Light and Matter interaction

February 6, 2020 6:49 PM

- [ ] Gauge Transformation
- [ ] Dipole Approx. and the Interaction Hamiltonian
- [ ] Semi-Classical & Rabi Oscillation
- [ ] Quantum Interaction & The Jaynes-Cumming model

$$H = \frac{1}{2m}\left(P - qA\right)^2 + q\phi + V(r)$$

$$H\psi(r,t) = i\hbar \frac{\partial}{\partial t}\psi(r,t) \qquad (1)$$

$\psi \to \psi e^{i\chi}$ → Gives the same state

But what if $\psi(r,t) \longrightarrow \mathcal{U}^{\circ}(r,t)\,\psi(r,t)$

with $\mathcal{U}^{\circ}(r,t) = \exp\left(-i\frac{e}{\hbar}\chi(r,t)\right)$.

$Pr(r,t)$ stays the same, but it would not work with eq. 1

For it to satisfy the eq 1

$$A \to A + \nabla\chi(r,t)$$

$$\phi \to \phi - \frac{\partial\chi(r,t)}{\partial t}$$

∂t

$\Rightarrow$ Gauge transformation would give a phase that does not change the probabilities.

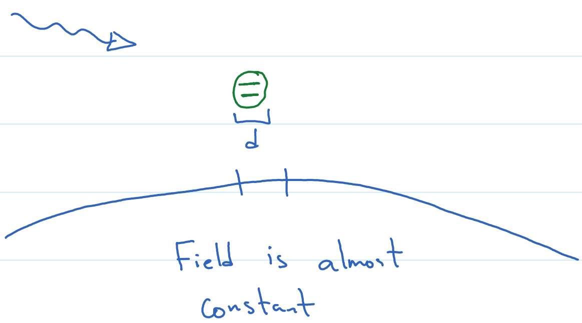
Hamiltonian $(q \to -e)$

We start with the Coulomb gauge $(\nabla A = 0, \phi = 0)$ and then add the additional $\chi(r,t)$:

$$H \xrightarrow{\text{Gauge}} H' = \frac{1}{2m}\left(\hat{P} + e\left[\vec{A}(r,t) + \nabla\chi(r,t)\right]\right) - e\left(\phi - \frac{\partial\chi(r,t)}{\partial t}\right) + V(r)$$

$$\Rightarrow \quad H' = \frac{1}{2m}\left(\hat{P} + e\left[\vec{A}(r,t) + \nabla\chi(r,t)\right]\right) + e\frac{\partial\chi(r,t)}{\partial t} + V(r)$$

Dipole Approximation



$\lambda \sim 500$ nm

$d \sim 10^{-10}$ m

$A(r,t) \rightsquigarrow A(r_0,t)$

r_0: Position of the atom.

Let's put $\chi(r,t) = -\vec{A}(r_0,t)\cdot\vec{r}$

This means

$$A(r,t) \to A(r_0,t) + \nabla\chi = 0$$

$$\mathcal{U}(r,t) \to 0 + \frac{\partial \chi(r,t)}{\partial t} = -\vec{r}\cdot \underbrace{\frac{\partial \vec{A}(r_0,t)}{\partial t}}_{E}$$

$$H' = \underbrace{\left(\frac{1}{2m} P^2 + V(r)\right)}_{H_0} - \underbrace{e\,\vec{r}\cdot\vec{E}}_{H_{int}}$$

$$H \rightsquigarrow \frac{\hbar^2}{2m}\left(\nabla - i\frac{e}{\hbar}\left(\vec{A}(r,t) - \vec{A}(r_0,t)\right)\right) + e\,\mathcal{U} - \hbar\,\dot{A}\cdot\vec{r} + V(r)$$

$$i\hbar\,\frac{\partial \psi(r,t)\,e^{i\chi(r,t)}}{\partial t} \longrightarrow e^{i\vec{A}(r_0,t)\cdot\vec{r}}\,\frac{\partial \psi(r,t)}{\partial t} - \hbar\,\dot{A}(r_0,t)\cdot r\; e^{iA(r_0,t)\cdot r}\,\psi(r,t)$$

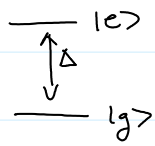
$$\Rightarrow \tilde{H} = H_0 - e\,\vec{r}\cdot \underset{\downarrow\, E}{\dot{A}} = \left(\frac{P^2}{2m} + V(r)\right) - \underbrace{e\,\vec{r}}_{\vec{d}}\cdot\vec{E}$$

$\vec{d}$ → <u>Dipole moment</u>

<u>Two-level</u>

$$H_0 = E_g |g\rangle\langle g| + E_e |e\rangle\langle e| = \frac{\Delta}{2}\sigma_z + \frac{E_g + E_e}{2}\mathbb{1}$$

$$|e\rangle \to \begin{pmatrix}1\\0\end{pmatrix}, \quad |g\rangle \to \begin{pmatrix}0\\1\end{pmatrix}, \quad \Delta = \frac{E_e - E_g}{2}$$



$$H_1 : \quad \vec{d} = g\,\sigma_x \qquad g = \langle e|\vec{d}|g\rangle$$

$$H_1 = \underbrace{g\,\sigma_x\;E(r,t)}$$

Semi-classical

$$E = E_0 \cos(\omega t)$$

Quantum

$$E \propto (\hat{a} - \hat{a}^\dagger)$$

Quantized field $A \to \hat{a}$

$$H_0 = (a^\dagger a + 1/2)\hbar\omega_D$$

Semi-Classical

$$H = \frac{\hbar\Delta}{2}\sigma_z - \hbar\Omega\,\sigma_x \cos(\omega_D t)$$

$$\Omega = \frac{g E_0}{\hbar}$$

Assignment: $\sigma_x \cos(\omega_D t) = \dfrac{e^{i\omega_D t\frac{\sigma_z}{2}}\sigma_x e^{-i\omega_D t\frac{\sigma_z}{2}} + e^{-i\omega_D t\frac{\sigma_z}{2}}\sigma_x e^{i\omega_D t\frac{\sigma_z}{2}}}{2}$ (*)

Hint: $\underbrace{e^{i\omega_D t\frac{\sigma_z}{2}}}\sigma_x e^{-i\omega_D t\frac{\sigma_z}{2}} = \cos(\omega_D t)\,\sigma_x + i\sin(\omega_D t)\,\sigma_y$

These are rotation operators in SU(2).

$\to R_z(\omega_D t)$

So, we can write the Hamiltonian as

$$H = \frac{\hbar\Delta}{2}\sigma_z + \frac{\Omega}{2}\Big(R_z(\omega_D t)\,\sigma_x\,R_z(-\omega_D t) + R_z(-\omega_D t)\,\sigma_x\,R_z(\omega_D t)\Big)$$

We switch to the interaction picture of $e^{i\frac{\Delta t\,\sigma_z}{2}} = R_z(\Delta t)$

$$H^{eff} = \hbar\Omega\, e^{i\frac{\Delta t}{2}\sigma_z} \left[\cos(\omega_D t)\, \sigma_x\right] e^{-i\Delta t \sigma_z/2}$$

Using ⊛

$$H^{eff} = \frac{\hbar\Omega}{2}\Big[\underbrace{e^{i(\Delta-\omega_D)t/2\,\sigma_z}\,\sigma_x\, e^{-i(\Delta-\omega_D)\frac{t}{2}\sigma_z}}_{A} + \underbrace{e^{i(\Delta+\omega_D)t/2\,\sigma_z}\,\sigma_x\, e^{-i(\Delta+\omega_D)\frac{t}{2}\sigma_z}}_{B}\Big]$$

For $\Delta \approx \omega_D$, $A \simeq \sigma_x$. But B is oscallating & the effect of $e^{i\hbar\Omega B} \simeq \mathbb{1}$.

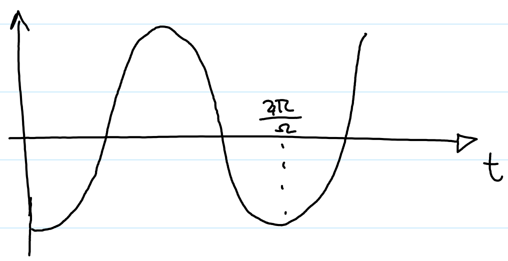
This is often refered to as the "rotating wave approx.".

For the plots, see https://github.com/sraeisi/QuantumOptics

$$\Longrightarrow H_{eff}^{RWA} = \frac{\hbar\Omega}{2}\,\sigma_x$$

So, if we start with $|g\rangle \to \begin{pmatrix}0\\1\end{pmatrix}$

$E \propto \langle\sigma_z\rangle$



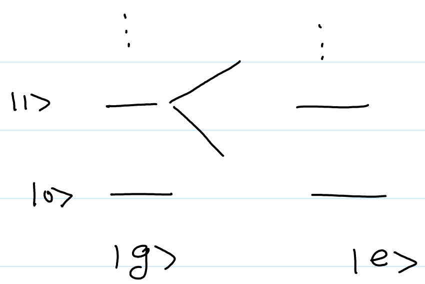
Quantize field

$$H = H_0 - \hbar \lambda \sigma_x (a - a^\dagger) + \hbar \omega_D a^\dagger a$$

$$\sigma_x = \sigma_+ + \sigma_-$$

And we switch to the RF of $\mathcal{U} = e^{i\frac{\Delta t}{2}\sigma_z} \otimes e^{i\omega_D t a^\dagger a} = R_z(\Delta t) \otimes e^{i\omega_D t N}$

$$H_{eff} = -\hbar \lambda \left(\sigma_+ e^{i\Delta t} + \sigma_- e^{-i\Delta t} \right)\left(a e^{-i\omega_D t} - a^\dagger e^{i\omega_D t} \right)$$



$|g\,n\rangle \to |e, n+1\rangle, \; |e, n-1\rangle$

$|e, n\rangle \to |g, n+1\rangle, \; |g, n-1\rangle$

Assumption

$$\Delta \sim \omega_D \xrightarrow{RWA} \hbar\lambda\left(\sigma_+ a \, e^{i(\Delta - \omega_D)t} + \sigma_- a^\dagger e^{-i(\Delta-\omega_D)t} \right)$$

↳ Going back to the original frame:

$$H = \frac{\hbar\Delta}{2}\sigma_z + \hbar\omega_D a^\dagger a + \underbrace{\hbar\lambda\left(\sigma_+ a + \sigma_- a^\dagger\right)}_{H_{int}}$$

Jaynes - Cumming Hamiltonian

It connects only two states

$$|g\, n\rangle \longleftrightarrow |e, n-1\rangle$$

$$H_{int}\,|g\,n\rangle = \hbar\lambda\sqrt{n}\,|e, n-1\rangle$$

$$H_{int}\,|e, n-1\rangle = \hbar\lambda\sqrt{n}\,|g, n\rangle$$

It would be similar to the Rabi Oscillation but

$$\Omega \to \Omega_n = 2\hbar\lambda\sqrt{n}$$

H in the subspace of $|g\,n\rangle$ & $|e, n-1\rangle$ is given by

$$= \begin{bmatrix} \frac{\hbar\Delta}{2} + \hbar\omega(n-1) & \hbar\lambda\sqrt{n} \\ \hbar\lambda\sqrt{n} & -\frac{\hbar\Delta}{2} + \hbar\omega n \end{bmatrix}$$

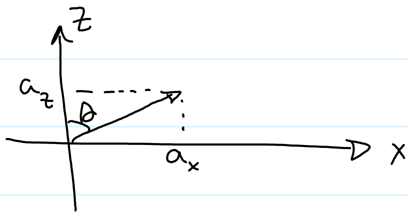
$$H_n = \underbrace{\left(\frac{\hbar\Delta}{2} - \frac{\hbar\omega}{2}\right)}_{a_z}\sigma_z + \hbar\omega\left(n - 1/2\right)\mathbb{1} + \underbrace{\hbar\lambda\sqrt{n}}_{a_x}\,\sigma_x$$

$$E_n = \hbar\omega\left(n - 1/2\right) \pm \hbar\sqrt{\left(\frac{\Delta-\omega}{2}\right)^2 + \lambda^2 n}$$

$|\psi_n^{\pm}\rangle$:

$$H_n = \underbrace{(a_x, 0, a_z)}_{\vec{a}} \cdot (\sigma_x, \sigma_y, \sigma_z)$$

$$\vec{a} = R_y(\theta)\begin{pmatrix}0\\0\\1\end{pmatrix} \qquad \theta = tg^{-1}\left(\frac{a_x}{a_z}\right)$$



$$|\psi_n^{+}\rangle = R_z(\theta)\,|e, n-1\rangle =$$

$$|\psi_n^+\rangle = R_z(\theta)\,|e, n-1\rangle =$$

$$\cos(\theta/2)\,|e, n-1\rangle + \sin(\theta/2)\,|g, n\rangle$$

$$|\psi_n^-\rangle = \cos(\theta/2)\,|g, n\rangle - \sin(\theta/2)\,|e, n-1\rangle$$
