{c}{n_2}$

$k_0 = \frac{2\pi}{\lambda_0}$

$$\Rightarrow k_1 = \frac{2\pi}{\lambda_1} = \frac{2\pi n_1}{\lambda_0} = k_0 n_1$$

& $k_2 = n_2 k_0$

From

$$\omega = k_I v_1 = k_R v_1 = k_T v_2$$

$$\Rightarrow k_I = k_R = k_1 = \frac{v_2}{v_1} k_T = \left(\frac{v_2}{v_1}\right) k_2$$

As $v_i = \frac{c}{n_i}$

$$\Rightarrow k_1 = \left(\frac{c/n_2}{c/n_1}\right) k_2 = \left(\frac{n_1}{n_2}\right) k_2$$

The total EM field in medium I

$$\tilde{\vec{E}}_{Tot\,1}(\bar{r},t) = \tilde{\vec{E}}_{In} + \tilde{\vec{E}}_R$$

$$\& \quad \tilde{\vec{B}}_{T1}(\bar{r},t) = \tilde{\vec{B}}_I + \tilde{\vec{B}}_R$$

Must match the total fields in 2

$$\tilde{\vec{E}}_{Tot\,2}(\bar{r},t) = \tilde{\vec{E}}_T(\bar{r},t)$$

$$\& \quad \tilde{\vec{B}}_{Tot_2}(\bar{r},t) = \tilde{\vec{B}}_T(\bar{r},t)$$

Using B.C (1-4), all shares the structure

$$(\quad)e^{i(k_I\cdot r - \omega t)} + (\quad)e^{i(k_R\cdot r - \omega t)} = (\quad)e^{i(k_T\cdot r - \omega t)}$$

BC's must hold for all (x,y) on the interface at $z=0$ and for all time.

As $e^{-i\omega t}$ is common in all terms. (freqs are equal)

$\Rightarrow$ Holds for all arbitrary time, t. ~~since~~

Following relation must hold for all (x, y) on interface at $z=0$

$$(\quad) e^{i\bar{k}_I \cdot \bar{r}} + (\quad) e^{i\bar{k}_R \cdot \bar{r}} = (\quad) e^{i\bar{k}_T \cdot \bar{r}}$$

when $z=0$ we must have

$$\vec{k}_I \cdot \bar{r} = \bar{k}_R \cdot \bar{r} = \bar{k}_T \cdot \bar{r}$$

more explicitly.

$$k_{I_x} x + k_{I_y} y = k_{R_x} x + k_{R_y} y = k_{T_x} x + k_{T_y} y \quad @ z=0$$

It holds for any arbitrary $(x, y, z=0)$ if & only if.

$$k_{I_x} = k_{R_x} = k_{T_x}$$

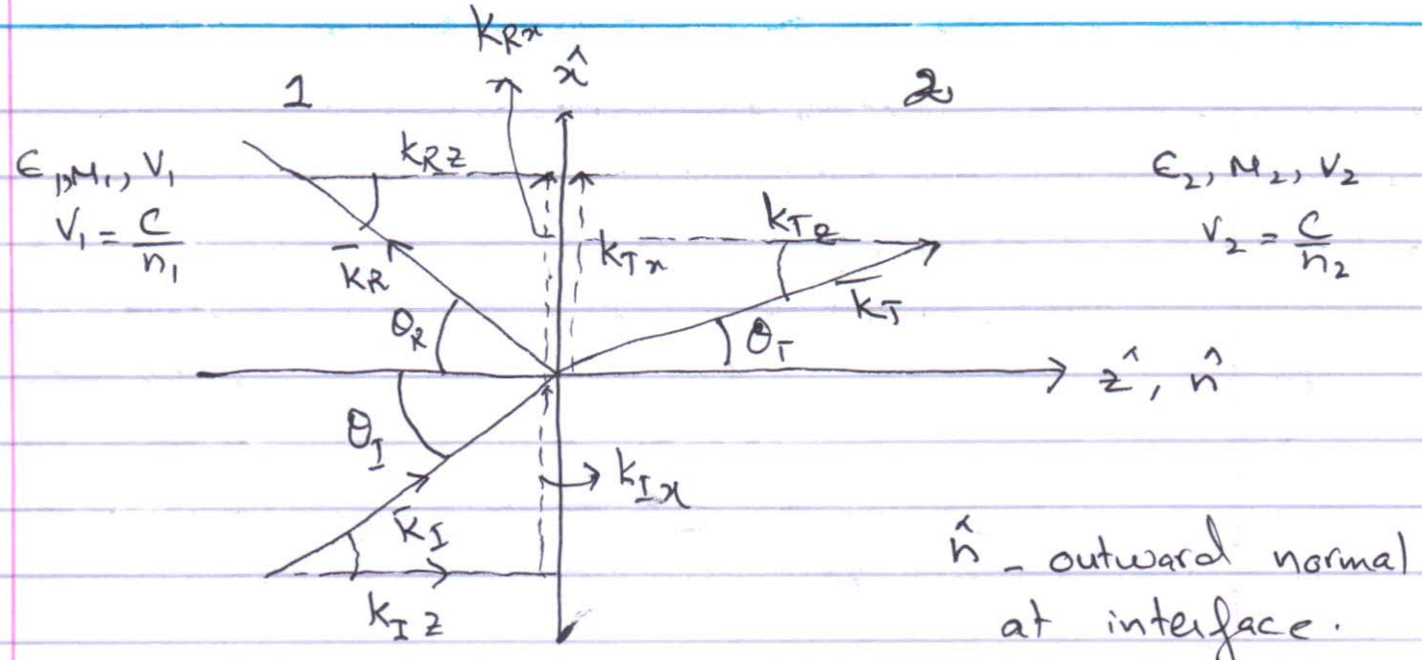
&

$$k_{I_y} = k_{R_y} = k_{T_y}$$

$\Rightarrow$ Problem has rotational symmetry (invariance about z-axis)

$\Rightarrow$ Make k_I to lies in x-z plane.

$$\Rightarrow \quad k_{I_y} = k_{R_y} = k_{T_y} = 0$$



$\hat{n}$ - outward normal at interface.

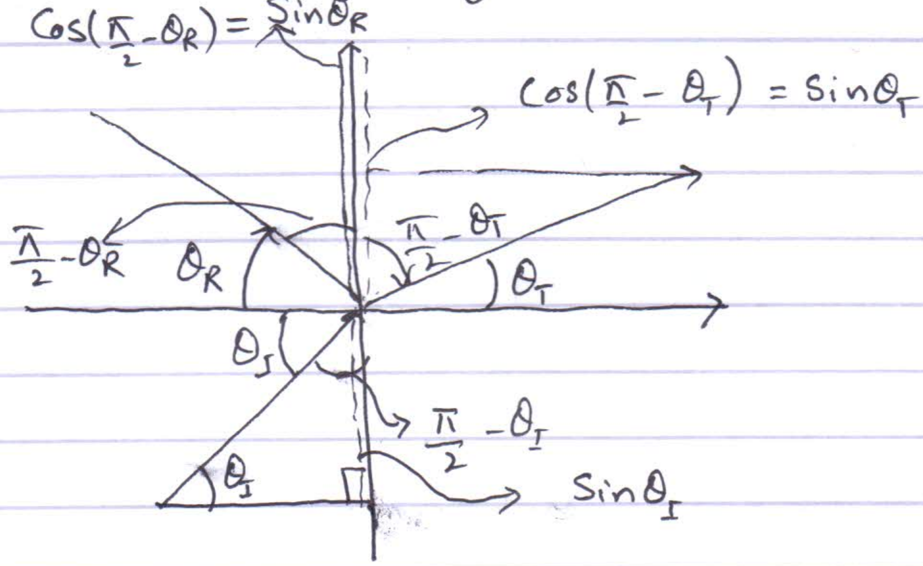
The First Law of Geometrical optics

⇒ The incident, reflected and transmitted wave vectors all lie in a plane known as plane of Incidence. (here the x-z plane)

⇒ The plane of incidence also includes the unit normal to the interface.

$$\hat{n} = +\hat{z}.$$

The second law of Geometrical optics
(Law of Reflection)



From figure we can see.

$$k_{Ix} = k_1 \sin\theta_I$$

$$k_{Rx} = k_R \sin\theta_R$$

$$k_{Tx} = k_T \sin\theta_T$$

But $k_I = k_R = k_1$ as $[(k_I)_x = (k_R)_x = (k_T)_x]$

$\Rightarrow \sin\theta_I = \sin\theta_R$

$\Rightarrow$ Angle of Incident = Angle of Reflection.

$$\theta_I = \theta_R$$

Law of Reflection.

The third law of Geometrical optics

Law of Reflection – Snell's law.

For transmitted angle θ_T

As $\quad k_I \sin\theta_I = k_T \sin\theta_T$

In medium 1: $\quad k_I = k_1 = \frac{\omega}{v_1} = \frac{n_1 \omega}{c} = n_1 k_0$

In medium 2: $\quad k_T = k_2 = \frac{\omega}{v_2} = \frac{n_2 \omega}{c} = n_2 k_0$

$$\Rightarrow \quad k_1 \sin\theta_I = k_2 \sin\theta_T$$

$$n_1 k_0 \sin\theta_I = n_2 k_0 \sin\theta_T$$

$$\Rightarrow \quad n_1 \sin\theta_I = n_2 \sin\theta_T$$

Transmitted

$$\frac{\sin\theta_T}{\sin\theta_I} = \frac{n_1}{n_2} \quad ; \quad \frac{\sin\theta_2}{\sin\theta_1} = \frac{n_1}{n_2}$$

Incident.

Snell's law

From three laws of Geometrical optics

$$\vec{k}_I \cdot \bar{r}\Big|_{z=0} = \bar{k}_R \cdot \bar{r}\Big|_{z=0} = \vec{k}_T \cdot \bar{r}\Big|_{z=0}$$

$$\Rightarrow e^{i(\bar{k}_I \cdot \bar{r} - \omega t)}\Big|_{z=0} = e^{i(\bar{k}_R \cdot \bar{r} - \omega t)}\Big|_{z=0} = e^{i(\bar{k}_T \cdot \bar{r} - \omega t)}\Big|_{z=0}$$

Everywhere on interface at $z = 0$

The B.C's BC: 1 — BC: 4, for a monochromatic plane EM wave incident ~~at z=0~~ on an interface at an oblique angle θ_I b/w two L/H/I media becomes:

BC: 1

$$\epsilon_1\left(\tilde{E}_{0I_z} + \tilde{E}_{0R_z}\right) = \epsilon_2 \tilde{E}_{0T_z}$$

BC: 2

$$\tilde{E}_{0I_{x,y}} + \tilde{E}_{0R_{x,y}} = \tilde{E}_{0T_{x,y}}$$

BC: 3 $\tilde{B}_{0I_z} + \tilde{B}_{0R_z} = \tilde{B}_{0T_z}$

BC: 4 $\frac{1}{\mu_1}\left[\tilde{B}_{0I_{x,y}} + \tilde{B}_{0R_{x,y}}\right] = \frac{1}{\mu_2}\tilde{B}_{0T_{xy}}$

with

$$\vec{\tilde{B}}_0 = \frac{1}{v}\hat{k} \times \vec{\tilde{E}}_0$$

For a monochromatic plane EM wave incident on a boundary b/w two linear/H/I media at an oblique angle of incidence, there are three possible polarization cases to consider:

CASE #1

$\vec{E}_I \perp$ to plane of incidence – known as Transverse Electric (TE) polarization. (S-polarization)

$\Rightarrow \bar{B}_I \parallel$ to plane of incidence.

CASE #2

$\vec{E}_I \parallel$ to plane of incidence – known as Transverse magnetic (TM) polarization. (P-polarization)

CASE #3

$\vec{E}_I$ is neither $\perp$ nor $\parallel$ to plane of incidence

{ B is neither $\parallel$ nor $\perp$ to plane of incidence)

Polarization for general case

$$\hat{n} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$$

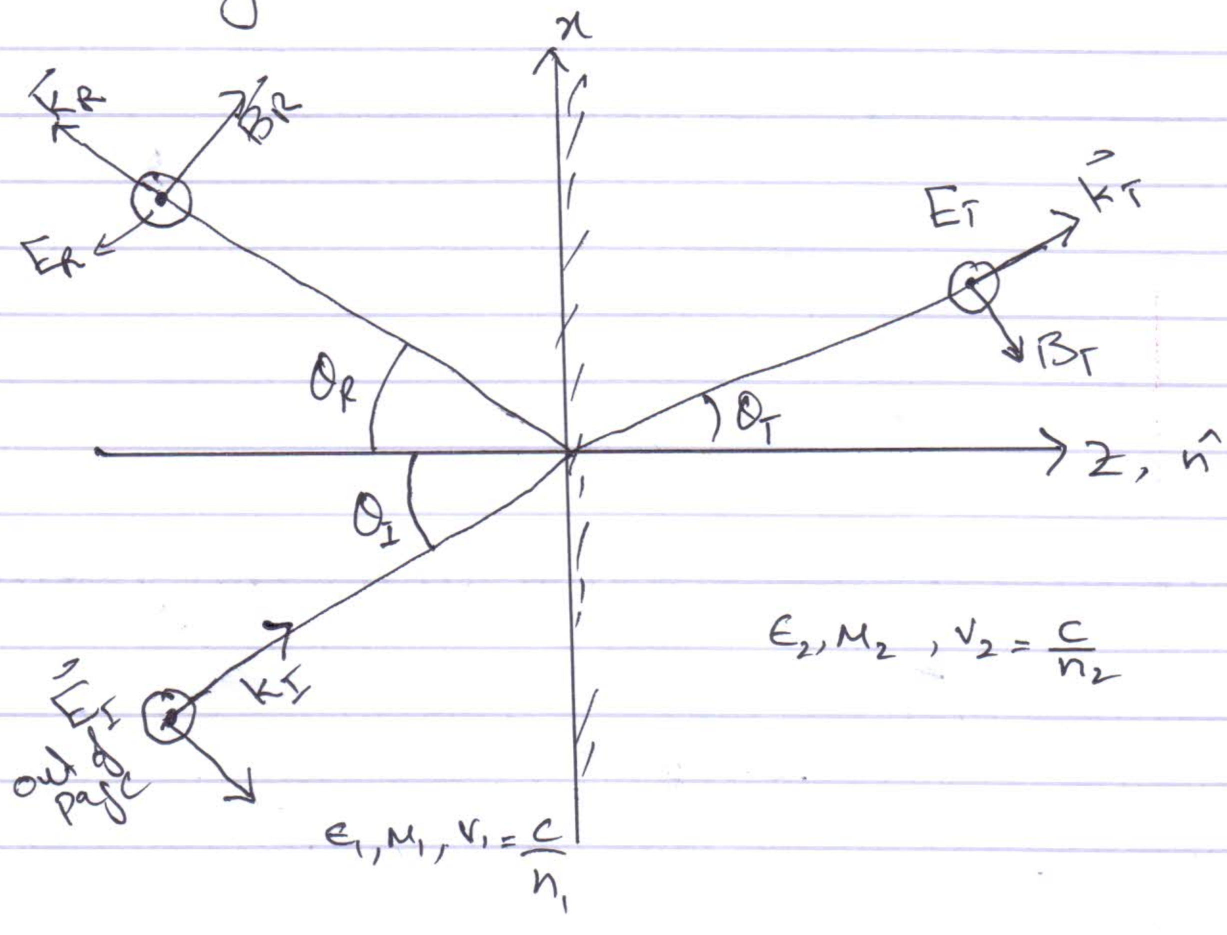
CASE I

Electric field vectors $\perp$ to the plane of incidence (TE polarization)

A monochromatic plane EM wave is incident on a boundary at $z=0$ in $x-y$ plane b/w two L/H/I media at an oblique angle of incidence.

$\Rightarrow$ EM is $\perp$ to plane of incidence (x-z in this case)

$\Rightarrow$ Plane of incidence contains the three wave-vectors $\vec{k}_I$, $\vec{k}_R$ & k_T and unit normal to interface.



$\Rightarrow$ All three E-fields are $\parallel$ to $\hat{y}$-axis and $\parallel$ to interface.

$\Rightarrow$ $\bar{B}$-field is related with E-field by

$$\vec{B} = \frac{1}{v} \hat{k} \times \vec{E}$$

$\Rightarrow$ $\bar{B}$-field vectors lie in the x-z plane.

The four B.C's on the complex $\tilde{\vec{E}}$ & $\tilde{\vec{B}}$ fields on the boundary at $z=0$ are

BC: 1
Normal compt (z, in this case) of $\vec{D}$ is continuous at $z=0$

$$\epsilon_1 \left(\tilde{E}_{0_{I_z}} + \tilde{E}_{0_{R_z}} \right) = \epsilon_2 \tilde{E}_{0_{T_z}}$$
$$0 \quad + \quad 0 \quad = \quad 0$$

BC: 2 : Tangential (x, y) compt. of $\bar{E}$ is continuous at $z=0$ ($\bar{E}$ has no x-comp)

$$\tilde{E}_{0_{I_y}} + \tilde{E}_{0_{R_y}} = \tilde{E}_{0_{T_y}}$$

BC: 3 Normal compt of B is continuous at $z=0$ (z - B $\perp$ comp)

$$\tilde{B}_{0_{I_z}} + \tilde{B}_{0_{R_z}} = \tilde{B}_{0_{T_z}}$$

As

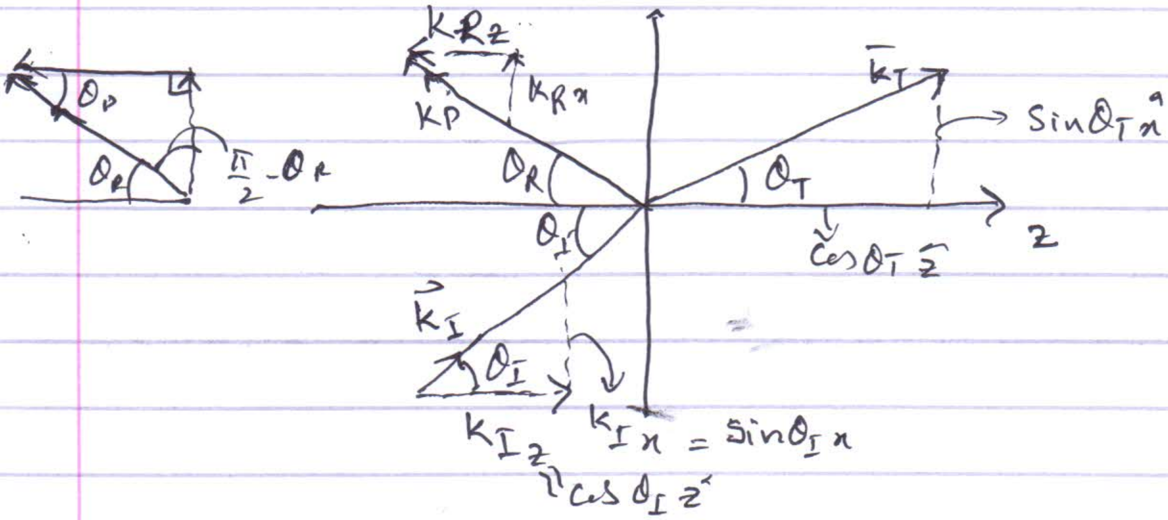
$$\vec{B}_o = \frac{1}{v}\hat{k} \times \vec{\tilde{E}}_o$$

And $\vec{k}$ lies in x-z plane

$$\hat{k}_I = \hat{k}_{Ix} + \hat{k}_{Iz} = \sin\theta_I \hat{x} + \cos\theta_I \hat{z}$$

$$\hat{k}_R = \hat{k}_{Rx} + \hat{k}_{Ry} = \sin\theta_R \hat{x} - \cos\theta_R \hat{z}$$

$$\hat{k}_T = \hat{k}_{Ix} + \hat{k}_{Ty} = \sin\theta_T \hat{x} + \cos\theta_T \hat{z}$$



$$\Rightarrow \tilde{B}_{oI_z}\hat{z} + \tilde{B}_{oR_z}\hat{z} = B_{oT_z}\hat{z}$$

$$\frac{1}{v_1}\left(\hat{k}_{Ix} \times \tilde{E}_{oI_y}\hat{y} + \hat{k}_{Rx} \times E_{oR_y}\hat{y}\right) = \frac{1}{v_2}\left(\hat{k}_{Tx} \times \tilde{E}_{oT_y}\hat{y}\right)$$

only $\hat{x}$ of $\bar{k}$ will contribute as $\hat{x} \times \hat{y} = \hat{z}$

$$\Rightarrow \frac{1}{v_1}\left[\tilde{E}_{oI}\sin\theta_I + \tilde{E}_{oR}\sin\theta_R\right]\hat{z} = \frac{1}{v_2}\tilde{E}_{oT}\sin\theta_T\hat{z}$$

BC: 4, Tangential compts (i,e x-y) of $\vec{H}$ are continous at $z=0$ (All B_y's are zero for TE - polarization)

$$\frac{1}{\mu_1}\left[\tilde{B}_{0I_x}\hat{x} + \tilde{B}_{0R_x}\hat{x}\right] = \frac{1}{\mu_2} B_{0T_x}\hat{x}$$

$$\Rightarrow \frac{1}{\mu_1 v_1}\left[-\tilde{E}_{0_I}\cos\theta_I + E_{0R}\cos\theta_R\right]\hat{x} = \frac{1}{\mu_2 v_2}\tilde{E}_{0R}(-\cos\theta_T)\hat{x}$$

As $\hat{z}\times\hat{y} = -\hat{x}$

As BC: 1 gives

$$0 + 0 = 0$$

BC: 2

$$\tilde{E}_{0I} + \tilde{E}_{0R} = \tilde{E}_{0T}$$

BC: 3 after using law of reflection

$$\theta_I = \theta_R$$

$$\frac{1}{v_1}\left[\tilde{E}_{0_I}\sin\theta_I + \tilde{E}_{0R}\sin\theta_I\right] = \frac{E_{0T}}{v_2}\sin\theta_T$$

$$\tilde{E}_{0_I} + \tilde{E}_{0R} = \left(\frac{v_1}{v_2}\frac{\sin\theta_T}{\sin\theta_I}\right)\tilde{E}_{0T}$$

From Snell's law.

$$\frac{n_1}{c}\sin\theta_I = \frac{n_2}{c}\sin\theta_T$$

$$\frac{1}{v_1}\sin\theta_I = \frac{1}{v_2}\sin\theta_T \Rightarrow \frac{v_1}{v_2}\frac{\sin\theta_T}{\sin\theta_I} = 1$$

$\Rightarrow$ BC: 3 reduces to BC: 1

$\Rightarrow$ $\tilde{E}_{0_I} + \tilde{E}_{0_R} = \tilde{E}_{0_T}$

From BC: 4

$$\tilde{E}_{0_I} - \tilde{E}_{0_R} = \frac{\mu_1 v_1}{\mu_2 v_2} \frac{\cos\theta_T}{\cos\theta_I} \tilde{E}_{0_T}$$

$\Rightarrow$ We have only two independent relations

1, $\tilde{E}_{0_I} + \tilde{E}_{0_R} = \tilde{E}_{0_T}$ ①

2, $\tilde{E}_{0_I} - \tilde{E}_{0_R} = \frac{\mu_1 v_1}{\mu_2 v_2} \frac{\cos\theta_T}{\cos\theta_I} \tilde{E}_{0_T}$ ②

As $\beta = \frac{\mu_1 v_1}{\mu_2 v_2}$ define $\alpha \equiv \frac{\cos\theta_T}{\cos\theta_I}$

with α & $\beta > 0$

Solving ① & ② we get.

$$\tilde{E}_{0_R} = \frac{(1-\alpha\beta)}{(1+\alpha\beta)} \tilde{E}_{0_I}$$

&

$$\tilde{E}_{0_T} = \left(\frac{2}{1+\alpha\beta}\right) \tilde{E}_{0_I}$$

(always in phase with incident wave for TE.

The Fresnel equations for TE Polarization

$$E_{0R} = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right) E_{0I}^{TE} \qquad \text{(For real fields)}$$

$$\alpha = \frac{\cos\theta_T}{\cos\theta_I}$$

and

$$E_{0T} = \frac{2}{(1+\alpha\beta)} E_{0I}$$

$$\beta = \frac{\mu_1 v_1}{\mu_2 v_2}$$

The Reflection & Transmission for TE polarization are:

$$R_{TE} = \frac{I_R}{I_I} = \frac{\frac{1}{2}\epsilon_1 v_1 E_{0R}^2 \cos\theta_I}{\frac{1}{2}\epsilon_1 v_1 E_{0I}^2 \cos\theta_I} = \left(\frac{E_{0R}}{E_{0I}}\right)^2$$

&

$$T_{TE} = \frac{I_T}{I_I} = \frac{\frac{1}{2}\epsilon_2 v_2 E_{0T}^2 \cos\theta_T}{\frac{1}{2}\epsilon_1 v_1 E_{0I}^2 \cos\theta_I} = \left(\frac{\epsilon_2 v_2}{\epsilon_1 v_1}\right)\left(\frac{\cos\theta_T}{\cos\theta_I}\right)\left(\frac{E_{0T}}{E_{0I}}\right)^2$$

As $\beta = \dfrac{\mu_1 v_1}{\mu_2 v_2} = \dfrac{\epsilon_2 v_2}{\epsilon_1 v_1}$ & $\alpha = \left(\dfrac{\cos\theta_T}{\cos\theta_I}\right)$

$$T_{TE} = \alpha\beta\left(\frac{E_{0T}}{E_{0I}}\right)^2$$

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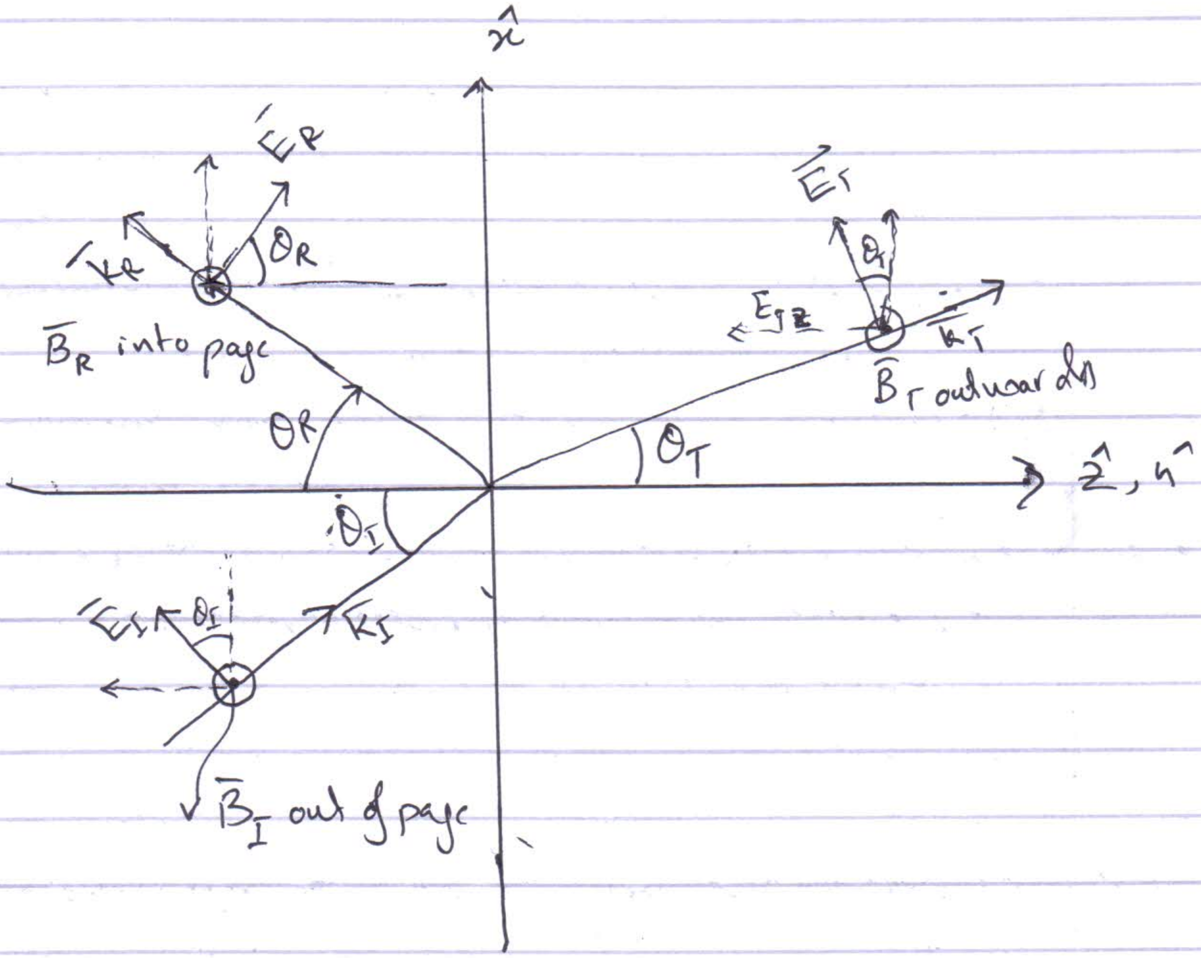
CASE II

Electric field vectors parallel to the plane of incidence: Transverse Magnetic TM Polarization

⇒ Electric field vector lies in plane of incidence

⇒ Magnetic field is ⊥ to plane of incidence and ∥ to interface.

⇒ (x−z) plane has all k-vectors and unit normal to the interface.



BC: 1 Normal (i,e z-) compt of D is continous at $z=0$

$$\epsilon_1\left[\tilde{E}_{0I}+\tilde{E}_{0R}\right]=\epsilon_2\tilde{E}_{0T}$$

z - component of $\bar{E}$ - field is

$$\epsilon_1\left[-\tilde{E}_{0I}\sin\theta_I+\tilde{E}_{0R}\sin\theta_R\right]=\epsilon_2\left[-E_{0T}\sin\theta_T\right]$$

BC: 2 Tangential (i,e x-y) compt of E is continous at $z=0$ (no y-compt for TM polar-ization)

$$\tilde{E}_{0I}\cos\theta_I+\tilde{E}_{0R}\cos\theta_R=\tilde{E}_{0T}\cos\theta_T$$

BC: 3 Normal (i,e z-compt) of $\bar{B}$ is continous at $z=0$

$$\tilde{B}_{0I_z}+\tilde{B}_{0R_z}=\tilde{B}_{0T_z}$$

$$0 + 0 = 0$$ no z-compt.

BC: 4 Tangential compt. (x,y) of $\vec{H}$ is continous at $z=0$ ($B_x=0$ for TM polariz

$$\frac{1}{\mu_1}\left[\tilde{B}_{0I_y}+\tilde{B}_{0R_y}\right]=\frac{1}{\mu_2}\tilde{B}_{0T_y}$$ using $\tilde{\bar{B}}=\frac{1}{v}\hat{k}\times\tilde{\bar{E}}$

$$\Rightarrow \frac{1}{\mu_1 v_1}\left[\tilde{E}_{0I_y}-\tilde{E}_{0R_y}\right]=\frac{1}{\mu_2 v_2}\tilde{E}_{0T_y}$$

BC: 1 & BC: 4 are same i,e

$$\tilde{E}_{0_I} - \tilde{E}_{0_R} = \frac{\mu_1 v_1}{\mu_2 v_2} \tilde{E}_{0_T} = \beta \tilde{E}_{0_T} \quad (1)$$

BC: 2

$$\tilde{E}_{0_I} + \tilde{E}_{0_R} = \tilde{E}_{0_T} \frac{\cos\theta_T}{\cos\theta_I} = \alpha \tilde{E}_{0_T} \quad -2$$

Solving we get.

$$\tilde{E}_{0_R} = \left(\frac{\alpha - \beta}{\alpha + \beta}\right) \tilde{E}_{0_I}$$

&

$$\tilde{E}_{0_T} = \left(\frac{2}{\alpha + \beta}\right) \tilde{E}_{0_I}$$

The Fresnel eqn's for TM polarization

$$E_{0_R} = \left(\frac{\alpha - \beta}{\alpha + \beta}\right) E_{0_I} \quad a,$$

&

$$E_{0_T} = \left(\frac{2}{\alpha + \beta}\right) E_{0_I} \quad b,$$

with $\alpha = \frac{\cos\theta_T}{\cos\theta_I}$ & $\beta = \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\epsilon_2 v_2}{\epsilon_1 v_1}$

Eqn's (a) & (b) are different from TE case.

The Fresnel Equations

TE

$$E_{0R} = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right) E_{0I}$$

$$E_{0T} = \left(\frac{2}{1+\alpha\beta}\right) E_{0I}$$

TM

$$E_{0R} = \frac{(\alpha-\beta)}{(\alpha+\beta)} E_{0I}$$

$$E_{0T} = \left(\frac{2}{\alpha+\beta}\right) E_{0I}$$

with $\alpha = \frac{\cos\theta_T}{\cos\theta_I}$ & $\beta = \frac{\mu_1 v_1}{\mu_2 v_2}$

$$\beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\epsilon_2 v_2}{\epsilon_1 v_1} = \frac{\mu_1 n_2}{\mu_2 n_1} = \frac{\epsilon_2 n_1}{\epsilon_1 n_2} \quad ; \quad v_1 = \frac{c}{n_1} = \frac{1}{\sqrt{\epsilon_1\mu_1}}$$

$$v_2 = \frac{c}{n_2} = \frac{1}{\sqrt{\epsilon_2\mu_2}}$$

For $\mu_1 \simeq \mu_2 \simeq \mu_2$

TE

$$\left(\frac{E_{0R}}{E_{0I}}\right) \simeq \frac{n_1\cos\theta_I - n_2\cos\theta_T}{n_1\cos\theta_I + n_2\cos\theta_T}$$

$$\left(\frac{E_{0T}}{E_{0I}}\right) \simeq \frac{2n_1\cos\theta_I}{n_1\cos\theta_I + n_2\cos\theta_T}$$

TM

$$\left(\frac{E_{0R}}{E_{0I}}\right) \simeq \frac{-n_2\cos\theta_I + n_1\cos\theta_T}{n_2\cos\theta_I + n_1\cos\theta_T}$$

$$\left(\frac{E_{0T}}{E_{0I}}\right) \simeq \frac{2n_1\cos\theta_I}{n_2\cos\theta_I + n_1\cos\theta_T}$$

Using Snell's law to eliminate θ_T [$n_I \sin\theta_i = n_T \sin\theta_T$)

$$\left(\frac{E_{0R}}{E_{0\perp}}\right)_R = \frac{\cos\theta_i - \sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_1}}{\cos\theta_I + \sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_1}}$$

2

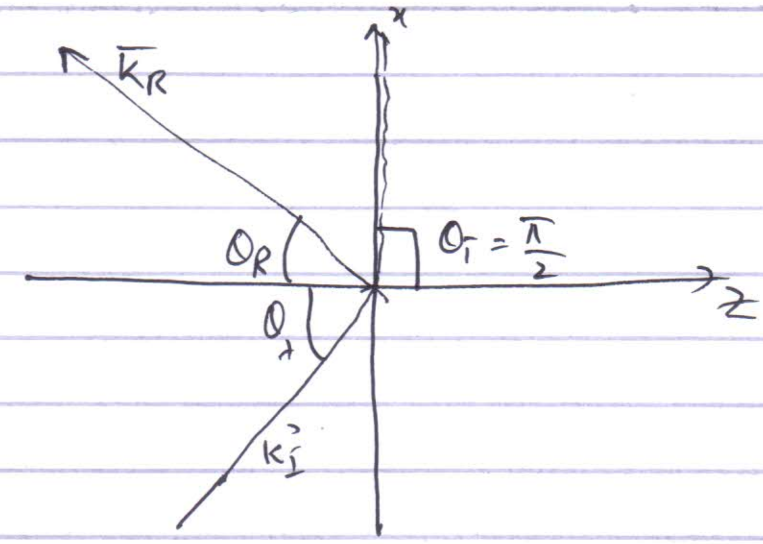
$$\frac{E'_{0T}}{E_{0I}} = \frac{2\cos\theta_i}{\cos\theta_{in} + \sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_I}}$$

For internal reflection ($n_1 > n_2$) there exists a critical angle of incidence $\theta_{I_{critical}}$ past which no transmitted beam exists for either TE or TEM polarization.

$$n_1 \sin\theta_{Ic} = n_2 \sin\theta_T^{max} = n_2 \sin\left(\frac{\pi}{2}\right) = n_2$$

$$\sin\theta_{Ic} = \frac{n_2}{n_1}$$

$$\theta_{I_c} = \sin^{-1}\left(\frac{n_2}{n_1}\right)$$



For $\theta_I \geq \theta_{I_c}$ — incident beam is totally internally reflected

For $\theta_I > \theta_{I_c}$

As $\frac{n_2}{n_1} < 1$

$$\sin\theta_I = \frac{n_2}{n_1}\sin\theta_T$$

as $\frac{n_2}{n_1}\cos\theta_T = \frac{n_2}{n_1}\sqrt{1 - \sin^2\theta_T} = \sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_T\left(\frac{n_2}{n_1}\right)^2}$

$$\sin\theta_T = \frac{n_1}{n_2}\sin\theta_I$$

$$\Rightarrow \frac{n_2}{n_1}\cos\theta_T = \sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_I} = i\sqrt{\sin^2\theta_I - \left(\frac{n_2}{n_1}\right)^2}$$

Evanescent Waves at an Interface

For transmitted wave

$$\vec{\tilde{E}}_T = \tilde{E}_{0_T} e^{i(\bar{k}_T \cdot \bar{r} - \omega t)}$$

As $\vec{k}_T$ lies in x-z plane (plane of incidence)

$$\vec{k}_T \cdot \bar{r} = (k_t \sin\theta_t \hat{x} + k_t \cos\theta_t \hat{z}) \cdot (x\hat{x} + z\hat{z})$$

$$= k_t (x \sin\theta_t + z \cos\theta_t)$$

But

$$\cos\theta_t = \sqrt{1 - \sin^2\theta_T} = \sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_i}$$

For $n_1 > n_2$

$$\Rightarrow \quad \cos\theta_T = i\sqrt{\sin^2\theta_i - \left(\frac{n_2}{n_1}\right)^2}$$

purely imaginary number

$$\Rightarrow \quad \bar{k}_T \cdot \bar{r} = k_t \left[x \sin\theta_i \left(\frac{n_1}{n_2}\right) + iz\sqrt{\sin^2\theta_i \left(\frac{n_1}{n_2}\right)^2 - 1} \right]$$

$$\Rightarrow \quad \tilde{E}_T(\bar{r}, t) = \tilde{E}_{0_T} e^{-\alpha z} e^{i\left(k_t x \sin\theta_i \left(\frac{n_1}{n_2}\right) - \omega t\right)}$$

where

$$\alpha = k_2 \sqrt{\left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_i - 1}$$

$e^{-\alpha z}$ — damping in z-axis.

while

$$e^{i\left(k_2 x \sin\theta_i \left(\frac{n_1}{n_2}\right) - \omega t\right)}$$ is

an oscillatory along interface in x-direction.

The evanescent wave amplitude will decay rapidly as it penetrates into the lower refractive index medium.

⇒ Differs from regular plane wave in that its amplituded exponentially decays in the directions ⊥ to its propagation direction