International Journal of Quantum Information
Vol. 7, Supplement (2009) 125–137

World Scientific
www.worldscientific.com

QUANTUM ESTIMATION FOR QUANTUM TECHNOLOGY

MATTEO G. A. PARIS

Dipartimento di Fisica dell'Università di Milano, I-20133 Milano, Italia

CNSIM, Udr Milano, I-20133 Milano, Italia

ISI Foundation, I-10133 Torino, Italia

Received 12 November 2008

Several quantities of interest in quantum information, including entanglement and purity, are nonlinear functions of the density matrix and cannot, even in principle, correspond to proper quantum observables. Any method aimed to determine the value of these quantities should resort to indirect measurements and thus corresponds to a parameter estimation problem whose solution, i.e. the determination of the most precise estimator, unavoidably involves an optimization procedure. We review local quantum estimation theory and present explicit formulas for the symmetric logarithmic derivative and the quantum Fisher information of relevant families of quantum states. Estimability of a parameter is defined in terms of the quantum signal-to-noise ratio and the number of measurements needed to achieve a given relative error. The connections between the optmization procedure and the geometry of quantum statistical models are discussed. Our analysis allows to quantify quantum noise in the measurements of non observable quantities and provides a tools for the characterization of signals and devices in quantum technology.

Keywords: Quantum estimation; Fisher information.

1. Introduction

Many quantities of interest in physics are not directly accessible, either in principle or due to experimental impediments. This is particolarly true for quantum mechanical systems where relevant quantities like entanglement and purity are nonlinear functions of the density matrix and cannot, even in principle, correspond to proper quantum observables. In these situations one should resort to indirect measurements, inferring the value of the quantity of interest by inspecting a set of data coming from the measurement of a different obeservable, or a set of observables. This is basically a parameter estimation problem which may be properly addressed in the framework of quantum estimation theory (QET),[1] which provides analytical tools to find the optimal measurement according to some given criterion. In turn, there are two main paradigms in QET: *Global* QET looks for the POVM minimizing a suitable cost functional, averaged over all possible values of the parameter to be estimated. The result of a global optimization is thus a single POVM, independent

on the value of the parameter. On the other hand, *local* QET looks for the POVM maximizing the Fisher information, thus minimizing the variance of the estimator, at a fixed value of the parameter.[2–6] Roughly speaking, one may expect local QET to provide better performances since the optimization concerns a specific value of the parameter, with some adaptive or feedback mechanism assuring the achievability of the ultimate bound.[7] Global QET has been mostly applied to find optimal measurements and to evaluate lower bounds on precision for the estimation of parameters imposed by unitary transformations. For bosonic systems these include single-mode phase,[8,9] displacement,[10] squeezing [11,12] as well as two-mode transformations, e.g. bilinear coupling.[13] Local QET has been applied to the estimation of quantum phase[14] and to estimation problems with open quantum systems and non unitary processes[15]: to finite dimensional systems,[16] to optimally estimate the noise parameter of depolarizing[17] or amplitude-damping,[18] and for continuous variable systems to estimate the loss parameter of a quantum channel[19–22] as well as the position of a single photon.[23] Recently, the geometric structure induced by the Fisher information itself has been exploited to give a quantitative operational interpretation for multipartite entanglement[24] and to assess quantum criticality as a resource for quantum estimation.[25]

In this paper we review local quantum estimation theory and present explicit formulas for the symmetric logarithmic derivative and the quantum Fisher information of relevant families of quantum states. We are interested in evaluating the ultimate bound on precision (sensitivity), i.e. the smallest value of the parameter that can be discriminated, and to determine the optimal measurement achieving those bounds. Estimability of a parameter will be then defined in terms of the quantum signal-to-noise ratio and the number of measurements needed to achieve a given relative error.

The paper is structured as follows. In the next Section we review local quantum estimation theory and report the solution of the optimization problem, *i.e.* the determination of the optimal quantum estimator in terms of the symmetric logarithmic derivative, as well as the ultimate bounds to precision in terms of the quantum Fisher information. General formulas for the symmetric logarithmic derivative and the quantum Fisher information are derived. In Sec. 3 we address the quantification of estimability of a parameter put forward the quantum signal-to-noise ratio and the number of measurements needed to achieve a given relative error as the suitable figures of merit. In Sec. 4 we present explicit formulas for sets of pure states and the generic unitary family. We also consider the multiparamer case and the problem of repametrization. In Sec. 5 we discuss the connections between estimability of a set of parameters, the optmization procedure and the geometry of quantum statistical models. Sec. 6 closes the paper with some concluding remarks.

2. Local Quantum Estimation Theory

The solution of a parameter estimation problem amounts to find an estimator, i.e. a mapping $\hat{\lambda} = \hat{\lambda}(x_1, x_2, \ldots)$ from the set χ of measurement outcomes into the

space of parameters. Optimal estimators in classical estimation theory are those saturating the Cramer-Rao inequality[26]

$$\mathrm{V}(\lambda) \geq \frac{1}{MF(\lambda)} \tag{1}$$

which establishes a lower bound on the mean square error $V(\lambda) = E_\lambda[(\hat{\lambda}(\{x\}) - \lambda)^2]$ of any estimator of the parameter λ. In Eq. (1) M is the number of measurements and $F(\lambda)$ is the so-called Fisher Information (FI)

$$F(\lambda) = \int dx p(x|\lambda) \left(\frac{\partial \ln p(x|\lambda)}{\partial \lambda} \right)^2 = \int dx \, \frac{1}{p(x|\lambda)} \left(\frac{\partial p(x|\lambda)}{\partial \lambda} \right)^2 . \tag{2}$$

where $p(x|\lambda)$ denotes the conditional probability of obtaining the value x when the parameter has the value λ. For unbiased estimators, as those we will deal with, the mean square error is equal to the variance $\mathrm{Var}(\lambda) = E_\lambda[\hat{\lambda}^2] - E_\lambda[\hat{\lambda}]^2$.

When quantum systems are involved any estimation problem may be stated by considering a family of quantum states ϱ_λ which are defined on a given Hilbert space $\mathcal{H}$ and labeled by a parameter λ living on a d-dimensional manifold $\mathcal{M}$, with the mapping $\lambda \mapsto \varrho_\lambda$ providing a coordinate system. This is sometimes referred to as a quantum statistical model. The parameter λ does not, in general, correspond to a quantum observable and our aim is to estimate its values through the measurement of some observable on ϱ_λ. In turn, a quantum estimator O_λ for λ is a selfadjoint operator, which describe a quantum measurement followed by any classical data processing performed on the outcomes. The indirect procedure of parameter estimation implies an additional uncertainty for the measured value, that cannot be avoided even in optimal conditions. The aim of quantum estimation theory is to optimize the inference procedure by minimizing this additional uncertainty.

In quantum mechanics, according to the Born rule we have $p(x|\lambda) = \mathrm{Tr}[\Pi_x \varrho_\lambda]$ where $\{\Pi_x\}$, $\int dx \, \Pi_x = \mathbb{I}$, are the elements of a positive operator-valued measure (POVM) and ϱ_λ is the density operator parametrized by the quantity we want to estimate. Introducing the Symmetric Logarithmic Derivative (SLD) L_λ as the selfadjoint operator satistying the equation

$$\frac{L_\lambda \varrho_\lambda + \varrho_\lambda L_\lambda}{2} = \frac{\partial \varrho_\lambda}{\partial \lambda} \tag{3}$$

we have that $\partial_\lambda p(x|\lambda) = \mathrm{Tr}[\partial_\lambda \varrho_\lambda \Pi_x] = \mathrm{Re}(\mathrm{Tr}[\varrho_\lambda \Pi_x L_\lambda])$. The Fisher Information (2) is then rewritten as

$$F(\lambda) = \int dx \, \frac{\mathrm{Re}\left(\mathrm{Tr}\left[\varrho_\lambda \Pi_x L_\lambda\right]\right)^2}{\mathrm{Tr}[\varrho_\lambda \Pi_x]}. \tag{4}$$

For a given quantum measurement, i.e. a POVM $\{\Pi_x\}$, Eqs. (2) and (4) establish the classical bound on precision, which may be achieved by a proper data processing, e.g. by maximum likelihood, which is known to provide an asymptotically efficient estimator. On the other hand, in order to evaluate the ultimate bounds to precision

we have now to maximize the Fisher information over the quantum measurements. Following Refs. 3–6 we have

$$F(\lambda) \leq \int dx \left| \frac{\mathrm{Tr}\left[\varrho_\lambda \Pi_x L_\lambda\right]}{\sqrt{\mathrm{Tr}[\varrho_\lambda \Pi_x]}} \right|^2 \tag{5}$$

$$= \int dx \left| \mathrm{Tr}\left[\frac{\sqrt{\varrho_\lambda}\sqrt{\Pi_x}}{\sqrt{\mathrm{Tr}\left[\varrho_\lambda \Pi_x\right]}} \sqrt{\Pi_x} L_\lambda \sqrt{\varrho_\lambda} \right] \right|^2$$

$$\leq \int dx\, \mathrm{Tr}\left[\Pi_x L_\lambda \varrho_\lambda L_\lambda\right] \tag{6}$$

$$= \mathrm{Tr}[L_\lambda \varrho_\lambda L_\lambda]$$

$$= \mathrm{Tr}[\varrho_\lambda L_\lambda^2]$$

The above chain of inequalities prove that the Fisher information $F(\lambda)$ of any quantum measurement is bounded by the so-called *Quantum Fisher Information* (QFI)

$$F(\lambda) \leq H(\lambda) \equiv \mathrm{Tr}[\varrho_\lambda L_\lambda^2] = \mathrm{Tr}[\partial_\lambda \varrho_\lambda L_\lambda] \tag{7}$$

leading the quantum Cramer-Rao bound

$$\mathrm{Var}(\lambda) \geq \frac{1}{MH(\lambda)} \tag{8}$$

to the variance of any estimator. The quantum version of the Cramer-Rao theorem provides an ultimate bound: it does depend on the geometrical structure of the quantum statistical model and does not depend on the measurement. Optimal quantum measurements for the estimation of λ thus corresponds to POVM with a Fisher information equal to the quantum Fisher information, i.e. those saturating both inequalities (5) and (6). The first one is saturated when $\mathrm{Tr}[\varrho_\lambda \Pi_x L_\lambda]$ is a real number $\forall\lambda$. On the other hand, Ineq. (6) is based on the Schwartz inequality $|\mathrm{Tr}[A^\dagger B]|^2 \leq \mathrm{Tr}[A^\dagger A]\mathrm{Tr}[B^\dagger B]$ applied to $A^\dagger = \sqrt{\varrho_\lambda}\sqrt{\Pi_x}/\sqrt{\mathrm{Tr}[\varrho_\lambda \Pi_x]}$ and $B = \sqrt{\Pi_x} L_\lambda \sqrt{\varrho_\lambda}$ and it is saturated when

$$\frac{\sqrt{\Pi_x}\sqrt{\varrho_\lambda}}{\mathrm{Tr}\left[\varrho_\lambda \Pi_x\right]} = \frac{\sqrt{\Pi_x} L_\lambda \sqrt{\varrho_\lambda}}{\mathrm{Tr}[\varrho_\lambda \Pi_x L_\lambda]} \quad \forall\lambda, \tag{9}$$

The operatorial condition in Eq. (9) is satisfied iff $\{\Pi_x\}$ is made by the set of projectors over the eigenstates of L_λ, which, in turn, represents the optimal POVM to estimate the parameter λ. Notice, however, that L_λ itself may not represent the optimal observable to be measured. In fact, Eq. (9) determines the POVM and not the estimator i.e. the function of the eigenvalues of L_λ. As we have already mentioned above, this corresponds to a classical post-processing of data aimed to saturate the Cramer-Rao inequality (1) and may be pursued by maximum likelihood, which is known to provide an asymptotically efficient estimator. Using the fact that $\mathrm{Tr}[\varrho_\lambda L_\lambda] = 0$ an explicit form for the optimal quantum estimator is

given by

$$O_\lambda = \lambda \mathbb{I} + \frac{L_\lambda}{H(\lambda)} \tag{10}$$

for which we have

$$\mathrm{Tr}[\varrho_\lambda O_\lambda] = \lambda, \quad \mathrm{Tr}[\varrho_\lambda O_\lambda^2] = \lambda^2 + \frac{\mathrm{Tr}[\varrho_\lambda L_\lambda^2]}{H^2(\lambda)}, \quad \text{and thus } \langle \Delta O_\lambda^2 \rangle = 1/H(\lambda).$$

Equation (3) is Lyapunov matrix equation to be solved for the SLD L_λ. The general solution may be written as

$$L_\lambda = 2\int_0^\infty dt\, \exp\{-\varrho_\lambda t\}\, \partial_\lambda \varrho_\lambda \exp\{-\varrho_\lambda t\} \tag{11}$$

which, upon writing ϱ_λ in its eigenbasis $\varrho_\lambda = \sum_n \varrho_n |\psi_n\rangle\langle\psi_n|$, leads to

$$L_\lambda = 2\sum_{nm} \frac{\langle \psi_m | \partial_\lambda \varrho_\lambda | \psi_n \rangle}{\varrho_n + \varrho_m} |\psi_m\rangle\langle\psi_n|, \tag{12}$$

where the sums include only terms with $\varrho_n + \varrho_m \neq 0$. The quantum Fisher information is thus given by

$$H(\lambda) = 2\sum_{nm} \frac{|\langle \psi_m | \partial_\lambda \varrho_\lambda | \psi_n \rangle|^2}{\varrho_n + \varrho_m}, \tag{13}$$

or, in a basis independent form,

$$H(\lambda) = 2\int_0^\infty dt\, \mathrm{Tr}[\partial_\lambda \varrho_\lambda \exp\{-\varrho_\lambda t\}\, \partial_\lambda \varrho_\lambda \exp\{-\varrho_\lambda t\}]. \tag{14}$$

Notice that the SLD is defined only on the support of ϱ_λ and that both the eigenvalues ϱ_n and the eigenvectors $|\psi_n\rangle$ may depend on the parameter. In order to separate the two contribution to the QFI we explicitly evaluate $\partial_\lambda \varrho_\lambda$

$$\partial_\lambda \varrho_\lambda = \sum_p \partial_\lambda \varrho_p |\psi_p\rangle\langle\psi_p| + \varrho_p |\partial_\lambda \psi_p\rangle\langle\psi_p| + \varrho_p |\psi_p\rangle\langle\partial_\lambda \psi_p| \tag{15}$$

The symbol $|\partial_\lambda \psi_n\rangle$ denotes the ket $|\partial_\lambda \psi_n\rangle = \sum_k \partial_\lambda \psi_{nk} |k\rangle$, where ψ_{nk} are obtained expanding $|\psi_n\rangle$ in arbitrary basis $\{|k\rangle\}$ independent on λ. Since $\langle \psi_n | \psi_m \rangle = \delta_{nm}$ we have $\partial_\lambda \langle \psi_n | \psi_m \rangle \equiv \langle \partial_\lambda \psi_n | \psi_m \rangle + \langle \psi_n | \partial_\lambda \psi_m \rangle = 0$ and therefore

$$\mathrm{Re}\langle \partial_\lambda \psi_n | \psi_m \rangle = 0 \quad \langle \partial_\lambda \psi_n | \psi_m \rangle = -\langle \psi_n | \partial_\lambda \psi_m \rangle = 0.$$

Using Eq. (15) and the above identities we have

$$L_\lambda = \sum_p \frac{\partial_\lambda \varrho_p}{\varrho_p} |\psi_p\rangle\langle\psi_p| + 2\sum_{n \neq m} \frac{\varrho_n - \varrho_m}{\varrho_n + \varrho_m} \langle \psi_m | \partial_\lambda \psi_n \rangle |\psi_m\rangle\langle\psi_n| \tag{16}$$

and in turn

$$H(\lambda) = \sum_p \frac{(\partial_\lambda \varrho_p)^2}{\varrho_p} + 2 \sum_{n \neq m} \sigma_{nm} \, |\langle \psi_m | \partial_\lambda \psi_n \rangle|^2 \tag{17}$$

where

$$\sigma_{nm} = \frac{(\varrho_n - \varrho_m)^2}{\varrho_n + \varrho_m} + \text{any antisymmetric term}, \tag{18}$$

as for example

$$\sigma_{nm} = 2\varrho_n \frac{\varrho_n - \varrho_m}{\varrho_n + \varrho_m} \quad \sigma_{nm} = 2\varrho_n \left(\frac{\varrho_n - \varrho_m}{\varrho_n + \varrho_m} \right)^2 \tag{19}$$

The first term in Eq. (17) represents the classical Fisher information of the distribution $\{\varrho_p\}$ whereas the second term contains the truly quantum contribution. The second term vanishes when the eigenvectors of ϱ_λ do not depend. In this case $[\varrho_\lambda, \partial_\lambda \varrho_\lambda] = 0$ and Eq. (11) reduces to $L_\lambda = \partial_\lambda \log \varrho_\lambda$.

Finally, upon substituting the above Eqs. in Eq. (10), we obtain the corresponding optimal quantum estimator

$$O_\lambda = \sum_p \left(\lambda + \frac{\partial_\lambda \varrho_p}{\varrho_p} \right) |\psi_p\rangle\langle\psi_p| + \frac{2}{H(\lambda)} \sum_{n \neq m} \frac{\varrho_n - \varrho_m}{\varrho_n + \varrho_m} \langle \psi_m | \partial_\lambda \psi_n \rangle |\psi_m\rangle\langle\psi_n|. \tag{20}$$

So far we have considered the case of a parameter with a fixed given value. A question arises on whether a bound for estimator variance may be established also for a parameter having an *a priori* distribution $z(\lambda)$. The answer is positive and given by the Van Trees inequality[28,29] which provides a bound for the average variance

$$\overline{\mathrm{Var}(\lambda)} = \int dx \int d\lambda z(\lambda) [\hat{\lambda}(\{x\}) - \lambda)]^2$$

of any unbiased estimator of the random parameter λ. Van Trees inequality states that

$$\overline{\mathrm{Var}(\lambda)} \geq \frac{1}{Z_F} \tag{21}$$

where the generalized Fisher information Z_F is given by

$$Z_F = \int dx \int d\lambda p(x, \lambda) \left[\partial_\lambda \log p(x, \lambda) \right]^2, \tag{22}$$

$p(x, \lambda)$ being the joint probability distribution of the outcomes and the parameter of interest. Upon writing the joint distribution as $p(x, \lambda) = p(x|\lambda) z(\lambda)$ Eq. (22) may be rewritten as

$$Z_F = \int d\lambda z(\lambda) F(\lambda) + M \int d\lambda z(\lambda) [\partial_\lambda \log z(\lambda)]^2. \tag{23}$$

Equation (23) says that the generalized Fisher information is the sum of two terms, the first is simply the average of the Fisher information over the *a priori* distribution

whereas the second term is the Fisher information of the priori distribution itself. As expected, in the asymptotic limit of many measurements the *a priori* distribution is no longer relevant. The quantity Z_F is upper bounded by the analogue expression Z_H where the average of the Fisher information is replaced by the average of the QFI $H(\lambda)$ The resulting quantum Van Trees bound may be easily written as

$$\overline{\mathrm{Var}(\lambda)} \geq \frac{1}{Z_H}. \tag{24}$$

3. Estimability of a Parameter

A large signal is easily estimated whereas a quantity with a vanishing value may be inferred only if the corresponding estimator is very *precise* i.e. characterized by a small variance. This intuitive statement indicates that in assessing the performances of an estimator and, in turn, the overall estimability of a parameter, the relevant figure of merit is the scaling of the variance with the mean value rather than its absolute value. This feature may be quantified by means of the signal-to-noise ratio (for a single measurement)

$$R_\lambda = \frac{\lambda^2}{\mathrm{Var}(\lambda)}$$

which is larger for better estimators. Using the quantum Cramer-Rao bound one easily derives that the signal-to-noise ratio of any estimator is bounded by the quantity

$$R_\lambda \leq Q_\lambda \equiv \lambda^2 H(\lambda)$$

which we refer to as the quantum signal-to-noise ratio. We say that a given parameter λ is effectively estimable quantum-mechanically when the corresponding Q_λ is large.

Upon taking into account repeated measurements we have that the number of measurements leading to a 99.9% (3σ) confidence interval corresponds to a relative error

$$\delta^2 = \frac{9\mathrm{Var}(\lambda)}{M\lambda^2} = \frac{9}{M}\frac{1}{Q_\lambda} = \frac{9}{M\lambda^2 H(\lambda)}$$

Therefore, the number of measurements needed to achieve a 99.9% confidence interval with a relative error δ scales as

$$M_\delta = \frac{9}{\delta^2}\frac{1}{Q_\lambda}$$

In other words, a vanishing Q_λ implies a diverging number of measurements to achieve a given relative error, whereas a finite value allows estimation with arbitrary precision at finite number of measurements.

4. Examples

In this section we provide explicit evaluation of the symmetric logarithmic derivative and the quantum Fisher information for relevant families of quantum states,

including sets of pure states and the generic unitary family. We also consider the multiparameter case and the problem of repametrization.

4.1. *Unitary families and the pure state model*

Let us consider the case where the parameter of interest is the amplitude of a unitary perturbation imposed to a given initial state ϱ_0. The family of quantum states we are dealing with may be expressed as $\varrho_\lambda = U_\lambda \varrho_0 U_\lambda^\dagger$ where $U_\lambda = \exp\{-i\lambda G\}$ is a unitary operator and G is the corresponding Hermitian generator. Upon expanding the unperturbed state in its eigenbasis $\varrho_0 = \sum \varrho_n |\varphi_n\rangle\langle\varphi_n|$ we have $\varrho_\lambda = \sum_n \varrho_n |\psi_n\rangle\langle\psi_n|$ where $|\psi_n\rangle = U_\lambda|\varphi_n\rangle$. As a consequence we have

$$\partial_\lambda \varrho_\lambda = i U_\lambda [G, \varrho_0] U_\lambda^\dagger.$$

and the SLD is may be written as $L_\lambda = U_\lambda L_0 U_\lambda^\dagger$ where L_0 is given by

$$\begin{aligned} L_0 &= 2i \sum_{n,m} \frac{\langle\varphi_m|[G,\varrho_0]|\varphi_n\rangle}{\varrho_n + \varrho_m} |\varphi_n\rangle\langle\varphi_m| \\ &= 2i \sum_{n\neq m} \langle\varphi_m|G|\varphi_n\rangle \frac{\varrho_n - \varrho_m}{\varrho_n + \varrho_m} |\varphi_n\rangle\langle\varphi_m|. \end{aligned} \tag{25}$$

The corresponding quantum Fisher information is independent on the value of parameter and may be written in compact form as

$$H = \mathrm{Tr}[\varrho_0 L_0^2] = \mathrm{Tr}[\varrho_0 [L_0, G]] = \mathrm{Tr}[L_0 [G, \varrho_0]] = \mathrm{Tr}[G [\varrho_0, L_0]]$$

or, more explicitly, as

$$H = 2 \sum_{n\neq m} \sigma_{nm} G_{nm}^2$$

where the elements σ_{nm} are given in Eq. (18), or equivalently (19), and $G_{nm} = \langle\varphi_n|G|\varphi_m\rangle = \langle\psi_n|G|\psi_m\rangle$ denote the matrix element of the generator G in either the eigenbasis of ϱ_0 or ϱ_λ.

For a generic family of pure states we have $\varrho_\lambda = |\psi_\lambda\rangle\langle\psi_\lambda|$. Since $\varrho_\lambda^2 = \varrho_\lambda$ we have $\partial_\lambda \varrho_\lambda = \partial_\lambda \varrho_\lambda \, \varrho_\lambda + \varrho_\lambda \partial_\lambda \varrho_\lambda$ and thus $L_\lambda = 2\partial_\lambda \varrho_\lambda = |\psi_\lambda\rangle\langle\partial_\lambda\psi_\lambda| + |\partial_\lambda\psi_\lambda\rangle\langle\psi_\lambda|$. Finally we have

$$H(\lambda) = 4[\langle\partial_\lambda\psi_\lambda|\partial_\lambda\psi_\lambda\rangle + (\langle\partial_\lambda\psi_\lambda|\psi_\lambda\rangle)^2] \tag{26}$$

For a unitary family of pure states $|\psi_\lambda\rangle = U_\lambda|\psi_0\rangle$ we have

$$\begin{aligned} |\partial_\lambda\psi_\lambda\rangle &= -iGU_\lambda|\psi_0\rangle = -iG|\psi_\lambda\rangle, \\ \langle\partial_\lambda\psi_\lambda|\partial_\lambda\psi_\lambda\rangle &= \langle\psi_0|G^2|\psi_0\rangle, \\ \langle\partial_\lambda\psi_\lambda|\psi_\lambda\rangle &= -i\langle\psi_0|G|\psi_0\rangle. \end{aligned}$$

The quantum Fisher information thus reduces to the simple form

$$H = 4\langle\psi_0|\Delta G^2|\psi_0\rangle \tag{27}$$

which is independent on λ and proportional to the fluctuations of the generator on the unperturbed state. Using Eq. (27) the quantum Cramer-Rao bound in (8) rewrites in the appealing form[27]

$$\mathrm{Var}(\lambda)\langle\Delta G^2\rangle \geq \frac{1}{4M}, \tag{28}$$

which represents a parameter-based uncertainty relation which applies also when the shift parameter λ in the unitary $U_\lambda = e^{-i\lambda G}$ *does not* correspond to the observable canonically conjugate to G. When the unperturbed state is not pure the QFI may be written as

$$H = 4\,\mathrm{Tr}[\Delta G^2 \varrho_0] + 4\sum_n \varrho_n \langle\varphi_n|\langle G\rangle^2 - 2GK^{(n)}G|\varphi_n\rangle \tag{29}$$

$$K^{(n)} = \sum_m \frac{\varrho_m}{\varrho_n + \varrho_m}|\varphi_m\rangle\langle\varphi_m| \overset{\varrho_0 \to |\varphi_0\rangle\langle\varphi_0|}{\longrightarrow} \frac{1}{2}|\varphi_0\rangle\langle\varphi_0| \tag{30}$$

and Eq. (28) becomes

$$\mathrm{Var}(\lambda)\langle\Delta G^2\rangle \geq \frac{1}{4M}\left[1 + \sum_n \varrho_n \langle\varphi_n|\langle G\rangle^2 - 2GK^{(n)}G|\varphi_n\rangle\right]^{-1}. \tag{31}$$

The second term in Eqs. (29) and (31) thus represents the *classical* contribution to uncertainty due to the mixing of the initial signal.

As we have seen, for unitary families of quantum states the QFI is independent on the value of the parameter. As a consequence the quantum signal-to-noise ratio Q_λ vanishes for vanishing λ and thus the number of measurements needed to achieve a relative error δ diverges as $M_\delta \sim (\delta\lambda)^{-2}$.

4.2. *Quantum operations*

Let us now consider a family of quantum states obtained from a given inital state ϱ_0 by the action of a generic quantum operation $\varrho_\lambda = \mathcal{E}_\lambda(\varrho_0) = \sum_k M_{k\lambda}\varrho_0 M_{k\lambda}^\dagger$. Upon writing the initial and the evolved states in terms of their eigenbasis $\varrho_0 = \sum_s \varrho_{0s}|\varphi_s\rangle\langle\varphi_s|$, $\varrho_\lambda = \sum_s \varrho_n|\psi_n\rangle\langle\psi_n|$ we may evaluate the SLD and the quantum Fisher information using Eqs. (12) and (13) where

$$\varrho_n = \sum_{ks} \varrho_{0s}|\langle\psi_n|M_{k\lambda}|\varphi_s\rangle|^2 \tag{32}$$

$$\begin{aligned}\langle\psi_m|\partial_\lambda\varrho_\lambda|\psi_n\rangle = \sum_{ks} \varrho_{0s}[&\langle\psi_m|\partial_\lambda M_{k\lambda}|\varphi_s\rangle\langle\varphi_s|M_{k\lambda}^\dagger|\psi_n\rangle \\ &+ \langle\psi_m|M_{k\lambda}|\varphi_s\rangle\langle\varphi_s|\partial_\lambda M_{k\lambda}^\dagger|\psi_n\rangle].\end{aligned} \tag{33}$$

For a pure state at the input $\varrho_0 = |\psi_0\rangle\langle\psi_0|$ the above equation rewrites without the sum over s.

4.3. *Multiparametric models and reparametrization*

In situations where more than one parameter is involved, the family of quantum states $\varrho_{\boldsymbol{\lambda}}$ depends on a set $\boldsymbol{\lambda} = \{\lambda_\mu\}$, $\mu = 1, \ldots, N$. In this cases the relevant object in the estimation problem is given by the so-called quantum Fisher information matrix, whose elements are defined as

$$\begin{aligned}\boldsymbol{H}(\boldsymbol{\lambda})_{\mu\nu} &= \mathrm{Tr}\left[\varrho_{\boldsymbol{\lambda}}\frac{L_\mu L_\nu + L_\nu L_\mu}{2}\right] = \mathrm{Tr}[\partial_\nu \varrho_{\boldsymbol{\lambda}} L_\mu] = \mathrm{Tr}[\partial_\mu \varrho_{\boldsymbol{\lambda}} L_\nu] \\ &= \sum_n \frac{(\partial_\mu \varrho_n)(\partial_\nu \varrho_n)}{\varrho_n} + \sum_{n\neq m} \frac{(\varrho_n - \varrho_m)^2}{\varrho_n + \varrho_m} \\ &\quad \times [\langle\psi_n|\partial_\mu\psi_m\rangle\langle\partial_\nu\psi_m|\psi_n\rangle + \langle\psi_n|\partial_\nu\psi_m\rangle\langle\partial_\mu\psi_m|\psi_n\rangle] \end{aligned} \tag{34}$$

where L_μ is the SLD corresponding to the parameter λ_μ. The Cramer-Rao theorem for multiparameter estimation says that the inverse of the Fisher matrix provides a lower bound on the covariance matrix $\mathrm{Cov}[\boldsymbol{\gamma}]_{ij} = \langle\lambda_i\lambda_j\rangle - \langle\lambda_i\rangle\langle\lambda_j\rangle$, i.e.

$$\mathrm{Cov}[\boldsymbol{\gamma}] \geq \frac{1}{M}\boldsymbol{H}(\boldsymbol{\lambda})^{-1}$$

The above relation is a matric inequality and the corresponding bound may not be achievable achievable in a multiparameter estimation. On the other hand, the diagonal elements of the inverse Fisher matrix provide achievable bounds for the variances of single parameter estimators *at fixed value* of the others, in formula

$$\mathrm{Var}(\lambda_\mu) = \gamma_{\mu\mu} \geq \frac{1}{M}(\boldsymbol{H}^{-1})_{\mu\mu}. \tag{35}$$

Of course, for a diagonal Fisher matrix $\mathrm{Var}(\lambda_\mu) \geq 1/\boldsymbol{H}_{\mu\mu}$.

Let us now suppose that the quantity of interest g is a known function $g(\boldsymbol{\lambda})$ of the parameters used to label the family of states. In this case we need to reparametrize the familiy with a new set of parameters $\widetilde{\boldsymbol{\lambda}} = \{\widetilde{\lambda}_j = \widetilde{\lambda}_j(\boldsymbol{\lambda})$ that includes the quantity of interest, e.g. $\widetilde{\lambda}_1 \equiv g(\boldsymbol{\lambda})$. Since $\widetilde{\partial}_\mu = \sum_\nu B_{\mu\nu}\partial_\nu$ where $B_{\mu\nu} = \partial\lambda_\nu/\partial\widetilde{\lambda}_\mu$ it is easy to prove that

$$\widetilde{L}_\mu = \sum_\nu B_{\mu\nu} L_\nu \quad \widetilde{\boldsymbol{H}} = \boldsymbol{B}\boldsymbol{H}\boldsymbol{B}^T.$$

The ultimate precision on the estimation of g at fixed values of the other parameters is thus given by

$$\mathrm{Var}(g) \geq \frac{1}{M}(\widetilde{\boldsymbol{H}}^{-1})_{11}$$

5. Geometry of Quantum Estimation

The estimability of a set of parameters labelling the family of quantum states $\{\varrho_{\boldsymbol{\lambda}}\}$ is naturally related to the distinguishability of the states within the quantum statistical model i.e. with the notions of distance. On the manifold of quantum states, however, different distances may be defined and a question arises on which of

them captures the notion of estimation measure. As it can be easily proved it turns out that the Bures distance[30–36] is the proper quantity to be taken into account. This may be seen as follows. The Bures distance between two density matrices is defined as $D_B^2(\varrho,\sigma) = 2[1-\sqrt{F(\varrho,\sigma)}]$ where $F(\varrho,\sigma) = (\mathrm{Tr}[\sqrt{\sqrt{\varrho}\sigma\sqrt{\varrho}}])^2$ is the fidelity. The Bures metric $g_{\mu\nu}$ is obtained upon considering the distance for two states obtained by an infinitesimal change in the value of the parameter

$$d_B^2 = D_B^2(\varrho_{\boldsymbol{\lambda}}, \varrho_{\boldsymbol{\lambda}+d\boldsymbol{\lambda}}) = g_{\mu\nu} d\lambda_\mu d\lambda_\nu .$$

By explicitly evaluating the Bures distance[37] one arrives at $g_{\mu\nu} = 1/4\boldsymbol{H}_{\mu\nu}(\boldsymbol{\lambda})$, i.e. the Bures metric is simply proportional to the QFI, which itself is symmetric, real and positive semidefinite, i.e. represents a metric for the manifold underlying the quantum statistical model. Indeed, a large QFI for a given λ implies that the quantum states $\varrho_{\boldsymbol{\lambda}}$ and $\varrho_{\boldsymbol{\lambda}+d\boldsymbol{\lambda}}$ should be statistically distinguishable more effectively than the analogue states for a value λ corresponding to smaller QFI. In other words, one confirms the intuitive picture in which optimal estimability (that is, a diverging QFI) corresponds to quantum states that are sent far apart upon infinitesimal variations of the parameters.

The structures described above are pictorially described in Fig. 1. The idea is that any measurement aimed to estimate the parameters $\boldsymbol{\lambda}$ turns the set of parameters into a statistical differential manifold endowed with the Fisher metric $\boldsymbol{F}_{\mu\nu}(\boldsymbol{\lambda})$. On the other hand, when the parameters are mapped into the manifold of quantum states the statistical distance is expressed in terms of the Bures metric. The connection between the two constructions is provided by the optimization of the estimation procedure over quantum measurements, which shows that the Quantum

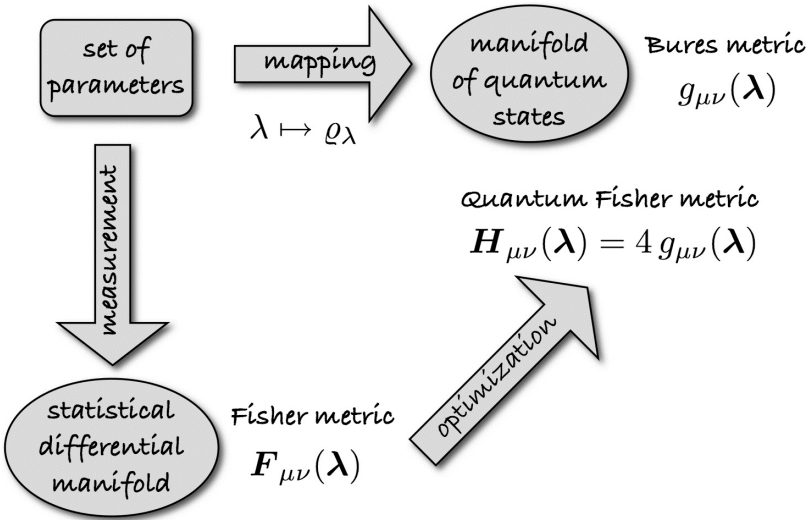

Fig. 1. Geometry of quantum estimation.

Fisher metric $\boldsymbol{H}_{\mu\nu}(\boldsymbol{\lambda})$ is the bound to $\boldsymbol{F}_{\mu\nu}(\boldsymbol{\lambda})$ and coincides, apart from a factor four, with the Bures metric.

6. Conclusion and Outlook

As a matter of fact, there are many quantities of interest that do not correspond to any quantum observable. Among these, we mention the amount of entanglement and the purity of a quantum state and the coupling constant of an interaction Hamiltonian or a quantum operation. In these situations, the values of the quantity of interest can be indirectly inferred by an estimation procedure, i.e. by measuring one or more proper observables, a quantum estimator, and then manipulating the outcomes by a suitable classical processing.

In this paper, upon exploiting the geometric theory of quantum estimation, we have described a general method to solve a quantum statistical model, i.e. to find the optimal quantum estimator and to evaluate the corresponding bounds to precision. To this aim we used the quantum Cramer-Rao theorem and the explicit evaluation of the quantum Fisher information matrix. We have derived the explicit form of the optimal observable in terms of the symmetric logarithmic derivative and evaluated the corresponding bounds to precision, which represent the ultimate bound posed by quantum mechanics to the precision of parameter estimation. For unitary families of quantum states the bounds may expressed in the form of a parameter-based uncertainty relation.

The analysis reported in this paper has a fundamental interest and represents a relevant tool in the design of realistic quantum information protocols. The approach here outlined is currently being applied to the estimation of entanglement[38] and the coupling constant of an interaction Hamiltonian.[25,39]

Acknowledgments

The author thanks Paolo Giorda, Alex Monras, Paolo Zanardi, Marco Genoni, Michael Korbman, Carmen Invernizzi and Stefano Olivares for stimulating discussions.

References

1. C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, New York, 1976); A. S. Holevo, *Statistical Structure of Quantum Theory*, Lect. Not. Phys. **61** (Springer, Berlin, 2001).
2. C. W. Helstrom, *Phys. Lett. A* **25** (1967) 1012.
3. H. P. Yuen, M. Lax, *IEEE Trans. Inf. Th.* **19** (1973) 740.
4. C. W. Helstrom, R. S. Kennedy, *IEEE Trans. Inf. Th.* **20** (1974) 16.
5. S. Braunstein and C. Caves, *Phys. Rev. Lett.* **72** (1994) 3439.
6. S. Braunstein, C. Caves and G. Milburn, *Ann. Phys.* **247** (1996) 135.
7. O. E. Barndorff-Nielsen, R. D. Gill, *J. Phys. A* **33** (2000) 4481.
8. A. S. Holevo, *Rep. Math. Phys.* **16** (1979) 385.

9. M. D'Ariano *et al.*, *Phys. Lett. A* **248** (1998) 103.
10. C. W. Helstrom, *Found. Phys.* **4** (1974) 453.
11. G. J. Milburn *et al.*, *Phys. Rev. A* **50** (1994) 801.
12. G. Chiribella *et al.*, *Phys. Rev. A* **73** (2006) 062103.
13. G. M. D'Ariano, M. G. A. Paris and P. Perinotti, *J. Opt. B* **3** (2001) 337.
14. A. Monras, *Phys. Rev. A* **73** (2006) 033821.
15. M. Sarovar and G. J. Milburn, *J. Phys. A* **39** (2006) 8487.
16. M. Hotta *et al.*, *Phys. Rev. A* **72** (2005) 052334; *J. Phys. A* **39** (2006).
17. A. Fujiwara, *Phys. Rev. A* **63** (2001) 042304; A. Fujiwara, H. Imai, *J. Phys. A* **36** (2003) 8093.
18. J. Zhenfeng *et al.*, preprint LANL quant-ph/0610060.
19. A. Monras and M. G. A. Paris, *Phys. Rev. Lett.* **98** (2007) 160401.
20. V. D'Auria *et al.*, *J. Phys. B* **39** (2006) 1187.
21. P. Grangier *et al.*, *Phys. Rev. Lett.* **59** (1987) 2153.
22. E. S. Polzik *et al.*, *Phys. Rev. Lett.* **68** (1992) 3020.
23. B. R. Frieden, *Opt. Comm.* **271** (2007) 7.
24. S. Boixo and A. Monras, *Phys. Rev. Lett.* **100** (2008) 100503.
25. P. Zanardi and M. G. A. Paris, arXiv:0708.1089.
26. H. Cramer, *Mathematical Methods of Statistics* (Princeton University Press, 1946).
27. L. Maccone, *Phys. Rev. A* **73** (2006) 042307.
28. H. L. Van Trees, *Detection, Estimation, Modulation Theory* (Wiley, New York, 1967).
29. R. D. Gill and B. Y. Levit, *Bernoulli* **1** (1995) 59.
30. D. J. C. Bures, *Trans. Am. Math. Phys.* **135** (1969) 199.
31. A. Uhlmann, *Rep. Math. Phys.* **9** (1976) 273.
32. R. Josza, *J. Mod. Opt.* **41** (1994) 2315.
33. M. Hübner, *Phys. Lett. A* **163** (1992) 239.
34. P. B. Slater, *J. Phys. A* **29** (1996) L271; *Phys. Lett. A* **244** (1998) 35.
35. M. J. W. Hall, *Phys. Lett. A* **242** (1998) 123.
36. J. Dittmann, *J. Phys. A* **32** (1999) 2663.
37. H-J. Sommers *et al.*, *J. Phys. A* **36** (2003) 10083.
38. M. G. Genoni, P. Giorda and M. G. A. Paris, preprint arXiv:0804.1705.
39. C. Invernizzi, M. Korbman, L. Campos and M. G. A. Paris, preprint arXiv:0807.3213.